Contemporary Mathematics
Volume **555**, 2011

A Note on Reductions of Monomial Ideals in $k[x, y]_{(x,y)}$

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We dedicate this paper to Professor Wolmer Vasconcelos.

Abstract. We consider monomial ideals in the two-dimensional localized polynomial ring $k[x, y]_{(x,y)}$ where k is an infinite field. The goal of this paper is to determine a sufficient condition under which an ideal containing $x^a y^b + x^c y^d$ is a reduction of an ideal containing $x^a y^b$ and $x^c y^d$. The main theorem states and provides a means to verify a condition that implies the integral closure of an ideal is a monomial ideal. As a corollary, we form an algorithm to obtain a minimal reduction of an arbitrary non-principal monomial ideal. We also demonstrate an application of the main results on the monomial ideals discussed in this paper to the computation of the Buchsbaum-Rim multiplicity of a module under certain conditions.

1. Introduction and Motivation

Let R be a commutative ring with identity and let $J \subseteq I$ be ideals of R. I is integral over J, if for every element u in I, there exist elements a_i in J^i such that

$$u^n + a_1 u^{n-1} + a_2 u^{n-2} + \cdots + a_{n-1} u + a_n = 0.$$

J is a reduction of I, if $I^{m+1} = JI^m$ for some positive integer m. When R is Noetherian, these two definitions are equivalent; in other words, I is integral over J if and only if J is a reduction of I (*c.f.* [**RS**, 1.1]). The reduction of ideals was first introduced by Northcott and Rees [**NR**] and Rees [**R**] extended the notion to modules. Since then the reduction of ideals and modules have been discussed extensively.

It is known that the ideals with reduction relation have the same integral closure and that they, if $\mathfrak{m}$-primary, have the same Hilbert-Samuel multiplicity. The main aim of this paper is to discuss the reductions of monomial ideals in a two-dimensional localized polynomial ring $R = k[x, y]_{(x,y)}$ over an infinite field. In particular, for a given ideal I in R, there is a monomial ideal I^* naturally arising from I (see definition below). Theorem 3.3 provides an algorithm to determine if

1991 *Mathematics Subject Classification.* Primary 13B22, Secondary 13A30, 13B25, 13F20.
Key words and phrases. monomidal ideals, reductions, minimal reductions, integral closures, Hilbert Samuel multiplicity, monomial modules, Buchsbaum-Rim multiplicity.
The first author was partially supported by FRCE Type B Grant #48780 and Early Career Investigator's Grant #C61368 of Central Michigan University.

I is a reduction of I^*. This is desirable since the integral closure and the Hilbert-Samuel multiplicity (if defined) both have graphical interpretations. So one can easily obtain this information for I if I^* is integral over I.

For the convenience of discussions in the latter sections, we elaborate here the graph associated to a monomial ideal and some relevant information one can read from it. Note that each monomial in a localized polynomial ring $k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ over a field k corresponds to a point in $\mathbb{Z}^d$. If $\mathfrak{a}$ is a monomial ideal in the above local ring, we define the *graph* of $\mathfrak{a}$, commonly called the *Newton polyhedron*, to be the set

$$G_{\mathfrak{a}} = \{(a_1, \ldots, a_d) \mid a_i \geq \alpha_i \ \forall i \text{ for some } x_1^{\alpha_1} \cdots x_d^{\alpha_d} \in \mathfrak{a}\} \subset \mathbb{R}^d.$$

Then, the integral closure of $\mathfrak{a}$ is again a monomial ideal (*c.f.* [**SH**, 1.4.2]) and is generated by those monomials corresponding to the integral lattice points in the convex hull of the graph of $\mathfrak{a}$ (*c.f.* [**SH**, 1.4.6]). If we further assume that $\mathfrak{a}$ is $\mathfrak{m}$-primary where $\mathfrak{m}$ is the unique maximal ideal in $k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$, then the Hilbert-Samuel multiplicity is

$$\text{(1.1)} \quad \mathrm{e}(\mathfrak{a}) = d! \times \text{the volume of the complement of the convex hull of } G_{\mathfrak{a}} \text{ in } \mathbb{R}^d_{\geq 0}.$$

(*c.f.* [**R2**, Exercise 2.8 (b)]).

In this paper, we assume that R is two-dimensional, *i.e.,* $R = k[x, y]_{(x,y)}$. Let I be the ideal generated by $f_i = \sum \mu_{ij} x^{a_{ij}} y^{b_{ij}}$ with $\mu_{ij} \neq 0$ and $i = 1, \ldots, m$. All the monomials occurring in the f_i for all i together generate a monomial ideal, denoted I^*. A simple exercise shows that such monomial ideal is independent from the choices of the generating set $\{f_i\}$. Indeed I^* is the smallest monomial ideal containing I (see also Remark 3.1). We focus on finding a sufficient condition under which I becomes a reduction of I^*. In such case, the integral closure and, if I is $\mathfrak{m}$-primary, the Hilbert-Samuel multiplicity of I can be obtained straightforwardly from the graph of I^* as stated in the previous paragraph.

A reduction of an ideal is *minimal* if it is minimal with respect to inclusion. In a Noetherian local ring $(R, \mathfrak{m})$, a reduction $\mathfrak{b}$ of an ideal $\mathfrak{a}$ is minimal if $\mathfrak{b}$ is generated by ℓ elements where ℓ is the *analytic spread* of $\mathfrak{a}$ (*i.e.*, the dimension of the fiber cone of $\mathfrak{a}$) (*c.f.* [**SH**, 8.3.5]). It is also known that $\mathrm{ht}\ \mathfrak{a} \leq \ell \leq \dim \mathrm{R}$ (*c.f.* [**SH**, 8.3.9]). Moreover when the residue field is infinite, then a reduction $\mathfrak{b}$ is minimal if and only if $\mathfrak{b}$ is minimally generated by ℓ elements (*c.f.* [**SH**, 8.3.5, 8.3.7]). In a Noetherian unique factorization local domain of dimension d with $d > 1$, all non-principal ideals have analytic spread $2 \leq \ell \leq d$. (An easy exercise shows that principal ideals in an UFD are integrally closed.) Most examples in this paper are taken from $k[x, y]_{(x,y)}$ which is a unique factorization local domain.

In particular, if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal in a local ring of dimension d with infinite residue field, then a reduction $\mathfrak{b}$ of $\mathfrak{a}$ is minimal if and only if $\mathfrak{b}$ is generated by d elements. In principal, $\mathfrak{b}$ can be chosen by taking d "general enough" combinations on the generators of $\mathfrak{a}$. Example 4.4, however, shows that false combinations exist easily. Let $R = k[x, y]_{(x,y)}$ with $|k| = \infty$. Deducing from the main result, Theorem 3.3, we give an algorithm of obtaining a minimal reduction of an arbitrary non-principal monomial ideal, not necessarily $\mathfrak{m}$-primary (Corollary 3.7). It was brought to our attention that V. C. Quiñonez has also achieved the same result as Corollary 3.7 in her research report [**Q**]. In Subsection 4.2, we briefly discuss the main theorem in [**Q**] and give a proof using Lemma 3.4

For an arbitrary monomial ideal $\mathfrak{a}$ in $R = k[x, y]_{(x,y)}$, it is known that the ideal $\mathfrak{b}$ generated by the monomials corresponding to the vertices of the graph $G_{\mathfrak{a}}$ of $\mathfrak{a}$ is a reduction of $\mathfrak{a}$. Singla [**S**, 2.1] proves that such $\mathfrak{b}$ is the *unique* minimal monomial reduction of $\mathfrak{a}$, where $\mathfrak{b}$ being a *minimal monomial reduction* means that no monomial ideal properly contained in $\mathfrak{b}$ is a reduction of $\mathfrak{a}$. Note that the minimal monomial reduction of a non-principal monomial ideal $\mathfrak{a}$ is almost never a minimal reduction unless $G_{\mathfrak{a}}$ contains only two vertices. For example, for the ideal $\mathfrak{a}_1 = (xy^5, x^2y^4, x^3y)$, the vertices of $G_{\mathfrak{a}_1}$ are $(1, 5)$ and $(3, 1)$, and so the ideal $\mathfrak{b}_1 = (xy^5, x^3y)$ is the unique minimal monomial reduction as well as a minimal reduction of $\mathfrak{a}_1$ since $\mathfrak{b}_1$ is generated by two elements. In fact from the discussion in the previous paragraphs, every non-principal ideal in $R = k[x, y]_{(x,y)}$ has analytic spread two, so a reduction is minimal if and only if it is generated by exactly two elements. For the ideal $\mathfrak{a}_2 = (xy^5, x^2y^2, x^3y)$, $\mathfrak{a}_2$ is its own minimal monomial reduction by inspecting $G_{\mathfrak{a}_2}$ but not a minimal reduction of itself since $\mathfrak{a}_2$ is minimally generated by three elements. On the other hand, $\mathfrak{b}_2 = (xy^5+x^3y, x^2y^2)$ is a reduction of $\mathfrak{a}_2$ by Corollary 3.7 and is not a minimal monomial reduction. In [**S**] Singla studies monomial reductions of monomial ideals in the polynomial ring $k[x_1, x_2, \ldots, x_n]$ and proves that every monomial ideal has a unique minimal monomial reduction. In this paper we intend to find, for a non-monomial ideal $\mathfrak{a}$, conditions that guarantee the existence of a monomial ideal which is integral over $\mathfrak{a}$.

The notion of the Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal is generalized to a module satisfying certain conditions by Buchsbaum and Rim in 1960s [**BR**]. It has attracted attention of both algebraists and geometers. Many nice properties of $\mathfrak{m}$-primary ideals are then extended to modules, especially those in the reduction theory (see Section 2). Buchsbaum-Rim multiplicity is also used to classify singularities of certain types (*c.f.* [**G**]).

Attempting to generalize the computation in (1.1) to Buchsbaum-Rim multiplicity, Jones [**J**] considers $R = k[x, y]_{(x,y)}$ and proves that the Buchsbaum-Rim multiplicity of so-called monomial modules (see definition in Section 5) of lower rank has a graphical computation. In particular, Jones breaks her computations into seven cases and for each case she gives a graphical interpretation. We present a formula that summarizes the seven individual cases in [**J**]. The multiplicity formula presented in this paper should be viewed as an improvement of the result in [**J**] instead of recovering. In fact, our formula is achieved by using Theorem 3.3, Corollary 3.7, and some computations in [**J**].

The paper is arranged as follows. Section 2 gives definitions of notation and reviews of some propositions that will be applied often. Section 3 proves the main result in monomial ideals. The proof of Theorem 3.3 utilizing properties on fiber cones is broken down to several lemmas since each lemma has its own independent interest. Section 4 mainly consists of examples and an application of Lemma 3.4 to Quiñonez's main theorem in [**Q**]. Indeed we suggest the readers first to look at the examples in Section 4 before jumping into the proof of Theorem 3.3. The examples in Section 4 are meant to provide better intuition for the condition in Theorem 3.3 which is very straightforward once visualized. Also we wish to point out that the condition in Theorem 3.3 is a sufficient condition. It is not clear, at least to the authors, whether or not it is possible to establish a necessary condition. These are discussed in Section 4 as well.

Finally, Section 5 is devoted to the relationship between the Buchsbaum-Rim multiplicity of a module and the Hilbert-Samuel multiplicity of certain ideals related to the module. The discussion in this section leads to the formula in Theorem 5.3 that is also proved in our joint work with B. Ulrich [**CLU**] in a more general setting. Here the authors would like to provide the original idea and the motivation of how such result is formulated using the reduction theory on monomial ideals discussed in Section 3. We refer the readers to [**CLU**] for more generalized results.

Acknowledgement. We would like to express our gratitude to William Heinzer for helpful discussions on the reduction of ideals and for his endless encouragement throughout the course of writing this paper.

2. Notation and Background

Assume R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Given an ideal $\mathfrak{a}$ of R, there are two rings that are used frequently in the study of reduction, namely the *Rees algebra* $R[\mathfrak{a}t]$ of $\mathfrak{a}$ and the *fiber cone* $R[\mathfrak{a}t]/\mathfrak{m}R[\mathfrak{a}t]$ of $\mathfrak{a}$. It can be seen directly from the definition that a subideal $\mathfrak{b} \subseteq \mathfrak{a}$ is a reduction if and only if the Rees algebra $R[\mathfrak{a}t]$ of $\mathfrak{a}$ is integral over the Rees algebra $R[\mathfrak{b}t]$ of $\mathfrak{b}$. In fact, a similar result holds for their fiber cones, *i.e.*, a subideal $\mathfrak{b} \subseteq \mathfrak{a}$ is a reduction if and only if the fiber cone of $\mathfrak{a}$ is integral over that of $\mathfrak{b}$ via the homomorphism induced by the inclusion $R[\mathfrak{b}t] \subseteq R[\mathfrak{a}t]$ (*c.f.* [**SH**, 8.2.4]). In the proof of our main result Theorem 3.3, we use this equivalent condition on the fiber cones to verify the reduction relation.

If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of R, then there exists a polynomial $P_{\mathfrak{a}}(n)$ of degree d, where d is the dimension of R, such that $P_{\mathfrak{a}}(n) = \ell(R/\mathfrak{a}^n)$ for all large n. This is called the *Hilbert-Samuel polynomial* and the coefficient of $\frac{n^d}{d!}$ is called the *Hilbert-Samuel multiplicity*, denoted $\mathrm{e}(\mathfrak{a})$. If we further assume that R is Cohen-Macaulay with infinite residue field, then given an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, it is known that an $\mathfrak{m}$-primary subideal $\mathfrak{b}$ of $\mathfrak{a}$ is a reduction of $\mathfrak{a}$ if and only if $\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\mathfrak{b})$. Furthermore, if $\mathfrak{b}$ is a minimal reduction of $\mathfrak{a}$, then

$$\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\mathfrak{b}) = \ell(R/\mathfrak{b}). \tag{2.1}$$

Let F be a free R-module of rank r and let M be a submodule of F with $\ell(F/M) < \infty$. Buchsbaum and Rim [**BR**, 3.4] prove that there exists a polynomial $\lambda(n)$ such that for all large $n \in \mathbb{N}$, $\lambda(n) = \ell(\mathcal{S}_n(F)/\mathcal{R}_n(M))$, where $\mathcal{S}(F) = \oplus_{n \geq 0}\mathcal{S}_n(F)$ is the symmetric algebra of F and $\mathcal{R}(M)$ is the R-subalgebra of $\mathcal{S}(F)$ generated by the image of M in $\mathcal{S}_1(F)$. It is also proved in [**BR**, 3.4] that the polynomial $\lambda(n)$ has degree $d+r-1$ unless $M = F$. The coefficient of $\frac{n^{d+r-1}}{(d+r-1)!}$ in this polynomial is defined to be the *Buchsbaum-Rim multiplicity* $\mathrm{br}(M)$. It is a positive integer whenever $M \neq F$ and only depends on F/M [**BR**, 3.3]. Note that if $r = 1$ and $M \neq F$, then M is an $\mathfrak{m}$-primary ideal of R and $\mathrm{br}(M) = \mathrm{e}(M)$, so the Buchsbaum-Rim multiplicity can be viewed as a generalization of the Hilbert-Samuel multiplicity.

Let U be a submodule of M. We write $\mathcal{R}(U)$ to be the R-subalgebra of $\mathcal{S}(F)$ generated by U. We say that U is a reduction of M if $\mathcal{R}(M)$ is integral over $\mathcal{R}(U)$ as rings. A free module has no proper reduction. Similar to the ideal case, a *minimal reduction* of M is a reduction that is minimal with respect to inclusion. When $M \neq F$ and the residue field of R is infinite, a reduction U of M is minimal if and only if its minimal number of generators is $d+r-1$ (*c.f.* [**BR**, 3.5], [**R**, 2.1 and 2.2], [**EHU**, page 707]).

Reductions of modules are closely related to Buchsbaum-Rim multiplicities. If U is a reduction of M then $\mathrm{br}(U) = \mathrm{br}(M)$ [**KT**, 6.3(i)], and the converse holds in case R is universally catenary and equidimensional with $d > 0$ (*c.f.* [**KR**, 4.11], [**KT**, 6.3(ii)], [**K**, 2.2], [**SUV1**, 5.5]). Furthermore, similar to the result (2.1) in ideals, if R is a Cohen-Macaulay local ring with infinite residue field and if U is a minimal reduction of M, then

$$\mathrm{br}(M) = \mathrm{br}(U) = \ell(F/U) \tag{2.2}$$

(*c.f.* [**BR**, 4.5(2)]).

Suppose M is a submodule of a free module F of rank r such that $F/M \cong I/J$ for two $\mathfrak{m}$-primary ideals I and J; for instance, if R is a Cohen-Macaulay ring of dimension at least two, one can choose I and J to be the Bourbaki ideal of F and M, respectively (*c.f.* [**B**, Chapter 7, no. 4], [**SUV2**, 3.2(a),(c)]). If $r \geq 2$ and if M is generated by three elements, then $M = F$ or $r = d = 2$ [**BR**, 3.5]. In this case, I and J can be chosen to be the unit ideal or an $\mathfrak{m}$-primary complete intersection. Since M is its own minimal reduction, by (2.2), we have the following equalities,

$$\mathrm{br}(M) = \ell(F/M) = \ell(R/J) - \ell(R/I) = \mathrm{e}(J) - \mathrm{e}(I).$$

From this we observe that not only does the Buchsbaum-Rim multiplicity generalize the Hilbert-Samuel multiplicity by definition and share parallel properties in the reduction theory as described earlier but also the two multiplicities are connected in such a special case. Such a relation was generalized in [**J**] when I and J are monomial ideals with small number of generators. In Section 5, we take the results in [**J**] one step further by formulating its outcome. The formula obtained in Theorem 5.3 also motivates the work in [**CLU**] for arbitrary modules over a two dimensional Gorenstein local ring.

3. Reductions of Monomial Ideals

For the rest of the paper, we let $R = k[x,y]_{(x,y)}$ be the polynomial ring $k[x,y]$ localized at the maximal ideal (x,y). We also assume that k is an infinite field. Note that by multiplying a suitable unit to a generator, we see that every ideal in R is generated by polynomials in $k[x,y]$. For every element f in $k[x,y]$, f can be written as a finite sum in distinct monomials; *i.e.*, $f = \sum \eta_{ij}x^iy^j$ with $\eta_{ij} \in k$ where we assume no repeated like terms in the expression. We use the following notation for the collection of the finitely many monomials occurring in f

$$\Gamma(f) = \{x^iy^j \mid f = \sum \eta_{ij}x^iy^j \text{ and } \eta_{ij} \neq 0\}.$$

REMARK 3.1. Let I be an ideal of R and suppose it is generated by $f_1, \ldots, f_m \in k[x,y]$. If I' is a monomial ideal containing I, then it is clear that I' contains $\Gamma(f_1) \cup \cdots \cup \Gamma(f_m)$. Hence the smallest monomial ideal containing I is generated by $\Gamma(f_1) \cup \cdots \cup \Gamma(f_m)$. We denote this monomial ideal by I^*.

We are interested in conditions under which I^* becomes integral over I.

QUESTION 3.2. Under what conditions is I a reduction of I^*?

Theorem 3.3 states a sufficient condition, in terms of monomials in $\Gamma(f_1)\cup\cdots\cup\Gamma(f_m)$, for I to be a reduction of I^*. As an application, Corollary 3.7 provides a minimal reduction of a given monomial ideal.

THEOREM 3.3. *Let $R = k[x,y]_{(x,y)}$ and $|k| = \infty$. Let I be an ideal of R generated by $f_1, \ldots, f_m \in k[x,y]$. Assume that the following is true: for all $i = 1,2,\ldots,m$ and for any two distinct monomials x^ay^b and x^cy^d in $\Gamma(f_i)$ with $c < a$ and $b < d$, there exists $x^ry^s \in \Gamma(f_j)$ for some j such that the point (r,s) lies on the left hand side of the line through (a,b) and (c,d). Then I is a reduction of I^*.*

Prior to proving this theorem, we discuss several supporting lemmas.

LEMMA 3.4. *Let $k[u_1,u_2,\ldots,u_n]$ be a k-algebra and consider its k-subalgebra $k[\eta_1u_1+\eta_2u_2+\cdots+\eta_nu_n]$ for nonzero $\eta_1,\ldots,\eta_n$ in k. For each $i = 1,2,\ldots,n$, suppose there are positive integers α_{ij},β_{ij} such that $u_i^{\alpha_{ij}}u_j^{\beta_{ij}} = 0$ for all $j \neq i$. Then u_i is integral over $k[\eta_1u_1+\eta_2u_2+\cdots+\eta_nu_n]$. Consequently, $k[u_1,\ldots,u_n]$ is integral over $k[\eta_1u_1+\eta_2u_2+\cdots+\eta_nu_n]$.*

PROOF. First, we show that the lemma can be reduced to proving the case where $\eta_1 = \eta_2 = \cdots = \eta_n = 1$. Note that $u_i^{\alpha_{ij}}u_j^{\beta_{ij}} = 0$ implies $(\eta_iu_i)^{\alpha_{ij}}(\eta_ju_j)^{\beta_{ij}} = 0$. Therefore, once the case where $\eta_1 = \eta_2 = \cdots = \eta_n = 1$ is proved, then by replacing u_ℓ by $\eta_\ell u_\ell$ for all $\ell = 1,2,\ldots n$, we have $k[\eta_1u_1,\ldots,\eta_nu_n]$ is integral over $k[\eta_1u_1+\eta_2u_2\cdots+\eta_nu_n]$. Since $\eta_1,\ldots,\eta_n$ are all units in k, $k[u_1,u_2,\ldots,u_n] = k[\eta_1u_1,\eta_2u_2,\ldots,\eta_nu_n]$. This gives the general case. Hence, it suffices to show that $k[u_1,u_2,\ldots,u_n]$ is integral over $k[u_1+u_2+\cdots+u_n]$ under the same hypothesis.

We prove the statement (with $\eta_1 = \cdots = \eta_n = 1$) by induction on n and assume $n \geq 2$ since the assertion is trivial for $n = 1$. We first show the base case $n = 2$. For simplicity on the notation, in this case we write $u_1^{\alpha_1}u_2^{\alpha_2} = 0$. This implies $u_1^{\alpha_1}u_2^{\alpha_2+1} = 0$ so we may assume that α_2 is odd. By a straightforward computation,

$$\begin{aligned} u_1^{\alpha_1+\alpha_2} &= u_1^{\alpha_1}(u_2^{\alpha_2}+u_1^{\alpha_2}) = u_1^{\alpha_1}\{[(u_1+u_2)-u_1]^{\alpha_2}+u_1^{\alpha_2}\} \\ &= u_1^{\alpha_1}[(u_1+u_2)^{\alpha_2}-\tbinom{\alpha_2}{1}(u_1+u_2)^{\alpha_2-1}u_1+\cdots \\ &\quad +\tbinom{\alpha_2}{\alpha_2-1}(u_1+u_2)u_1^{\alpha_2-1}]. \end{aligned}$$

We conclude

$$\begin{aligned} u_1^{\alpha_1+\alpha_2} - \tbinom{\alpha_2}{\alpha_2-1}(u_1+u_2)u_1^{\alpha_1+\alpha_2-1} + \cdots \\ +\tbinom{\alpha_2}{1}(u_1+u_2)^{\alpha_2-1}u_1^{\alpha_1+1} - (u_1+u_2)^{\alpha_2}u_1^{\alpha_1} = 0. \end{aligned}$$

Thus u_1 is integral over $k[u_1+u_2]$ and so is u_2.

Assume $n \geq 3$ and suppose the assertion holds for all k-algebras with $n-1$ or less generators. For the k-algebra $k[u_1,u_2,\ldots,u_n]$ with n generators, choose $\alpha = \max_{i,j}\{\alpha_{ij},\beta_{ij}\}$, then we have $u_i^\alpha u_j^\alpha = 0$ for all $i \neq j$. Consider the k-algebras

$$k[u_1+\cdots+u_{n-1}+u_n] \subseteq k[u_1+\cdots+u_{n-1},u_n] \subseteq k[u_1,\ldots,u_{n-1},u_n].$$

We claim that

(1) $k[u_1,\ldots,u_{n-1},u_n]$ is integral over $k[u_1+\cdots+u_{n-1},u_n]$, and
(2) $k[u_1+\cdots+u_{n-1},u_n]$ is integral over $k[u_1+\cdots+u_{n-1}+u_n]$.

For (1), since $u_i^\alpha u_j^\alpha = 0$ for all $i \neq j$, by the induction hypothesis, we have that $k[u_1,\ldots,u_{n-1}]$ is integral over $k[u_1+u_2+\cdots+u_{n-1}]$ and so $k[u_1,\ldots,u_{n-1},u_n]$ is integral over $k[u_1+u_2+\cdots+u_{n-1},u_n]$. For (2), consider the element $(u_1+u_2+\cdots+$

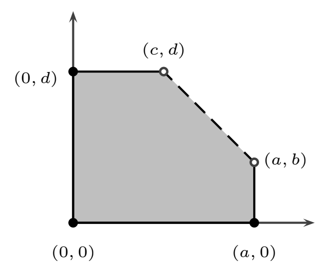
$u_{n-1})^{(n-1)\alpha}u_n^{\alpha}$. Note that after expanding $(u_1+u_2+\cdots+u_{n-1})^{(n-1)\alpha}$, all terms are of the form $u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_{n-1}^{\alpha_{n-1}}$ with $\alpha_1+\alpha_2+\cdots+\alpha_{n-1}=(n-1)\alpha$. Hence at least one of the α_i is larger than or equal to α. Therefore, $(u_1+u_2+\cdots+u_{n-1})^{(n-1)\alpha}u_n^{\alpha}=0$. Thus, by the case of $n=2$, $k[u_1+\cdots+u_{n-1},u_n]$ is integral over $k[u_1+\cdots+u_{n-1}+u_n]$. At last, it follows from (1) and (2) that $k[u_1,\ldots,u_n]$ is integral over $k[u_1+\cdots+u_n]$ and the proof is complete. □

The following two lemmas are deduced from algebraic inequalities regarding relative positions of a point and a line. They both play important roles in the proof of Theorem 3.3. Although only one implication in Lemma 3.5 will be needed, we prove an equivalent statement for completion.

LEMMA 3.5. *Let $(a,b),(c,d),(e,f)\in\mathbb{Z}^2_{\geq 0}$ with $a>c$ and $b<d$. Then the point (e,f) lies within the convex region with vertices (a,b), (c,d), $(0,d)$, $(0,0)$, $(a,0)$, including all boundaries except the line segment connecting (a,b) and (c,d), if and only if there exist nonnegative integers $\alpha,\beta,\gamma,\delta$ such that γ,δ are not both zero and that*

$$\left(x^ay^b\right)^{\alpha}\left(x^cy^d\right)^{\beta}=x^{\gamma}y^{\delta}\left(x^ey^f\right)^{\alpha+\beta}.$$

Simply speaking, the above monomial equality holds if and only if (e,f) is in the following shaded region:



PROOF. Suppose that (e,f) is in the assumed region. If $0\leq e\leq c<a$ and $0\leq f\leq d$ and $(e,f)\neq(c,d)$, then x^cy^d is a proper multiple of x^ey^f, so we may set $\alpha=0$, $\beta=1$, $\gamma=c-e$, and $\delta=d-f$ to achieve the monomial equality. Symmetrically, if $0\leq e\leq a$ and $0\leq f\leq b<d$ and $(e,f)\neq(a,b)$, we may take $\alpha=1$, $\beta=0$, $\gamma=a-e$, and $\delta=b-f$. The remaining case is when (e,f) lies in the interior region of the triangle with vertices (a,b), (c,d), (c,b). The fact that (e,f) is on the left hand side of the line through (a,b) and (c,d) implies

$$f(a-c)<b(e-c)+d(a-e).$$

By setting $\alpha=e-c$, $\beta=a-e$ and $\delta=b(e-c)+d(a-e)-f(a-c)$, we have

$$(x^ay^b)^{\alpha}(x^cy^d)^{\beta}=y^{\delta}(x^ey^f)^{\alpha+\beta}$$

in which α,β,δ are all positive integers for $c<e<a$.

Conversely, assume the existence of α,β,γ and δ. Then we have

$$(a-e)\alpha+(c-e)\beta=\gamma\quad\geq\quad 0 \tag{3.1}$$

$$(b-f)\alpha+(d-f)\beta=\delta\quad\geq\quad 0 \tag{3.2}$$

and γ and δ are not both zero. This implies α and β are not both zero either. First we claim that $e\leq a$. Suppose not, *i.e.*, $e>a$; this implies $(a-e)\alpha+(c-e)\beta<0$

since α and β are nonnegative and not both zero. This contradicts (3.1). Similarly, one can show that $f \leq d$. Therefore, to complete the proof, it suffices to show that (e, f) does not lie in the upper right triangular region (including the boundaries) with vertices $(a, b), (c, d)$ and (a, d). Suppose the contrary, then

$$c \leq e \leq a$$

$$b \leq f \leq d$$

$$af + bc + de - ad - be - cf \geq 0. \tag{3.3}$$

First, if $e = a$, then (3.1) becomes $(c - e)\beta = \gamma$ and this implies $\beta = \gamma = 0$ since $c - e = c - a < 0$ and $\beta, \gamma \geq 0$. Consequently, (3.2) becomes $(b - f)\alpha = \delta$ and this implies $\delta = 0$. This contradicts to the condition that γ and δ are not both zero. Therefore, $e \lneq a$. Suppose $f = b$. From (3.3) one conclude $(a - e)(b - d) \geq 0$. But this cannot be true because $b < d$ and $e < a$. So $f \gneq b$. Symmetrically, $f < d$ and $e > c$. Therefore, $c < e < a$ and $b < f < d$. Now, multiplying (3.2) by $(a - e)$, we have

$$(a - e)(b - f)\alpha + (a - e)(d - f)\beta = (a - e)\delta. \tag{3.4}$$

Replacing $(a - e)\alpha$ in (3.4) using (3.1), (3.4) becomes

$$(b - f)(e - c)\beta + (a - e)(d - f)\beta = (a - e)\delta + (f - b)\gamma. \tag{3.5}$$

Since $e < a$ and $b < f$ and since δ and γ are nonnegative and not both zero, the right hand side of (3.5) is positive. Thus, $(b - f)(e - c) + (a - e)(d - f) > 0$ since $\beta \geq 0$. This contradicts (3.3). Hence the point (e, f) must be in the desired region. □

LEMMA 3.6. *Let $(a, b), (c, d), (e, f) \in \mathbb{Z}^2_{\geq 0}$ with $a > c$ and $b < d$. If the point (e, f) lies within the triangular region bounded by the x-axis, the line $x = a$, and the line through (c, d) and (a, b) excluding the vertical side and hypotenuse, i.e., the shaded region in Figure 3.6.1,*

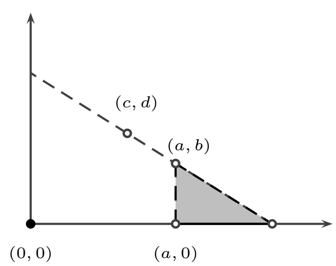


Figure 3.6.1.

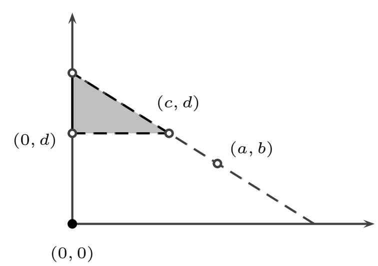


Figure 3.6.2.

then there exist positive integers α, β, γ such that

$$\left(x^a y^b\right)^{\alpha+\beta} = x^\gamma \left(x^e y^f\right)^\alpha \left(x^c y^d\right)^\beta.$$

Symmetrically, if the point (e, f) is in the shaded region in Figure 3.6.2, then there exist positive integers α, β, δ such that

$$\left(x^c y^d\right)^{\alpha+\beta} = y^\delta \left(x^a y^b\right)^\alpha \left(x^e y^f\right)^\beta.$$

PROOF. Note that if (e,f) is in the shaded region in Figure 3.6.1, then (a,b) is on the right hand side of the line through (c,d) and (e,f). This implies

$$a(d-f) > c(b-f) + e(d-b).$$

Take $\alpha = d-b$, $\beta = b-f$ and $\gamma = a(d-f) - c(b-f) - e(d-b)$, then we have the desired equality. Symmetrically, if (e,f) is in the shaded region in Figure 3.6.2, then (c,d) is on the right hand side of the line through (e,f) and (a,b). We take $\alpha = c-e$, $\beta = a-c$, and $\delta = d(a-e) - b(c-e) - f(a-c)$ to obtain the desired equality. □

Now, we are ready to prove the main theorem of this section, Theorem 3.3.

PROOF. We express the generators of I as the following: $f_1 = \sum_{j=1}^{n_1} \eta_{1j} x^{a_{1j}} y^{b_{1j}}$, $f_2 = \sum_{j=1}^{n_2} \eta_{2j} x^{a_{2j}} y^{b_{2j}}$, $\ldots$, $f_m = \sum_{j=1}^{n_m} \eta_{mj} x^{a_{mj}} y^{b_{mj}}$ with $\eta_{ij} \neq 0$ in k. Then by Remark 3.1 I^* is the ideal generated by $x^{a_{ij}} y^{b_{ij}}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Consider the polynomial ring $R[U_{ij}] = R\big[U_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n_i\big]$ and the ring homomorphism

$$\begin{array}{cccc} \varphi & : & R[U_{ij}] & \longrightarrow & R[I^*t] \\ & & U_{ij} & \longmapsto & x^{a_{ij}} y^{b_{ij}} t \end{array}$$

Then the Rees algebra of I^* is $R[I^*t] \cong R[U_{ij}]/\ker\varphi$ and the fiber cone is

$$\frac{R[I^*t]}{\mathfrak{m}R[I^*t]} \cong \frac{R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)} \tag{3.6}$$

where $\mathfrak{m} = (x,y)R$ is the maximal ideal of R.

Let u_{ij} denote the homomorphic image of U_{ij} in $R[U_{ij}]/(\mathfrak{m}R[U_{ij}] + \ker\varphi)$. In order to show that I is a reduction of I^*, it suffices to show that the fiber cone $R[I^*t]/\mathfrak{m}R[I^*t]$ of I^* is integral over the fiber cone $R[It]/\mathfrak{m}R[It]$ of I. This is equivalent to showing that the k-algebra $k[u_{ij}]$ is integral over $k\big[\sum_{j=1}^{n_1} \eta_{1j} u_{1j}, \cdots, \sum_{j=1}^{n_m} \eta_{mj} u_{mj}\big]$ (*c.f.* Section 2 and [**SH**, 8.2.4]). Therefore by Lemma 3.4, for each u_{ij}, it is enough to prove that for all $\ell \neq j$, $u_{ij}^{\alpha_\ell} u_{i\ell}^{\beta_\ell} = 0$ for some positive integers α_ℓ and β_ℓ. Note that U_{ij} (*resp.* $U_{i\ell}$) corresponds to $x^{a_{ij}} y^{b_{ij}} t$ (*resp.* $x^{a_{i\ell}} y^{b_{i\ell}} t$) in the isomorphism (3.6). Without loss of generality, we assume $a_{ij} \geq a_{i\ell}$. If $a_{ij} > a_{i\ell}$ and $b_{ij} \geq b_{i\ell}$, then $x^{a_{ij}} y^{b_{ij}} t = x^{a_{ij}-a_{i\ell}} y^{b_{ij}-b_{i\ell}} (x^{a_{i\ell}} y^{b_{i\ell}} t)$. This shows $U_{ij} - x^{a_{ij}-a_{i\ell}} y^{b_{ij}-b_{i\ell}} U_{i\ell} \in \ker\varphi$ and so $U_{ij} \in \mathfrak{m}R[U_{ij}] + \ker\varphi$. That is $u_{ij} = 0$ and so is true $u_{ij} u_{i\ell} = 0$. Similarly for $a_{ij} = a_{i\ell}$ and $b_{ij} < b_{i\ell}$ (*resp.* $b_{ij} > b_{i\ell}$), one can show $u_{i\ell} = 0$ (*resp.* $u_{ij} = 0$) and so $u_{ij} u_{i\ell} = 0$.

The last case is that $a_{ij} > a_{i\ell}$ and $b_{ij} < b_{i\ell}$. By the assumption of the theorem, there exists $x^{a_{hs}} y^{b_{hs}} \in \Gamma(f_h)$ such that (a_{hs}, b_{hs}) lies on the left hand side of the line through (a_{ij}, b_{ij}) and $(a_{i\ell}, b_{i\ell})$ as shown in the following shaded region:

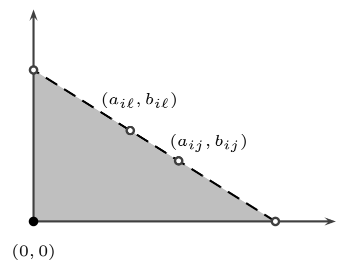


Figure 3.3.1.

We divide Figure 3.3.1 into the following three parts:

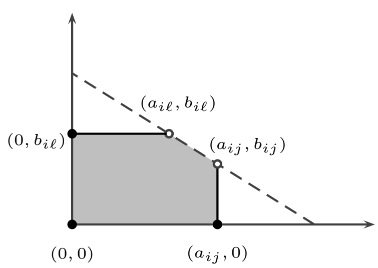


Figure 3.3.2.

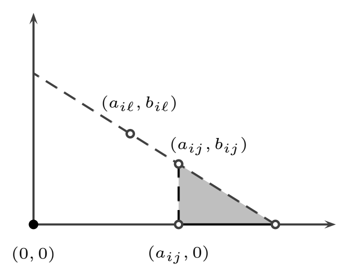


Figure 3.3.3.

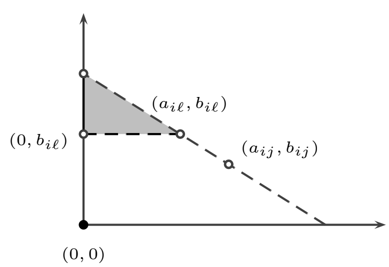


Figure 3.3.4.

If (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.3.2, then, by Lemma 3.5, there exist nonnegative integers $\alpha, \beta, \gamma, \delta$ with γ, δ not both zero such that

$$(x^{a_{ij}}y^{b_{ij}}t)^{\alpha}(x^{a_{i\ell}}y^{b_{i\ell}}t)^{\beta} = x^{\gamma}y^{\delta}(x^{a_{hs}}y^{b_{hs}}t)^{\alpha+\beta}.$$

Thus, we have $U_{ij}^{\alpha}U_{i\ell}^{\beta} - x^{\gamma}y^{\delta}U_{hs}^{\alpha+\beta} \in \ker\varphi$. That is $U_{ij}^{\alpha}U_{i\ell}^{\beta} \in \mathfrak{m}R[U_{ij}] + \ker\varphi$ and $u_{ij}^{\alpha}u_{i\ell}^{\beta} = 0$. If (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.3.3, then, by Lemma 3.6, there exist positive integers α, β, γ such that

$$\left(x^{a_{ij}}y^{b_{ij}}\right)^{\alpha+\beta} = x^{\gamma}\left(x^{a_{hs}}y^{b_{hs}}\right)^{\alpha}\left(x^{a_{i\ell}}y^{b_{i\ell}}\right)^{\beta},$$

and this implies $u_{ij}^{\alpha+\beta} = 0$ as above. Similarly, if (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.3.4, we may apply Lemma 3.6 and get $u_{i\ell}^{\alpha+\beta} = 0$ for some positive integers α, β. These prove that u_{ij} is integral over $k\big[\sum_{j=1}^{n_1}\eta_{1j}u_{1j}, \cdots, \sum_{j=1}^{n_m}\eta_{mj}u_{mj}\big]$ for all i, j. Hence I is a reduction of I^*. The proof is completed. $\square$

As an immediate application of Theorem 3.3, we find a minimal reduction of a given monomial ideal I. Note that every monomial ideal has a unique minimal monomial generating set. The monomials corresponding to the vertices of the convex hull of the graph of I are part of the irredundant monomial generators of I. Indeed, if we let H denote the ideal generated by the monomials corresponding to the vertices of the convex hull of the graph of I, then all monomial generators of I are integral over H and thus H is a reduction of I. In order to find a minimal reduction of I, it is enough to find a minimal reduction of H. The ideal H is called *least monomial reduction* in [**Q**] and *minimal monomial reduction* in [**S**], but it is not necessarily a minimal reduction as pointed out in Introduction.

Note that every principal ideal is integrally closed and is its own minimal reduction. So we are interested in monomial ideals minimally generated by at least two elements. Although the following corollary was mentioned in [**Q**] and [**SH**], it was among the initial results of our project proved in 2001. We later extend the study on the monomial ideals and obtain Theorem 3.3 in its current form. Corollary 3.7 and a special case of Lemma 3.4 were quoted and utilized in Lu's master thesis ([**L**], 2003) under the second author Liu's supervision.

COROLLARY 3.7. Let $R = k[x,y]_{(x,y)}$ and $|k| = \infty$. Let I be a monomial ideal minimally generated by at least two elements. Suppose the convex hull of the graph of I has vertices (a_1,b_1), (a_2,b_2), $\ldots$, (a_n,b_n) with $a_1 > a_2 > \cdots > a_n$ and $b_1 < b_2 < \cdots < b_n$. Then

$$K = \Big(\sum_{\substack{1\le i\le n \\ i \text{ is odd}}} x^{a_i} y^{b_i}, \sum_{\substack{1\le i\le n \\ i \text{ is even}}} x^{a_i} y^{b_i} \Big)$$

is a minimal reduction of I.

PROOF. Notice that for any two odd (*resp.* even) indices $j < k$, there exists an even (*resp.* odd) index i such that $j < i < k$. Since (a_1,b_1), (a_2,b_2), $\ldots$, (a_n,b_n) are vertices of a convex graph, (a_i,b_i) is on the left of the line through (a_j,b_j) and (a_k,b_k). By Theorem 3.3, K is a reduction of K^*. Since $K^* \subseteq I$ and since their graphs have the same convex hull, K^* is a reduction of I. Thus, K is a reduction of I. Furthermore, as pointed out in Introduction, since I is non-principal, a reduction of I is minimal if it is generated by two elements. Hence K is indeed a minimal reduction of I. □

Independently, V. C. Quiñonez also obtains the result of Corollary 3.7 by proving a more general theorem on reductions of ideals. In Section 4, we recover Quiñonez's main theorem as an application of Lemma 3.4.

4. Examples and Remarks

4.1. Examples. This section is mainly dedicated to discuss the conditions that make I^* an integral extension of I.

Let $R = k[x,y]_{(x,y)}$ and $|k| = \infty$. We first give an example to illustrate Theorem 3.3 and Corollary 3.7. Theorem 3.3 provides a sufficient condition under which the ideal I^* is integral over I. Additional remarks are made here on the condition in Theorem 3.3. Examples 4.2 and 4.3 show that this is not a necessary condition for I to be a reduction of I^*. Example 4.3 indicates that a seemingly most intuitive extension is still not necessary. In Example 4.4 we discuss another extension which may likely make the condition necessary but we do not have neither a proof nor enough evidence to make such conjecture.

Example 4.1. Consider the ideal $I = (x^{11} + x^{10}y + x^2y^4 + xy^7, x^{10}y^2 + x^3y^3 + y^8, x^5y^2 + xy^7)$. Note that the monomials occurring in the first generator of I are $x^{11}, x^{10}y, x^2y^4, xy^7$. From Figure 4.1.1, we see that for every line through two of the above 4 monomials, either x^5y^2 or y^8 is on the left of the line; for example, x^5y^2 is on the left of the line through x^{11} and x^2y^4, and y^8 is on the left of the line through x^2y^4 and xy^7. Similarly, from Figure 4.1.2, we see that for each of the three lines determined by the three monomials occurring in the second generator of I, either x^5y^2 or x^2y^4 is on the left of the line. From Figure 4.1.3, x^3y^3 is on the left of the line through those two monomials occurring in the third generator of I. Hence, by Theorem 3.3, $I^* = (x^{11}, x^{10}y, x^5y^2, x^3y^3, x^2y^4, xy^7, y^8)$ is integral over I.

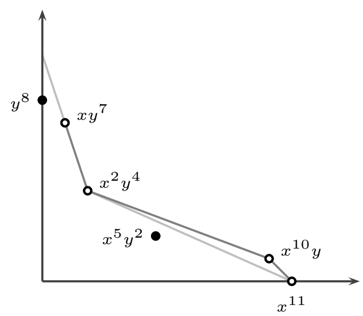


Figure 4.1.1.

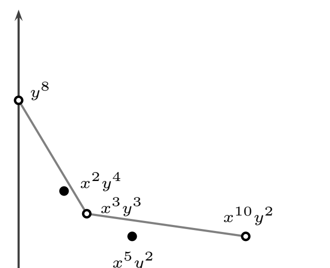


Figure 4.1.2.

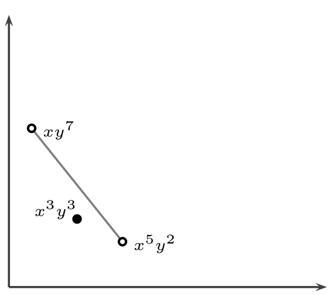


Figure 4.1.3.

On the other hand, from Figure 4.1.4, we see that the monomials corresponding to the vertices of the convex hull of the graph of I^* are x^{11}, x^5y^2, x^3y^3, x^2y^4, y^8. Hence, by Corollary 3.7, $(x^{11} + x^3y^3 + y^8, x^5y^2 + x^2y^4)$ is a minimal reduction of I^*.

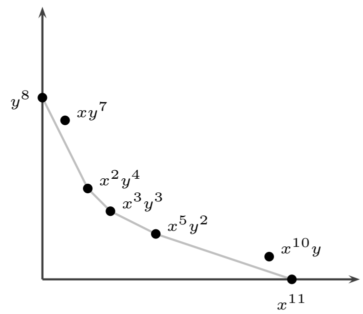


Figure 4.1.4.

EXAMPLE 4.2. Consider the ideal $I = (x^4, x^3y + xy^3, y^4)$. Although I does not satisfy the assumption in Theorem 3.3, I is indeed a reduction of I^*. In fact, because the convex hull of the graph of I^* has vertices $(4,0)$ and $(0,4)$, the ideal $K = (x^4, y^4)$ is a reduction of I^*. Thus, I^* is integral over I. This example tells us that the hypothesis given in Theorem 3.3 is not a necessary condition for I being a reduction of I^*.

EXAMPLE 4.3. Consider two ideals $J_1 = (x^3 + x^2y + xy^2, y^3)$ and $J_2 = (x^3 + x^2y, xy^2 + y^3)$. Note that ${J_1}^* = {J_2}^* = \mathfrak{m}^3$ and it is integrally closed. Because R is a two-dimensional regular local ring, $\mathfrak{m}^3$ has reduction number one (*c.f.* [**H**, 5.1]). So an ideal K is a reduction of $\mathfrak{m}^3$ if and only if $K\mathfrak{m}^3 = \mathfrak{m}^6$. One can check that $J_1\mathfrak{m}^3 = \mathfrak{m}^6$ and $J_2\mathfrak{m}^3 \subsetneq \mathfrak{m}^6$. Hence, J_1 is a reduction of ${J_1}^*$ but J_2 is not a reduction of ${J_2}^*$. Note that both J_1 and J_2 satisfy the following condition: for all $i = 1, 2, \ldots, m$ and for any two distinct monomials x^ay^b and x^cy^d in $\Gamma(f_i)$ with $c < a$ and $b < d$, there exists $x^ry^s \in \Gamma(f_j)$ for some j such that the point (r, s) lies on the line through (a, b) and (c, d), where $f_1, \ldots, f_m$ generate the ideal. This example tells us that one should not expect to generalize the condition in Theorem 3.3 to including the line connecting (a, b) and (c, d).

EXAMPLE 4.4. Consider the ideal $I = (x^4, x^2y + xy^2, y^4)$. Then the ideal $I^* = (x^4, x^2y, xy^2, y^4)$ and I^* has reduction number one, again since it is integrally closed in the two-dimensional regular local ring R (*c.f.* [**H**, 5.1]). It can be checked directly t hat $II^* \subsetneq (I^*)^2$. Thus, I is not a reduction of I^*. On the other hand, we know that $K_1 = (x^4 + y^4, x^2y + xy^2)$ is a reduction of I, also through a direct computation: $K_1I^2 = I^3$. We observe that x^4, y^4 are in $\Gamma(x^4 + y^4)$ and the term $x^2y + xy^2$ is a combination of two monomials corresponding to two points on the left hand side of the line through $(4,0)$ and $(0,4)$. It would be interesting to determine whether or not Theorem 3.3 can be extended to state "if the left side of the line contains points whose combination is in I, then K_1 is a reduction of I." However, the authors are not able to verify this statement.

By Corollary 3.7, $K_2 = (x^4 + xy^2, x^2y + y^4)$ is a minimal reduction of I^* while K_1 is not a reduction of I^*. Although the set consisting of $(a_1 : \cdots : a_4 : b_1 : \cdots : b_4)$, with the fact that $a_1x^4 + a_2x^2y + a_3xy^2 + a_4y^4$ and $b_1x^4 + b_2x^2y + b_3xy^2 + b_4y^4$ generate a minimal reduction of I^*, form a dense open set in $\mathbb{P}^3 \times \mathbb{P}^3$, this example shows there are ample exceptions.

4.2. An application of Lemma 3.4. In [**Q**] Quiñonez discusses minimal reductions of monomial ideals in $k[[x, y]]$ and proves the following Theorem 4.5 on the reduction of ideals in general. It states that if a nice partition on a generating set of an ideal (not necessarily monomial) is available to result certain relationships among the generators, then the partition provides a reduction. The existence of such a partition may not be determined easily in general but in two-dimensional local rings such as $k[x,y]_{(x,y)}$ and $k[[x,y]]$, the partition consisting of even indices and odd indices as stated in Corollary 3.7 satisfies the conditions required in Theorem 4.5 and hence a minimal reduction of the ideal is obtained. In the following, we recover Quiñonez's main theorem as an application of Lemma 3.4 in the previous section.

THEOREM 4.5 (Quiñonez, [**Q**, 3.3]). *Let* $I = (m_i)_{0\le i\le r}$ *be an ideal in a Noetherian local ring* $(R,\mathfrak{m})$*. Assume that there exists a partition on* r *elements:* $\{0,\dots,r\} = \bigcup_{0\le\alpha\le s} S_\alpha$*, where* $s \le r$*, such that if* $i,j\in S_\alpha$*,* $i\ne j$*, then* $m_i^l m_j^l \in I^{2l}\mathfrak{m}$ *for some integer* l*. Let* $J = \left(\sum_{i\in S_\alpha} m_i\right)_{0\le\alpha\le s}$*, then* J *is a reduction of* I*.*

With the same strategy as proving Theorem 3.3, this theorem can be proved as a corollary of Lemma 3.4. Precisely, consider the polynomial ring $R[U_0,U_1,\dots,U_r]$ and the ring homomorphism

$$\begin{array}{cccc} \varphi & : & R[U_0,U_1,\dots,U_r] & \longrightarrow & R[It] \\ & & U_i & \longmapsto & m_i t \end{array}.$$

Then the Rees algebra of I is $R[It] \cong R[U_0,\dots,U_r]/\ker\varphi$ and the fiber cone of I is

$$\frac{R[It]}{\mathfrak{m}R[It]} \cong \frac{R[U_0,U_1,\dots,U_r]}{(\mathfrak{m}R[U_0,U_1,\dots,U_r]+\ker\varphi)}.$$

Let u_i denote the homomorphic image of U_i in $R[U_0,\dots,U_r]/(\mathfrak{m}R[U_0,\dots,U_r]+\ker\varphi)$. In order to show that J is a reduction of I, it suffices to show that the fiber cone $R[It]/\mathfrak{m}R[It]$ of I is integral over the fiber cone $R[Jt]/\mathfrak{m}R[Jt]$ of J. This is equivalent to showing that the k-algebra $k[u_0,\dots,u_r]$ is integral over $k\left[\sum_{i\in S_\alpha} m_i t \mid 0\le\alpha\le s\right]$, where $k = R/\mathfrak{m}$ is the residue field of R. Note that with the assumption that if $i,j\in S_\alpha$, $i\ne j$, then $m_i^l m_j^l \in I^{2l}\mathfrak{m}$ for some integer l, we have $(m_i t)^l (m_j t)^l \in \mathfrak{m}(It)^{2l}$ and this implies $u_i^l u_j^l = 0$. Now, apply Lemma 3.4 to complete the proof.

5. Application to Multiplicities

In this section, we apply Theorem 3.3 to revisit the computations of Buchsbaum-Rim Multiplicity in Jones [**J**]. We also present a formula that summarizes the seven individual cases concluded in [**J**]. As before, $R = k[x,y]_{(x,y)}$ with $|k| = \infty$. The modules under consideration are finitely generated over R and arise from the following Setting 5.1. These are called *monomial modules* (*c.f.* [**R1**, Section 4]).

SETTING 5.1. Let $I = (x^s, y^t)$ and $J = (x^{s+i}, y^{j+t}, x^d y^{e+t})$ be monomials in R. Let F be a free module of rank two with free basis e_1, e_2. Consider the homomorphism $\phi : F \longrightarrow I$ defined by $\phi(e_1) = x^s$ and $\phi(e_2) = y^t$. Then ϕ induces a short exact sequence

$$0 \longrightarrow M \longrightarrow F \xrightarrow{\widetilde{\phi}} I/J \longrightarrow 0 \tag{5.1}$$

where M is the kernel of the induced map. In this case, M can be identified with the submodule of F generated by the columns of the matrix

$$\begin{pmatrix} x^i & 0 & 0 & -y^t \\ 0 & y^j & x^d y^e & x^s \end{pmatrix}.$$

Clearly, $F/M \cong I/J$ in Setting 5.1. We call the above matrix a *presenting matrix* of M and denote it by $\widetilde{M}$ by abusing the notation since presenting matrices of a module are not unique. However, the Fitting ideal of F/M, used most often in the present discussion, does not depend on the choice of the presenting matrix of M.

In [**J**], Jones computes the Buchsbaum-Rim multiplicity $\mathrm{br}(M)$ of M as in Setting 5.1 and compares $\mathrm{br}(M)$ and $\mathrm{e}(J) - \mathrm{e}(I)$. Applying reduction theory on modules, Jones classifies the module M into seven cases and gives graphical interpretation of the relation between $\mathrm{br}(M)$ and $\mathrm{e}(J) - \mathrm{e}(I)$. However, the algebraic meaning of some crucial graphs is lacking. In this section, we make use of the Fitting ideals of modules to revisit Jones's classification and reinterpret the Buchsbaum-Rim multiplication of M as the Hilbert-Samuel multiplicities of I, J, and certain zeroth Fitting ideals of modules closely related to M.

The following result of Rees will be used frequently in our approach. This theorem enables us to transfer reduction relations between modules to those between ideals and vice versa.

THEOREM 5.2. (Rees, [**R**, 1.2]) Let $N \subseteq M \subseteq F \cong R^r$ with $\ell(F/N) < \infty$. Then N is a reduction of M if and only if $\mathrm{Fitt}_0(F/N)$ is a reduction of $\mathrm{Fitt}_0(F/M)$.

Recall that the zeroth Fitting ideal $\mathrm{Fitt}_0(F/M)$ is the ideal generated by all 2×2-minors of the matrix $\widetilde{M}$ as in Setting 5.1, so

$$\mathrm{Fitt}_0(F/M) = (x^i y^j, x^{i+s}, x^{i+d} y^e, y^{t+j}, x^d y^{e+t}) \ .$$

Note that this is a monomial ideal so its Hilbert-Samuel multiplicity is easy to compute using (1.1). Also, given a submodule N of M or a submodule L of F containing M, Theorem 3.3 and Theorem 5.2 provide means to determine if N is a reduction of M or if L is integral over M.

THEOREM 5.3. *Let $R = k[x,y]_{(x,y)}$ with $|k| = \infty$. Let I, J, F and M be as in Setting* 5.1 *such that $F/M \cong I/J$. Then,*

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) - [\mathrm{e}(\mathrm{Fitt}_0(F/N)) - \mathrm{e}(\mathrm{Fitt}_0(F/M))] \,,$$

where N is the submodule of F lifted from either (x^{s+i}, y^{t+j}) or $(x^{s+i}+y^{t+j}, x^d y^{t+e})$ whichever is a minimal reduction of J.

PROOF. As in Section 3, we continue to use the correspondence between monomials and lattice points of $\mathbb{Z}^2$ in $\mathbb{R}^2$. Recall that J is minimally generated by three elements x^{s+i}, y^{t+j} and $x^d y^{t+e}$. By Corollary 3.7, depending on the relative position of the point $(d, e+t)$ and the line segment connecting $(s+i, 0)$ and $(0, j+t)$, either (x^{s+i}, y^{t+j}) or $(x^{s+i} + y^{t+j}, x^d y^{t+e})$ is a minimal reduction of the monomial ideal J.

Before we start the discussion, it should be pointed out that the four points corresponding to the four monomials y^{t+j}, $x^d y^{t+e}$, $x^{d+i} y^e$ and $x^i y^j$ form a parallelogram in all cases in this proof. This parallelogram is applied to classify various situations. We will show how the formula is achieved via several steps.

Step 1. We discuss the case where $K_1 = (x^{s+i}, y^{j+t})$ is a minimal reduction of J. In other words, we have the following graph:

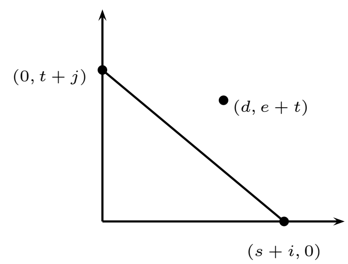


Figure A.

Recall the map ϕ in Setting 5.1 and notice that the pre-image of K_1 under ϕ is the submodule N_1 of F with the following presenting matrix

$$\widetilde{N_1} = \begin{pmatrix} x^i & 0 & -y^t \\ 0 & y^j & x^s \end{pmatrix}.$$

The Fitting ideal $\mathrm{Fitt}_0(F/N_1)$ is monomial and

$$\mathrm{Fitt}_0(F/M) = \mathrm{Fitt}_0(F/N_1) + (x^{i+d}y^e, x^d y^{e+t}).$$

It is already clear that $x^d y^{e+t}$ is integral over $(x^{i+s}, y^{j+t}) \subseteq \mathrm{Fitt}_0(F/N_1)$. Thus whether or not N_1 is a minimal reduction of M depends on whether or not $x^{i+d}y^e$ is integral over $\mathrm{Fitt}_0(F/N_1)$ by Theorem 5.2.

Step 1.1. Depending on the location of the point (i, j), the shape of the convex hull of the graph of $\mathrm{Fitt}_0(F/N_1)$ is one of the following two:

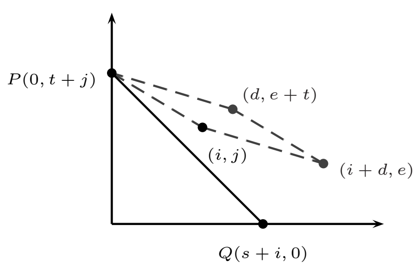


Figure A1.

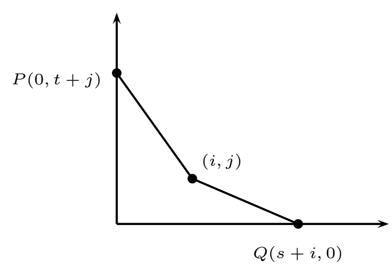


Figure A2.

If it is the case of Figure $A1$, the entire parallelogram with vertices (i, j), $(0, t+j)$, $(d, e+t)$, $(i+d, e)$ is on the right hand side of the line segment $\overline{PQ}$ and so $x^{i+d}y^e$ is integral over $\mathrm{Fitt}_0(F/N_1)$. We note that $A1$ also includes the case where the positions of (i, j) and $(d, e+t)$ are exchanged. In the case of Figure $A2$, there are two possibilities depending on the point $(i+d, e)$ being on the right or left of the line segment connecting (i, j) and $(s+i, 0)$:

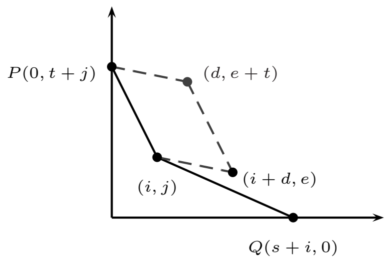


*Figure A*2.1.

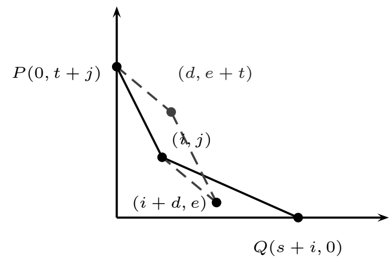


*Figure A*2.2.

In the case of Figure $A2.1$, $x^{i+d}y^e$ is also integral over $\mathrm{Fitt}_0(F/N_1)$. If it is the case of Figure $A2.2$, then $x^{i+d}y^e$ is not integral over $\mathrm{Fitt}_0(F/N_1)$ and thus N_1 is not a reduction of M. We deal with this case in the next step. Therefore, we conclude that in the cases of Figure $A1$ and Figure $A2.1$, N_1 is a minimal reduction of M and by (2.2)

$$\begin{aligned}\mathrm{br}(M) = \mathrm{br}(N_1) = \ell(F/N_1) &= \ell(I/K_1) \\ &= \ell(R/K_1) - \ell(R/I) \\ &= \mathrm{e}(J) - \mathrm{e}(I)\end{aligned}$$

Step 1.2. For the case of Figure $A2.2$, the convex hull of $\mathrm{Fitt}_0(F/M)$ is as in Figure B.

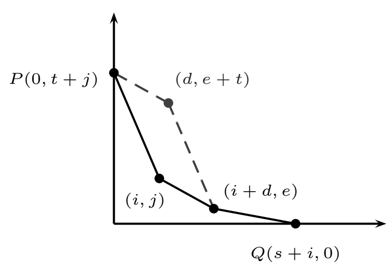


Figure B.

The result in [**J**] shows that the multiplicity of such module M is

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) - 2(\text{dark area})$$

where the *dark area* refers to the triangular area in both Figures C1 and C2.

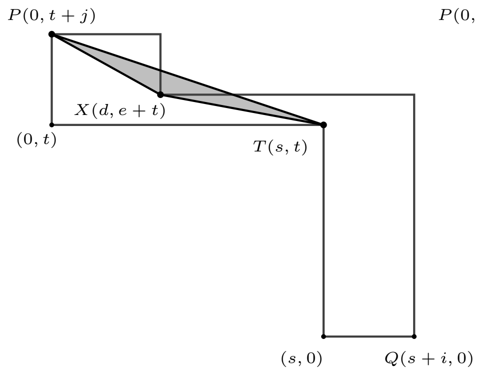


Figure C1.

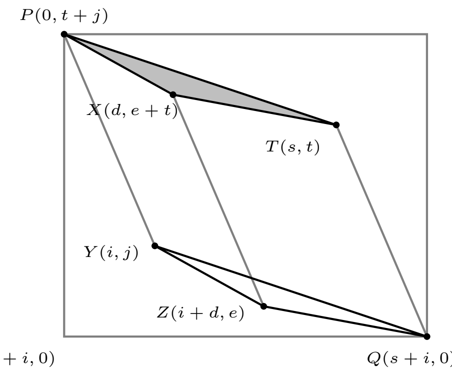


Figure C2.

Figure C1 is an original graph in [**J**]. We add a few more points corresponding to the generators of the convex hull of the graph of $\mathrm{Fitt}_0(F/M)$ to obtain an alternative graph as shown in C2. We utilize parallel lines in Figure C2 and observe that $\triangle PXT$ and $\triangle YZQ$ have the same area. By (1.1), it is clear that

$$\begin{aligned} 2(\text{dark area}) &= \mathrm{e}(x^{s+i}, y^{t+j}, x^i y^j) - \mathrm{e}(x^{s+i}, y^{t+j}, x^i y^j, x^{d+i} y^e) \\ &= \mathrm{e}(\mathrm{Fitt}_0(F/N_1)) - \mathrm{e}(\mathrm{Fitt}_0(F/M)). \end{aligned} \tag{5.2}$$

The second equality is straightforward since $x^d y^{e+t}$ is integral over $(x^{s+i}, y^{t+j}, x^i y^j, x^{d+i} y^e)$ by the assumption. Thus, we have

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) - [\mathrm{e}(\mathrm{Fitt}_0(F/N_1)) - \mathrm{e}(\mathrm{Fitt}_0(F/M))]. \tag{5.3}$$

Note that (5.3) also holds for the cases in the previous step since N_1 in Step 1.1 is a minimal reduction of M.

Recall that $K_1 = (x^{s+i}, y^{t+j})$ is assumed to be a minimal reduction of J in this current Step 1. We would like to emphasize that the module N_1 in (5.3) is a submodule in the rank two free module lifted by a minimal reduction (x^{s+i}, y^{t+j}) of J. This observation is a key to the conclusion of the next step.

Step 2. We discuss the case where $K_2 = (x^{s+i} + y^{j+t}, x^d y^{e+t})$ is a minimal reduction of J. In other words, we have the following graph:

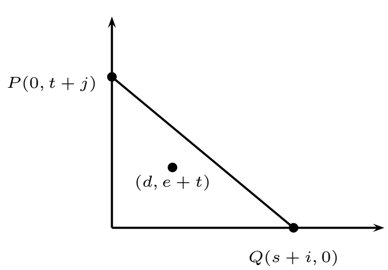


Figure D.

Note that the pre-image of K_2 under ϕ is the submodule N_2 of F with a presenting matrix

$$\widetilde{N_2} = \begin{pmatrix} x^i & 0 & -y^t \\ y^j & x^d y^e & x^s \end{pmatrix}$$

The zeroth Fitting ideal $\mathrm{Fitt}_0(F/N_2)$ of F/N_2 is $(x^{i+d}y^e, x^{i+s}+y^{j+t}, x^dy^{e+t})$. Notice that the point x^dy^{e+t} is on the left hand side of the line $\overline{PQ}$, so by Theorem 3.3, the monomial ideal $\mathfrak{a} = (x^{i+d}y^e, x^{i+s}, y^{j+t}, x^dy^{e+t})$ is integral over $\mathrm{Fitt}_0(F/N_2)$. Moreover, $\mathrm{Fitt}_0(F/M) = \mathfrak{a} + (x^iy^j)$. Thus, in order to determine if N_2 is a minimal reduction of M, i.e., $\mathrm{Fitt}_0(F/N_2)$ is a reduction of $\mathrm{Fitt}_0(F/M)$, it is equivalent to determining if $\mathfrak{a}$ is a reduction of $\mathrm{Fitt}_0(F/M)$. And it suffices to check if x^iy^j is integral over $\mathfrak{a}$. Depending on the location of the point $(i+d, e)$, the shape of the convex hull of the graph of $\mathfrak{a}$ is one of the following three:

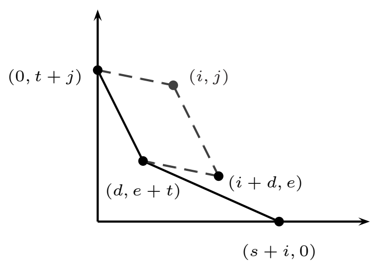


*Figure D*1.

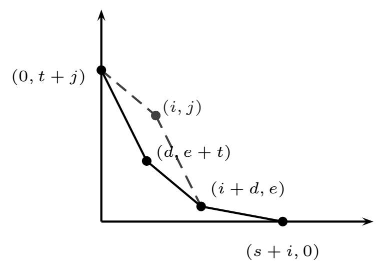


*Figure D*2.

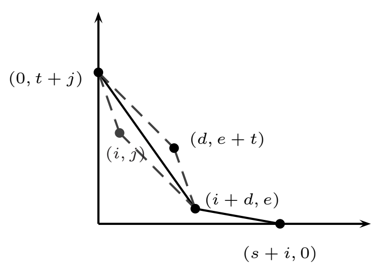


*Figure D*3.

Step 2.1. Note that in the cases of Figures $D1$ and $D2$, the entire parallelogram with vertices (i,j), $(0,t+j)$, $(d,e+t)$, $(i+d,e)$ is inside the convex hull of the graph of $\mathfrak{a}$, so x^iy^j is integral over $\mathfrak{a}$ and hence N_2 is a minimal reduction of M. Therefore again by (2.2) and the fact that $F/N_2 \cong I/K_2$,

$$\begin{aligned} \mathrm{br}(M) = \mathrm{br}(N_2) = \ell(F/N_2) &= \ell(I/K_2) \\ &= \ell(R/K_2) - \ell(R/I) \\ &= \mathrm{e}(K_2) - \mathrm{e}(I) = \mathrm{e}(J) - \mathrm{e}(I). \end{aligned}$$

The last two equalities hold due to the fact that K_2 is a minimal reduction of J. This shows

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I).$$

Step 2.2. In the case of Figure $D3$, using the same parallelogram as above, the point (i,j) is not in the convex hull of the graph of $\mathfrak{a}$. So N_2 is not a minimal reduction of M. As Step 1.2, we quote the result in [**J**] which shows $\mathrm{br}(M)$ as follows:

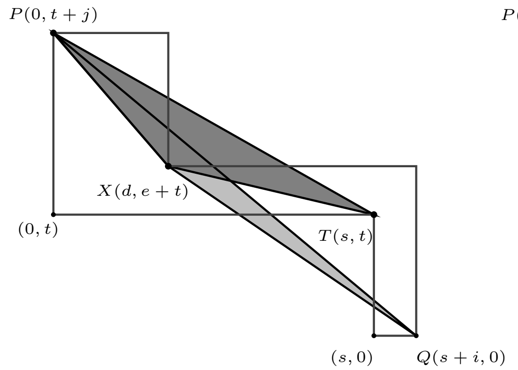


Figure E1.

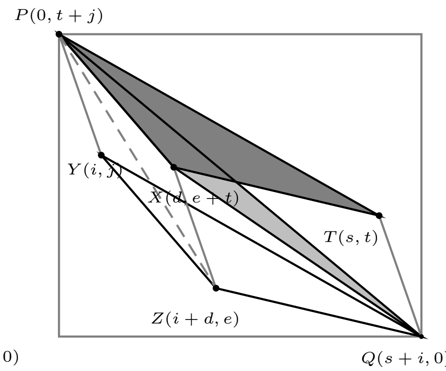


Figure E2.

$$\text{br}(M) = \text{e}(J) - \text{e}(I) - 2(\text{dark area}) + 2(\text{light area}) \tag{5.4}$$

where the dark area is the area of $\triangle PXT$ and the light area is that of $\triangle PXQ$ in both Figures E1 and E2. Similar to the discussion in Step 1.2, by moving $\triangle PXT$ to its similar triangle $\triangle YZQ$, we now have

(5.5)

$$\text{br}(M) = \text{e}(J) - \text{e}(I) - [\text{e}(x^{s+i}, y^{t+j}, x^i y^j) - \text{e}(\text{Fitt}_0(F/M))] + [\text{e}(x^{s+i}, y^{t+j}) - \text{e}(J)].$$

We would like to take a moment to look at (5.5) closer. Recall that in the formula (5.2) in Step 1.2, $(x^{s+i}, y^{t+j}, x^i y^j)$ is the Fitting ideal of the submodule N_1 of F lifted from the ideal $K_1 = (x^{s+i}, y^{t+j})$ which is a minimal reduction of J in the case of Step 1.2. However, in the present case, the ideals (x^{s+i}, y^{t+j}) and J have no reduction relation. Formulas (5.3) and (5.5) together inspire us to take a minimal reduction of J and take its lifting in F into consideration. More precisely, in the right hand side of (5.5), we consider K_2 in place of (x^{s+i}, y^{t+j}) and replace $(x^{s+i}, y^{t+j}, x^i y^j)$ by $\text{Fitt}_0(F/N_2)$. Then, we compare $\text{e}(J) - \text{e}(I) - \text{br}(M)$ and $\text{e}(\text{Fitt}_0(F/N_2)) - \text{e}(\text{Fitt}_0(F/M))$. Note that by (5.4), we have

$$\text{e}(J) - \text{e}(I) - \text{br}(M) = 2(\text{the area of } \triangle PXT - \text{ the area of } \triangle PXQ).$$

On the other hand, although the ideal $\text{Fitt}_0(F/N_2)$ is not monomial, we recall that the monomial ideal $\mathfrak{a}$ is integral over $\text{Fitt}_0(F/N_2)$; thus, $\text{e}(\text{Fitt}_0(F/N_2)) = \text{e}(\mathfrak{a})$. Moreover, the difference between the convex hull of the graph of $\mathfrak{a}$ and that of $\text{Fitt}_0(F/M)$ is the triangular region $\triangle PYZ$. Hence

$$\begin{aligned} \text{e}(\text{Fitt}_0(F/N_2)) - \text{e}(\text{Fitt}_0(F/M)) &= \text{e}(\mathfrak{a}) - \text{e}(\text{Fitt}_0(F/M)) \\ &= 2(\text{the area of } \triangle PYZ). \end{aligned}$$

Two quantities in comparison become 2(the area of $\triangle PXT -$ the area of $\triangle PXQ$) and 2(the area of $\triangle PYZ$). By straightforward computation, we find that both quantities are equal to $td - i(j - e)$ and we have

$$\text{br}(M) = \text{e}(J) - \text{e}(I) - [\text{e}(\text{Fitt}_0(F/N_2)) - \text{e}(\text{Fitt}_0(F/M))]\,. \tag{5.6}$$

We note that (5.6) is also satisfied for the cases discussed in Step 2.1 where N_2 is indeed a minimal reduction of M.

Hence, the desired formula follows either Step 1 or Step 2 depending on which minimal reduction J possesses. □

Theorem 5.3 describes the Buchsbaum-Rim multiplicity of a monomial module as the Hilbert-Samuel multiplicities of certain ideals closely related to the module. The graphical argument applied in the proof presented here is limited to the monomial modules of rank two with small number of generators as in Setting 5.1. For modules of higher rank or with bigger number of generators, although one can try to do similar observation and work as done in [**J**], one has to discuss a lot more cases and their classification is much more complicated. For instance, in the case where the module $M \cong I/J$ has rank two as in Setting 5.1 but with J generated by four monomials instead of three, there are more than 50 cases (see [**L**]). This also shows a formulated result such as Theorem 5.3 is desirable in order to extend the outcome to modules of higher ranks. One also notice that the terms involved in the formula in Theorem 5.3 are defined even if I and J are not monomial ideals. For modules of higher rank, and not being restricted to monomial quotients, we refer to [**CLU**, 2.4 and Section 3] where linkage theory of ideals are utilized.

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