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AN INTERSECTION MULTIPLICITY IN TERMS OF Ext-MODULES

C-Y. JEAN CHAN

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ABSTRACT. The main aim of this paper is to discuss the relation between Serre's intersection multiplicity and the Euler form. The Euler form is defined to be an alternating sum of the length of Ext-modules and is used by Mori and Smith to develop intersection theory over noncommutative rings. We show that they differ by a sign and that this relation is closely related to Serre's vanishing theorem.

1. INTRODUCTION

Assume A is a regular local ring. Let M and N be finitely generated A-modules such that $M \otimes N$ has finite length. In the 1950s, Serre [16] defined the intersection multiplicity of M and N as

$$\chi(M,N) = \sum (-1)^i \operatorname{length}(\operatorname{Tor}_i(M,N)).$$

To develop intersection theory over noncommutative rings, Mori and Smith attempted to generalize intersection multiplicity using the *Euler form*

$$\xi(M,N) = \sum (-1)^i \operatorname{length}(\operatorname{Ext}^i(M,N))$$

for two arbitrary finitely generated Noetherian right modules. The intersection multiplicity is defined as

$$(-1)^{\dim N} \xi(M,N),$$

where the dimension of modules over a noncommutative ring is defined by using the Euler form (cf. [9]). It should be pointed out that the tensor product of two right modules is not well-defined. Therefore the Tor-modules cannot be applied in noncommutative cases. Mori and Smith established Bézout's Theorem for noncommutative spaces for this intersection multiplicity. We refer the readers to Mori and Smith's paper [9] for complete details.

Since the idea is to generalize results in commutative algebraic geometry, there should be relations between these two intersection multiplicities. In [8], Mori conjectured

$$\chi(M,N) = (-1)^{\operatorname{codim} M} \xi(M,N).$$

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This paper presents the study of the relation between $\chi(M,N)$ and $\xi(M,N)$ over commutative Noetherian rings. More generally, we will see that the formula relating $\chi(M,N)$ and $\xi(M,N)$ is closely related to Serre's vanishing theorem.

In section 2, we prove that the above formula holds over a regular local ring. Our proof relies on Serre's vanishing theorem.

Theorem 1 (Vanishing, Roberts [11], Gillet and Soulé [6])**.** *Assume M and N are finitely generated modules over a complete intersection A such that M and N both have finite projective dimension and $M \otimes N$ has finite length. If* $\dim M + \dim N < \dim A$*, then* $\chi(M,N) = 0$.

However, as we try to generalize the result over a complete intersection, the original proof cannot be applied in this generality. The properties of complete intersections and the formula itself suggest that we apply an argument similar to the one used in Roberts' proof of the vanishing theorem. This will be discussed in the third section. We further prove the formula for a Gorenstein ring of dimension ≤ 5. This is also inspired by the vanishing theorem under the same condition [12].

From the discussion in the earlier sections, it seems likely that a proof of the vanishing theorem will also give a proof of the formula that we are interested in:

$$\chi(M,N) = (-1)^{\operatorname{codim} M}\xi(M,N). \tag{1}$$

In the last section, we investigate the relation between the vanishing theorem and this formula. We will see that they are equivalent over a regular local ring. The equivalence is not known in general.

It is known that all finitely generated modules over a regular local ring have finite projective dimension, so the sum in the definitions of both multiplicities have finitely many terms involved. Over a more general ring, even if the two modules satisfy the same conditions as in Serre's definition, it is not necessary that the expressions for $\chi(M,N)$ and $\xi(M,N)$ are only finite sums. There are many ways to fix this. We mainly discuss the case with the condition that both modules have finite projective dimension added to the assumption, except the discussion in section 4.

After the earlier version of this paper was done, the author found that S. P. Dutta has similar results. In [2] and [3] Dutta proved the vanishing theorem for Gorenstein rings of dimension ≤ 5 by proving that a formula holds true. This formula is equivalent to Formula (1) under certain conditions. We will discuss this in section 2 and section 4.

2. The relation between $\chi(M,N)$ and $\xi(M,N)$ over a regular local ring

We first state the main theorem in this section.

Theorem 2. *Let A be a regular local ring and let M and N be finitely generated A-modules such that $M \otimes N$ has finite length. Let* $\operatorname{codim} M$ *denote* $\dim A - \dim M$*. Then,*

$$\chi(M,N) = (-1)^{\operatorname{codim} M}\xi(M,N). \tag{2}$$

Remark. Let A be a regular local ring and let M and N satisfy the hypotheses of Theorem 2. Serre [16] proved that

$$\dim M + \dim N \leq \dim A.$$

When the ring is not regular, whether or not this inequality is true remains unknown if both modules have finite projective dimension.

To relate Serre's intersection multiplicity and the Euler form, we write $\chi(M, N)$ as the Euler characteristic of the total complex of the tensor product of the finite resolutions of both modules. Then, we claim that $\xi(M, N)$ is equal to the Euler characteristic described above with the resolution of M replaced by its dual. By reducing M to the case of the ring modulo a prime ideal and computing the latter Euler characteristic, the theorem will be proven. Before proving the theorem, we explain the above idea in detail starting with some background on homological algebra.

Recall that a *perfect* complex, $G_\bullet$, is a bounded complex of finitely generated free modules. Let $\operatorname{Supp}(G_\bullet)$ denote the support of the complex $G_\bullet$, which is the set $\{\mathfrak{p} \in \operatorname{Spec}(A) : (G_\bullet)_\mathfrak{p} \text{ is not exact}\}$. We define the dimension of a complex to be the length of largest ℓ such that $\mathfrak{p}_0, \ldots, \mathfrak{p}_\ell$ are distinct in $\operatorname{Supp}(G_\bullet)$ and $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_\ell$. If $G_\bullet$ is a complex supported only at the maximal ideal, let $\chi(G_\bullet) = \sum_i (-1)^i \operatorname{length}(H_i(G_\bullet))$, where $H_i(G_\bullet)$ is the *i-th* homology of $G_\bullet$. We call $\chi(G_\bullet)$ the *Euler characteristic* of $G_\bullet$. One can use this to interpret Serre's intersection multiplicity. Let M and N be assumed as above and let $E_\bullet$ and $F_\bullet$ be finite free resolutions of M and N respectively. Then

$$\chi(M, N) = \chi(E_\bullet \otimes F_\bullet), \tag{3}$$

where $E_\bullet \otimes F_\bullet$ denotes the total complex of the tensor product of the two complexes.

We recall two well-known facts from homological algebra. They imply an important property which allows us to consider the Euler form in terms of Euler characteristics in the later discussion.

Proposition 1. *Let A be a commutative Noetherian local ring. Let N be a finitely generated module of finite projective dimension and let $G_\bullet$ be a bounded below complex of finitely generated A-free modules such that $H_i(G_\bullet) \otimes N$ has finite length for all i. Then,*

$$\chi(G_\bullet \otimes N) = \sum (-1)^i \chi(H_i(G_\bullet), N).$$

Proposition 2. *Let A be a commutative Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$. Let M and N be finitely generated A-modules with finite projective dimension such that $M \otimes N$ has finite length. Let $E_\bullet$ and $F_\bullet$ be finite free resolutions of M and N respectively. Then,*

$$\xi(M, N) = \chi(E_\bullet^* \otimes F_\bullet) = \sum_{i=0}^{s} (-1)^i \chi(\operatorname{Ext}^i(M, A), N). \tag{4}$$

Proposition 1 is the consequence of the fact that spectral sequences preserve Euler characteristics. Let $G_\bullet$ in Proposition 1 be $\operatorname{Ext}_\bullet(E_\bullet, A)$ in Proposition 2. Proposition 2 follows from Proposition 1 and the fact that $\operatorname{Hom}(E_i, A) \otimes N = \operatorname{Hom}(E_i, N)$ for all i. We will use Proposition 2 in the proof of Theorem 2 and the following sections.

We now begin to prove Theorem 2.

Proof. By taking a filtration of M,

$$0 = M_0 \subset \cdots \subset M_r = M,$$

such that $M_j/M_{j-1} \cong A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$, we have

$$\chi(M,N) = \sum \chi(A/\mathfrak{p}_j, N)$$

and

$$\xi(M,N) = \sum \xi(A/\mathfrak{p}_j, N).$$

Moreover, for any finitely generated A-module P, the vanishing theorem implies that

$$\chi(P,N) = \sum_{\dim A/\mathfrak{p} = \dim P} \operatorname{length}(P_{\mathfrak{p}})\chi(A/\mathfrak{p}, N).$$

We first assume M is of the form $A/\mathfrak{q}$, where $\mathfrak{q}$ is a prime ideal. Using Formula (4), we conclude

$$\begin{aligned} \xi(M,N) &= \sum_{i=0}^{s}(-1)^i \operatorname{length}(\operatorname{Ext}^i(M,N)) \\ &= \sum_{i=0}^{s}(-1)^i \chi(\operatorname{Ext}^i(M,A),N) \\ &= \sum_{i=0}^{s}(-1)^i \operatorname{length}(\operatorname{Ext}^i_A(M,A)\otimes A_{\mathfrak{q}})\chi(A/\mathfrak{q},N), \end{aligned}$$

since $\operatorname{Supp}(\operatorname{Ext}^i(E_\bullet, A)) \subset \operatorname{Supp}(M)$. It is known that, if $M = A/\mathfrak{q}$, then

$$\operatorname{Ext}^i_A(M,A)\otimes A_{\mathfrak{q}} = \begin{cases} (A/\mathfrak{q})_{\mathfrak{q}} & i = \dim A - \dim M, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

which can be reduced by localization to showing that

$$\operatorname{Ext}^i_A(k,A) = \begin{cases} k & i = \dim A, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

(This is a special case of more general results in Bass[1]; moreover, Formula (6) is often taken as a definition of Gorenstein rings.) Thus,

$$\chi(A/\mathfrak{q}, N) = (-1)^{\operatorname{codim} A/\mathfrak{q}}\xi(A/\mathfrak{q}, N).$$

This also shows the vanishing theorem for the Euler form when the first module is in the form of $A/\mathfrak{q}$; namely,

$$\xi(A/\mathfrak{q}, N) = 0$$

if $\dim A/\mathfrak{q} + \dim N < \dim A$.

In general, $\chi(M,N)$ is a sum of $\chi(A/\mathfrak{p}_j, N)$ taken over all $\mathfrak{p}_j$ occurring in a filtration of M. Since $\dim M + \dim N \le \dim A$, by the vanishing theorem, those terms with prime ideals of dimension less than $\dim M$ must vanish; similarly for $\xi(M,N)$. From the result of the special case proved above, we obtain

$$\chi(M,N) = (-1)^{\operatorname{codim} M}\xi(M,N).$$

□

Let A be a Gorenstein ring of dimension d and let M be a finitely generated module of dimension m. We define

$$\check{M} = \operatorname{Ext}^{d-m}(M,A).$$

We say that M is perfect if and only if M is Cohen-Macaulay and of finite projective dimension; in this case, proj.dim$M$ = codim M. If M is perfect, then $\check{M}$ is also perfect of dimension equal to $\dim M$ and $\check{\check{M}} \cong M$.

There used to be a general version of the vanishing conjecture, which we call *the generalized vanishing conjecture* for simplicity, that states that if either M or N has finite projective dimension and length$(M \otimes N) < \infty$, then $\dim M + \dim N < \dim A$ implies $\chi(M, N) = 0$. The generalized vanishing conjecture is known to be false due to Dutta, Hochster, and McLaughlin's example [4]. We recall a theorem in Dutta [2].

Theorem 3 (Dutta)**.** *Let A be a Gorenstein ring. The generalized vanishing conjecture is true if and only if for any perfect module M and any Cohen-Macaulay module N with* $\dim M + \dim N = \dim A$*,*

$$\text{length}(M \otimes N) = \text{length}(M \otimes \check{N}). \tag{7}$$

Since the vanishing theorem is true if either M or N is of the form $A/(x_1, \dots, x_k)$, where $x_1, \dots, x_k$ form a regular sequence (see Serre [16]), it is not hard to show that $\chi(M, N) = (-1)^{\operatorname{codim} M} \xi(M, N)$ in this case. Using finitely many short exact sequences with modules of this form, one can reduce Theorem 2 to the case in which M is perfect and N is Cohen-Macaulay. In this case, we have

$$\text{length}(M \otimes N) = \text{length}(\check{M} \otimes \check{N})$$

and

$$\text{length}(M \otimes \check{N}) = \text{length}(\check{\check{M}} \otimes \check{N}) = \text{length}(\check{M} \otimes N).$$

It is not hard to see that $\xi(M, N) = (-1)^{\operatorname{codim} M}\, \text{length}(\check{M} \otimes N)$ and

$$\chi(M, N) = \text{length}(M \otimes N)$$

if M is perfect and N is Cohen-Macaulay with $\dim M + \dim N = \dim A$. Thus, using Theorem 3, this approach gives another proof of Theorem 2. For details we refer to Dutta [2] and [3]; at the time these papers were written the vanishing theorem was still a conjecture.

3. The relation between $\chi(M, N)$ and $\xi(M, N)$ over a complete intersection

The relation presented by Formulas (4) and (5), and the vanishing theorem are key points in the proof of Theorem 2. Over a complete intersection, these three statements are still true. However, not all modules, in particular the type of the ring modulo a prime ideal, have finite projective dimension. One may not apply the vanishing theorem as over a regular local ring. Following the proof of the vanishing theorem (see Roberts [11]), we use the theory of local Chern characters to do further generalizations.

Assume A is a Noetherian local ring. Let $\mathrm{A}_*(A)_{\mathbb{Q}} = \bigoplus \mathrm{A}_k(A)_{\mathbb{Q}}$ be the Chow group on Spec A tensored with rational number field. The *local Chern character* of a perfect complex $G_\bullet$ is

$$\mathrm{ch}(G_\bullet) = \mathrm{ch}_0(G_\bullet) + \mathrm{ch}_1(G_\bullet) + \cdots$$

where, for each i and k, $\mathrm{ch}_i(G_\bullet)$ is a map

$$\mathrm{ch}_i(G_\bullet) : \mathrm{A}_k(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_{k-i}(\mathrm{Supp}(G_\bullet))_{\mathbb{Q}}$$

and $A_*(\mathrm{Supp}(G_\bullet))$ is the Chow group on $\mathrm{Supp}(G_\bullet)$. The *Todd class* of $G_\bullet$, $\tau(G_\bullet)$, is an element in $\mathrm{A}_*(\mathrm{Supp}(G_\bullet))$.

In particular, if $G_\bullet$ is supported at the maximal ideal $\mathfrak{m}$, then

$$\tau(G_\bullet) = \chi(G_\bullet)[A/\mathfrak{m}].$$

In this case, we will identify the image of the Todd class with a number in $\mathbb{Q}$ since $\mathrm{A}_*(\mathrm{Supp}(G_\bullet))_\mathbb{Q} \cong \mathbb{Q}$. The local Riemann-Roch formula (see Roberts [10, Theorem 12.6.1]) implies that

$$\tau(G_\bullet) = \mathrm{ch}(G_\bullet)(\tau(A)),$$

which relates local Chern characters with Euler characteristics

$$\chi(G_\bullet) = \mathrm{ch}(G_\bullet)(\tau(A)).$$

For more details on the definitions and properties, we refer to Fulton [5] and Roberts [10].

Theorem 4. *Let A be a complete intersection and let M and N be finitely generated A-modules with finite projective dimension such that $M \otimes N$ has finite length and* $\dim M + \dim N \leq \dim A$. *Then,*

$$\chi(M, N) = (-1)^{\mathrm{codim}\, M}\xi(M, N). \tag{8}$$

Proof. Assume $\dim A = d$, $\dim M = m$ and $\dim N = n$. Let $E_\bullet$ and $F_\bullet$ be finite free resolutions of M and N respectively. From Formulas (3) and (4), $\xi(M, N) = \chi(E_\bullet^* \otimes F_\bullet)$. It thus suffices to show that

$$\chi(E_\bullet \otimes F_\bullet) = (-1)^{\mathrm{codim}\, M}\chi(E_\bullet^* \otimes F_\bullet). \tag{9}$$

It is known that $\tau(A) = [A]$ when A is a complete intersection. Since $\mathrm{Supp}(E_\bullet^* \otimes F_\bullet)$ contains the maximal ideal only, by the local Riemann-Roch formula, we have

$$\chi(E_\bullet^* \otimes F_\bullet) = \mathrm{ch}(E_\bullet^* \otimes F_\bullet)(\tau(A)) = \mathrm{ch}_d(E_\bullet^* \otimes F_\bullet)([A]).$$

From the multiplicativity of local Chern characters and the duality property,

$$\mathrm{ch}_i(E_\bullet^*) = (-1)^i \,\mathrm{ch}_i(E_\bullet)$$

(see Roberts [10, Theorem 9.7.2 and Theorem 12.3.1]), we have

$$\begin{aligned} \mathrm{ch}_d(E_\bullet^* \otimes F_\bullet)([A]) &= \sum_{i+j=d} \mathrm{ch}_i(E_\bullet^*)\,\mathrm{ch}_j(F_\bullet)([A]) \\ &= \sum_{i+j=d} (-1)^i \,\mathrm{ch}_i(E_\bullet)\,\mathrm{ch}_j(F_\bullet)([A]). \end{aligned}$$

For any $\alpha \in \mathrm{A}_d(A)$, $\mathrm{ch}_j(F_\bullet)(\alpha) \in \mathrm{A}_{d-j}(\mathrm{Supp}(F_\bullet))$. Since $F_\bullet$ is a finite free resolution of N, $\dim F_\bullet = \dim N = n$. Those j with $\mathrm{ch}_j(F_\bullet)(\alpha)$ possibly nonzero are $j \geq d-n$, which implies $i \leq n$ for $i+j = d$. On the other hand, the multiplicativity property, $\mathrm{ch}_i(E_\bullet)\,\mathrm{ch}_j(F_\bullet)(\alpha) = \mathrm{ch}_j(F_\bullet)\,\mathrm{ch}_i(E_\bullet)(\alpha)$, implies $i \geq d - m$ by the same argument and we thus have $d - m \leq i \leq n$. By the assumption that $m + n \leq d$, we conclude that $\mathrm{ch}_i(E_\bullet)\,\mathrm{ch}_j(F_\bullet)(\alpha) = 0$ except possibly when $m + n = d$, $i = d - m$ and $j = d-n$. This argument was first pointed out in the proof of Serre's vanishing theorem in Roberts [11]. In Serre's vanishing theorem, $m + n < d$, so all terms vanish. Similarly,

$$\begin{aligned} \chi(E_\bullet \otimes F_\bullet) &= \textstyle\sum_{i+j=d} \mathrm{ch}_i(E_\bullet)\,\mathrm{ch}_j(F_\bullet)([A]) \\ &= \mathrm{ch}_{d-m}(E_\bullet)\,\mathrm{ch}_{d-n}(F_\bullet)([A]). \end{aligned}$$

Hence,

$$\chi(E_\bullet^* \otimes F_\bullet) = (-1)^{d-m} \operatorname{ch}_{d-m}(E_\bullet) \operatorname{ch}_{d-n}(F_\bullet)([A]) = (-1)^{d-m}\chi(E_\bullet \otimes F_\bullet).$$

This proves Formula (9) and completes the proof of the theorem. □

The key point of the above proof, as in Roberts [11], is to write both $\chi(M,N)$ and $\xi(M,N)$ in terms of local Chern characters. The condition that the ring is a complete intersection implies that most terms vanish and the remaining ones are easy to determine. In [13] Roberts proved that $\operatorname{ch}_1(G_\bullet) = 0$ if $G_\bullet$ is a perfect complex with support of codimension greater than or equal to 2. Therefore, the vanishing conjecture holds for Gorenstein rings of dimension less than or equal to 5 (see Roberts [12]); namely, $\chi(M,N) = 0$ if $\dim M + \dim N < \dim A$ and so is $\xi(M,N)$. This case was also proved by Dutta [2] and [3] using homological methods. We will come back to Dutta's proof in the next section. From the proof of Theorem 4, only one term possibly does not vanish if $\dim M + \dim N = \dim A$ and A is a complete intersection. Over a Gorenstein ring, more terms remain undetermined in both $\chi(M,N)$ and $\xi(M,N)$. However, they agree with signs. We have the following theorem.

Theorem 5. *Let A be a Gorenstein ring of dimension less than or equal to 5. Let M and N be as in Theorem 4. Then,*

$$\chi(M,N) = (-1)^{\operatorname{codim} M}\xi(M,N).$$

Proof. Let $E_\bullet$ and $F_\bullet$ be finite free resolutions of M and N respectively and let d, m, and n be as in the previous proof. We prove the case when $d = \dim A = 5$. The proof for lower dimensional rings is similar. Since A is a Gorenstein ring, $\tau_{d-i}(A) = 0$ if i is odd (see [10, Proposition 12.4.4]). Therefore

$$\xi(M,N) = \chi(E_\bullet^* \otimes F_\bullet) = \sum_{s=1,3,5} C(s),$$

where $C(s)$ denotes $ch_s(E_\bullet^* \otimes F_\bullet)(\tau_s(A))$ and is equal to

$$C(s) = \sum_{i+j=s} (-1)^i \operatorname{ch}_i(E_\bullet) \operatorname{ch}_j(F_\bullet)(\tau_s(A)).$$

If $s = 5$, $\tau_5(A)$ is the top term of the Todd class; then as in the proof for complete intersections,

$$C(5) = \begin{cases} (-1)^{5-m} ch_{5-m}(E_\bullet) ch_{5-n}(F_\bullet)([\operatorname{Spec} A]) & \text{if } m+n=5, \\ 0 & \text{otherwise.} \end{cases}$$

If $s = 3$ or 1, each term has either zeroth Chern character or first Chern character. The zeroth Chern character can be identified with the alternating sum of the ranks of the free modules of the complex and so vanishes if the complex is not supported at a minimal prime ideal, namely, if the complex has codimension greater than zero. Also, the first Chern character vanishes if the complex has codimension ≥ 2 as was pointed out earlier [13]. As it was shown in Roberts [12], the local Chern characters in $C(s)$ all vanish if $\dim M + \dim N < \dim A$. We summarize the possibly nonvanishing cases when the sum of the dimensions reaches $\dim A$ and leave the details to the reader:

$$\begin{aligned} C(3) &= \begin{cases} \operatorname{ch}_2(E_\bullet)\operatorname{ch}_1(F_\bullet)(\tau_3(A)) & \text{if } \dim E_\bullet = 1 \text{ and } \dim F_\bullet = 4, \\ -\operatorname{ch}_3(E_\bullet)\operatorname{ch}_0(F_\bullet)(\tau_3(A)) & \text{if } \dim E_\bullet = 0 \text{ and } \dim F_\bullet = 5\ , \\ 0 & \text{otherwise,} \end{cases} \\ C(1) &= 0. \end{aligned}$$

The same argument holds for $\chi(E_\bullet \otimes F_\bullet)$ and there are exactly the same terms left with opposite signs as we wish to have. Therefore, we have

$$\chi(E_\bullet \otimes F_\bullet) = (-1)^{\operatorname{codim} M} \xi(E_\bullet^* \otimes F_\bullet).$$

This completes the proof of the theorem. □

A similar argument shows the vanishing theorem for the Euler form. One may also prove it as a corollary to Theorem 1 (Serre's vanishing theorem) and Theorem 4 or Theorem 5.

Corollary. *Let A, M, and N be as in Theorem 4 or Theorem 5. If* $\dim M + \dim N < \dim A$, *then* $\xi(M, N) = 0$.

4. Discussion

In this section, we further investigate the relation between the vanishing theorem and the formula relating $\xi(M, N)$ to $\chi(M, N)$.

We remark that an example in Roberts [10, Chap. 13, Section 3] shows that Theorem 4 is false over a Cohen-Macaulay ring when both modules have finite projective dimension. So, the result of Theorem 4 remains unknown only over a Gorenstein ring of dimension greater than 5.

Over a higher dimensional Gorenstein ring, whether or not the vanishing theorem holds is still an open problem. Also, we do not know whether or not Theorem 4 is true over an arbitrary Gorenstein ring. However, in the following theorem, we are able to show that the fact that the formula holds implies the vanishing theorem.

Let A be a Gorenstein ring and let M and N be finitely generated A-modules. We say *M and N have Property A* if they satisfy the following conditions:

1. $\dim M + \dim N \leq \dim A$.
2. $M \otimes N$ has finite length.
3. Either M or N has finite projective dimension.

We note that condition 2 implies that $\operatorname{Ext}^i(M, N)$ and $\operatorname{Tor}_i(M, N)$ have finite length and condition 3 ensures that both expressions for $\chi(M, N)$ and $\xi(M, N)$ have finite sums. In particular, if A is Gorenstein, a finitely generated module has finite projective dimension if and only if it has finite injective dimension. Therefore, with condition 2, it suffices to have one of the two modules having finite projective dimension in order to have both $\chi(M, N)$ and $\xi(M, N)$ well-defined.

The idea of the proof of the following theorem is due to I. Mori.

Theorem 6. *Let A be a Gorenstein ring of dimension d. If the formula*

$$\chi(M, N) = (-1)^{\operatorname{codim} M} \xi(M, N)$$

holds for any two modules M and N which have Property A, then the vanishing theorem holds; namely, if any two modules P and Q have Property A, then $\dim P + \dim Q < \dim A$ *implies*

$$\chi(P, Q) = 0.$$

Proof. Let P and Q be two modules which have Property A and $\dim P + \dim Q < \dim A$. Suppose there exists an A-module P_1 such that P_1 and Q have Property A and $\dim P_1 = \dim P + 1$. Therefore, $P \oplus P_1$ and Q also have Property A and the formula holds by the assumption. Thus, we may apply $\chi(\cdot, Q)$ and $\xi(\cdot, Q)$ on $P \oplus P_1$, which has dimension equal to $\dim P_1$, and get

$$\begin{aligned}
\chi(P,Q) + \chi(P_1,Q) &= \chi(P \oplus P_1, Q) \\
&= (-1)^{\operatorname{codim} P_1}\xi(P \oplus P_1, Q) \\
&= -(-1)^{\operatorname{codim} P}\xi(P,Q) + (-1)^{\operatorname{codim} P_1}\xi(P_1,Q) \\
&= -\chi(P,Q) + \chi(P_1,Q).
\end{aligned}$$

This shows that $\chi(P,Q) = 0$.

We now prove the existence of the module P_1. Assume $\dim P = \ell - 1$. There exists a regular sequence on A with length $d - \ell$, $x_1, \ldots, x_{d-\ell}$, such that the module $Q/(x_1, \ldots, x_{d-\ell})$ has finite length. Following a standard commutative algebra argument, such a regular sequence may be constructed by avoiding finitely many prime ideals at each step. Therefore, $P_1 = A/(x_1, \ldots, x_{d-\ell})$ has dimension ℓ as required. Moreover, the Koszul complex on $x_1, \ldots, x_{d-\ell}$ is a resolution of P_1 so P_1 has finite projective dimension. All three conditions in Property A hold. $\square$

In section 3, we have seen that the vanishing theorem and Formula (8) over a complete intersection or a small Gorenstein ring share essentially the same proof. It is worth pointing out that Theorem 2 together with Theorem 6 shows the equivalence of the vanishing theorem and the formula over a regular local ring as Theorem 3 also proves. It is not clear that this equivalence is true in general. Roberts in [14] further proved the vanishing theorem when the singular locus of $\operatorname{Spec} A$ has dimension at most one. In this generality, there will be a lot more terms that remain undetermined than those in the proof of Theorem 5 and it is not obvious that the formula will hold. However, we do know that the formula holds if A is a Gorenstein ring with singular locus of dimension at most one. It is unknown whether or not the vanishing conjecture and the formula are equivalent over arbitrary Gorenstein rings.

Although the generalized vanishing conjecture has a counterexample, Theorem 3 can be revised so that the vanishing conjecture over a Gorenstein ring is a sufficient condition for Formula (7) (see [3]). Dutta proved the vanishing theorem for Gorenstein rings of dimension ≤ 5 by proving Formula (7) is true. As the referee pointed out, Theorem 6 can be derived from either Theorem 3 or its revised form since the conditions of M and N in Theorem 3 all satisfy Property A and the formulas in these two theorems are equivalent under the conditions in which M is perfect and N is Cohen-Macaulay.

Dutta, Hochster and McLaughlin constructed an example over a complete intersection which shows that the generalized vanishing theorem is not true [4]; that is, one of the modules has infinite projective dimension. Over the same ring, there exist two modules which have Property A, but the formula is not satisfied. These modules may be constructed by going through Theorem 6 based on Dutta, Hochster and McLaughlin's example.

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Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, Utah 84112

Current address: Department of Mathematics, Purdue University, 1395 Mathematical Sciences Building, West Lafayette, Indiana 47907-1395

E-mail address: cyjan@math.utah.edu