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Filtrations of Modules, the Chow Group, and the Grothendieck Group

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The aim of this paper is to show that a finitely generated module over a Noetherian ring defines a unique cycle class in the components with codimension zero and one of the Chow group of the ring. The main theorem generalizes a classical result over integrally closed domains and implies the isomorphism between the Chow group and the Grothendieck group under certain conditions. We also discuss the difference between the map constructed in this paper and the Riemann–Roch map.

1. INTRODUCTION AND MOTIVATION

In this paper, all rings are assumed to be finitely generated algebras over a regular ring with finite Krull dimension; hence they are commutative Noetherian rings with identity. All modules are finitely generated.

Let A be a Noetherian ring and let M be a finitely generated module over A. There is a finite filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$$

of M such that for each $i = 1, \dots, n$, $F_i/F_{i-1} \cong A/\mathfrak{p}_i$, for some prime ideal $\mathfrak{p}_i$. A filtration of this type is usually not unique. Let $\mathscr{S}_M$ denote the set of all filtrations as described and denote elements in $\mathscr{S}_M$ with script letters, say $\mathscr{F}$. Throughout this paper, filtrations of a module will be those in $\mathscr{S}_M$ unless otherwise stated.

For any finitely generated A-module M, if $\mathfrak{p}$ is a minimal prime ideal in the support of M, then the number of times $A/\mathfrak{p}$ occurs in any filtration of M is exactly equal to the length of $M_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$. Apart from this,

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neither the prime ideals which occur nor the number of times they occur is unique. However, each filtration gives a cycle defined by the prime ideals occurring in it.

The main aim of this paper is to show that in certain cases the cycle is unique up to rational equivalence and to investigate the extent to which the cycle is unique up to rational equivalence. The uniqueness, as we will show in the second section, holds when the cycle is defined by the prime ideals of codimension zero and one, but it is not true in general. The motivation for this comes from the divisor class of a module over an integrally closed domain, which will be stated in the following.

Throughout this paper, the dimension of modules and rings is as defined in Roberts [6, Chap. 4] or Fulton [2, Chap. 20]. Readers who are not familiar with this material may consider the dimension as the Krull dimension. In fact, they are the same in most cases.

We first define the Chow group of a ring and rational equivalence. Let $Z_i(A)$ be the free Abelian group generated by prime ideals of dimension i. The dimension of a prime ideal $\mathfrak{p}$ is defined to be the dimension of $A/\mathfrak{p}$ as an A-module. If $\mathfrak{p}$ is a prime ideal with $\dim A/\mathfrak{p} = i$, we denote the generator in $Z_i(A)$ corresponding to $\mathfrak{p}$ by $[\operatorname{Spec} A/\mathfrak{p}]$. The *group of cycles* of A is defined by $Z_*(A) = \oplus Z_i(A)$.

Let $\mathfrak{p}$ be a prime ideal of dimension $i+1$ and let x be an element not in $\mathfrak{p}$. Define

$$\operatorname{div}(\mathfrak{p}, x) = \sum_{\dim A/\mathfrak{q} = i} \operatorname{length}_{A_\mathfrak{q}}\big((A/\mathfrak{p}/x(A/\mathfrak{p}))_\mathfrak{q}\big)[\operatorname{Spec} A/\mathfrak{q}].$$

Let $\operatorname{Rat}_i(A)$ be the free Abelian subgroup of $Z_i(A)$ generated by all cycles of the form $\operatorname{div}(\mathfrak{p}, x)$ with $\mathfrak{p}$ a prime ideal of dimension $i+1$ and x not in $\mathfrak{p}$. Two cycles are *rationally equivalent* if their difference lies in $\operatorname{Rat}_i(A)$. This equivalence relation is called *rational equivalence*. Denote $Z_i(A)/\operatorname{Rat}_i(A)$ by $A_i(A)$. The *Chow group* of A is the direct sum of all groups $A_i(A)$ and is denoted by $A_*(A)$. Let $[\operatorname{Spec} A/\mathfrak{p}]$ also denote the generator of $A_i(A)$ corresponding to the prime ideal $\mathfrak{p}$. We refer to Roberts [6] for details and basic facts.

If the ring A is an integral domain and if the zero ideal is in the support of a module M, then it is the only minimal prime ideal of M. We consider the number of times the zero ideal occurs in a filtration. It is not hard to see that this number is the rank of M, which is zero if M is a torsion module. If A is a Noetherian integrally closed domain, each finitely generated module can be uniquely assigned an element $(r(M), -c(M))$ in $\mathbb{Z} \oplus \operatorname{Cl}(A)$, where $\operatorname{Cl}(A)$ is the divisor class group of A, $r(M)$ is the rank of M, and $c(M)$ is the divisor class of M defined in $\operatorname{Cl}(A)$ (see Bourbaki [1, Chap. VII, Sect. 4, No. 7]). On the other hand, whereas A is a

Noetherian integrally closed domain of dimension d, there exists an isomorphism

$$\mathbb{Z} \oplus \mathrm{Cl}(A) \overset{i}{\cong} A_d(A) \oplus A_{d-1}(A)$$

(see Roberts [6, Chap. 1]). For each module M and any filtration $\mathscr{F} \in \mathscr{S}_M$, let $\mathscr{C}_{\mathscr{F}}$ be the cycle in $Z_*(A)$ obtained by summing all prime ideals of dimension at least $d-1$ occurring in the filtration $\mathscr{F}$. The cycle class in $A_d(A) \oplus A_{d-1}(A)$ defined by $\mathscr{C}_{\mathscr{F}}$ corresponds to the element $(r(M), -c(M))$ that M is assigned to in $\mathbb{Z} \oplus \mathrm{Cl}(A)$. This gives the following proposition.

PROPOSITION. *If A is a Noetherian integrally closed domain and M is a finitely generated A-module, then $\mathscr{C}_{\mathscr{F}_1}$ is rationally equivalent to $\mathscr{C}_{\mathscr{F}_2}$ for any two filtrations $\mathscr{F}_1$ and $\mathscr{F}_2$ in $\mathscr{S}_M$.*

This proposition is generalized by Theorem 1 in Section 2. The third section presents an application of Theorem 1 to the relation between the Chow group of a ring and the Grothendieck group of finitely generated modules over that ring, for which the Riemann−Roch theorem has given some results. This leads us to think about the difference between the Riemann−Roch map and the map σ defined in Section 3 and to make a further observation to a possible generalization of the map σ in a special example. We will discuss these in the last section. An example is also given in the third section to explain that not all cycles are rationally equivalent in general if they contain lower dimensional prime ideals.

2. THE MAIN THEOREM

Let A be a Noetherian ring of dimension d.

DEFINITIONS. For any finitely generated A-module M, define

$$M_i := \{m \in M : \dim Am \leq d - i\}$$

$$r_i(M) := \sum_{\substack{\mathfrak{p} \in \mathrm{Spec}\, A \\ \dim A/\mathfrak{p} = d-i}} \mathrm{length}_{A_{\mathfrak{p}}}(M_i \otimes A_{\mathfrak{p}})$$

$$R(M) := (r_0(M), r_1(M)).$$

$R(M)$ is called the *rank* of M.

If A is an integral domain, then $r_0(M)$ is the rank of M defined in the usual sense. If $M_i = M_{i-1}$, then $r_i(M) = 0$. We note that $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{d-1} \supseteq M_d \supseteq 0$.

Let $\mathscr{S}_M$ and $\mathscr{C}_{\mathscr{F}}$ be as defined in the last section.

If a module M is isomorphic to A modulo a prime ideal $\mathfrak{p}$ of dimension greater than or equal to $d-1$, then $\mathfrak{p}$ is the only associated prime ideal of M. For any nonzero element x in $A/\mathfrak{p}$ and $\mathscr{E} \in \mathscr{S}_{(A/\mathfrak{p})/Ax}$,

$$\mathscr{C}_{\mathscr{E}} = \begin{cases} 0, & \text{if } \dim A/\mathfrak{p} = d-1 \\ \operatorname{div}(\mathfrak{p}, x), & \text{if } \dim A/\mathfrak{p} = d. \end{cases}$$

Therefore, if $M \cong A/\mathfrak{p}$, then for any $\mathscr{F}$ in $\mathscr{S}_M$, $\mathscr{C}_{\mathscr{F}}$ is rationally equivalent to $[\operatorname{Spec} A/\mathfrak{p}]$. From now on, $\mathscr{C}_{\mathscr{F}}$ is replaced by $[\operatorname{Spec} A/\mathfrak{p}]$ modulo rational equivalence for all $\mathscr{F}$ in $\mathscr{S}_M$ with M of the type of modules as described in this paragraph.

THEOREM 1. *Assume M is a finitely generated A-module. Then, for any $\mathscr{F}, \mathscr{G} \in \mathscr{S}_M$, $\mathscr{C}_{\mathscr{F}}$ and $\mathscr{C}_{\mathscr{G}}$ are rationally equivalent, which is written as $\mathscr{C}_{\mathscr{F}} \sim \mathscr{C}_{\mathscr{G}}$. These equivalent cycles determine a unique class in $A_*(A)$, denoted $\{M\}$, and we call $\{M\}$ the cycle class associated to M.*

It follows from the theorem that these classes are *additive*; that is, $\{M'\} + \{M''\} = \{M\}$ in $A_*(A)$ whenever $0 \to M' \to M \to M'' \to 0$ is a short exact sequence. This is because a filtration of M' combined with one of M'' gives a filtration of M.

Each filtration in $\mathscr{S}_M$ must start with an associated prime ideal. First we would like to show that it suffices to prove the theorem for those filtrations starting with an associated prime ideal of dimension at least $d-1$.

LEMMA. *Let N be a submodule of M such that N has dimension at most $d-2$. Then $\{C_{\mathscr{F}} \in Z_*(A) : \mathscr{F} \in \mathscr{S}_M\}$ coincides with $\{C_{\mathscr{R}} \in Z_*(A) : \mathscr{R} \in \mathscr{S}_{M/N}\}$.*

Proof. Since N is a submodule of M, there is a short exact sequence $0 \to N \to M \to M/N \to 0$. Any filtration of M/N combined with one of N gives a filtration of M. So, the collection of prime ideals of dimension $\geq d-1$ occurring in such a filtration of M gives exactly the same cycle as the one from M/N.

Conversely, for a given filtration $\mathscr{F} \in \mathscr{S}_M$,

$$\mathscr{F} : 0 \subseteq F_1 \subseteq \cdots \subseteq F_{i-1} \subseteq F_i \subseteq \cdots \subseteq F_n = M,$$

with $F_i/F_{i-1} \cong A/\mathfrak{p}_i$,

$$0 \subseteq \frac{F_1}{F_1 \cap N} \subseteq \cdots \subseteq \frac{F_{i-1}}{F_{i-1} \cap N} \subseteq \frac{F_i}{F_i \cap N} \subseteq \cdots \subseteq \frac{F_n}{F_n \cap N} = M/N,$$

defines a filtration of M/N. This is not a filtration in $\mathscr{S}_{M/N}$. However, we may take filtrations of the quotients of each two successive modules and get a filtration of M/N in $\mathscr{S}_{M/N}$, denoted by $\overline{\mathscr{F}}$. Obviously, $\overline{\mathscr{F}}$ may not be unique. Since we only care about the cycle defined by prime ideals of dimension d and $d-1$, namely, $\mathscr{C}_{\overline{\mathscr{F}}}$, we will show, in the next step, that $\mathscr{C}_{\overline{\mathscr{F}}}$ depends only on $\mathscr{F}$ and is unique as a cycle in $Z_*(A)$.

If $\dim F_i/F_{i-1} \geq d-1$, then

$$\left(\frac{F_i}{F_i \cap N}\right) \Big/ \left(\frac{F_{i-1}}{F_{i-1} \cap N}\right) \cong F_i/F_{i-1} \cong A/\mathfrak{p}_i.$$

This follows from the fact that there is an onto map

$$F_i/F_{i-1} \xrightarrow{\phi} \left(\frac{F_i}{F_i \cap N}\right) \Big/ \left(\frac{F_{i-1}}{F_{i-1} \cap N}\right) \cong \frac{F_i}{F_{i-1} + (F_i \cap N)}.$$

If $\bar{x} \in F_i/F_{i-1}$ and $\phi(\bar{x}) = \bar{x} = 0 \in F_i/(F_{i-1} + (F_i \cap N))$, then $x \in F_{i-1} + (F_i \cap N)$. This implies that $\bar{x} \in (N \cap F_i)/(F_{i-1} \cap N) \subseteq N/(F_{i-1} \cap N)$, which has dimension at most $d-2$. However, if $\bar{x} \neq 0$, $\operatorname{annih}(\bar{x}) = \mathfrak{p}_i$ and $A/\mathfrak{p}_i$ with dimension at least $d-1$ is embedded into $N/(F_{i-1} \cap N)$, which is a contradiction. Thus, $\bar{x} = 0$ and this shows ϕ is an isomorphism. The cycle obtained from this quotient,

$$\left(\frac{F_i}{F_i \cap N}\right) \Big/ \left(\frac{F_{i-1}}{F_{i-1} \cap N}\right) \cong A/\mathfrak{p}_i,$$

is [Spec $A/\mathfrak{p}_i$].

If F_i/F_{i-1} has dimension at most $d-2$, then so does

$$\left(\frac{F_i}{F_i \cap N}\right) \Big/ \left(\frac{F_{i-1}}{F_{i-1} \cap N}\right).$$

The prime ideals occurring in any filtration of this part map to zero in $Z_*(A)$. Thus, the prime ideals of dimension at least $d-1$ occurring in any filtration $\overline{\mathscr{F}}$ induced by $\mathscr{F}$ are exactly the same as those occurring in $\mathscr{F}$; in other words, $\mathscr{C}_{\overline{\mathscr{F}}} = \mathscr{C}_{\mathscr{F}}$. This proves the lemma. ∎

From the definition, one can see that $(M/M_2)_2 = 0$. Thus, by the lemma, we may replace M by M/M_2 since $\dim M_2 \leq d-2$ and assume M has no associated prime ideals of dimension less than $d-1$.

Thus, we may assume that the first nonzero submodule in the filtration has dimension at least $d-1$. Recall that $R(M) = (r_0(M), r_1(M))$. Given two finitely generated modules, N_1 and N_2, we will write $R(N_1) < R(N_2)$

according to the lexicographic order, which means either $r_0(N_1) < r_0(N_2)$, or $r_0(N_1) = r_0(N_2)$ and $r_1(N_1) < r_1(N_2)$.

Proof of the Theorem. We do induction on the rank of modules as defined earlier. If $r_0(M) = 0$, then $\dim M \le d - 1$. The number of times that minimal prime ideals occur in any filtration of M in $\mathscr{S}_M$ is uniquely determined, so $C_{\mathscr{F}} = C_{\mathscr{G}}$ in $Z_*(A)$ for any two filtrations in $\mathscr{S}_M$.

Suppose the theorem is true for modules with rank less than $R(M)$; namely, if N is a finitely generated A-module with $R(N) < R(M)$, then $\{N\}$ is well defined. Let $\mathscr{F}$ and $\mathscr{G}$ both be in $\mathscr{S}_M$. Let $F_1 \cong A/\mathfrak{p}$ and $G_1 \cong A/\mathfrak{q}$, where there exist m and m' in M such that $\operatorname{annih}(m) = \mathfrak{p}$ and $\operatorname{annih}(m') = \mathfrak{q}$ with $\dim A/\mathfrak{p} \ge d - 1$ and $\dim A/\mathfrak{q} \ge d - 1$.

We claim that $R(M/Am)$ is less than $R(M)$, and so is $R(M/Am')$. From the short exact sequence

$$0 \to Am \to M \to M/Am \to 0,$$

if $\dim Am = d$, then $r_0(M/Am) = r_0(M) - 1$ and this implies $R(M/Am) < R(M)$.

If $\dim Am = d - 1$, then $r_0(M/Am) = r_0(M)$ and we have to show $r_1(M/Am) < r_1(M)$ in order to show $R(M/Am) < R(M)$. By definition, there is a canonical injection from M_1/Am to $(M/Am)_1$. Assume $\bar{x} \in (M/Am)_1$; i.e., $\bar{x} \in (M/Am)$ and $\dim \bar{x} \le d - 1$. We claim that x must be in M_1. If so, then $M_1/Am \cong (M/Am)_1$. Hence, $r_1(M/Am) = r_1(M_1/Am) = r_1(M) - 1$. This shows $R(M/Am) < R(M)$. That $x \in M_1$ is true for if $Ax \cap Am = 0$, then $\operatorname{annih}(\bar{x}) = \operatorname{annih}(x)$, which implies $\dim Ax \le d - 1$ and $x \in M_1$. If $Ax \cap Am \ne 0$, from the short exact sequence

$$0 \to Ax \cap Am \to Ax \to Ax/(Ax \cap Am) \cong A\bar{x} \to 0,$$

we know that $\dim Ax = \sup\{\dim(Ax \cap Am), \dim Ax/(Ax \cap Am)\} \le d - 1$. Therefore, x is also in M_1.

Since $R(M/Am) < R(M)$, by the induction hypothesis, $\{M/Am\}$ is well defined, and the same is true for M/Am'. We have $C_{\mathscr{F}} \sim [\operatorname{Spec} A/\mathfrak{p}] + \{M/Am\}$ and $C_{\mathscr{G}} \sim [\operatorname{Spec} A/\mathfrak{q}] + \{M/Am'\}$.

Suppose

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence with nonzero modules N', N, and N''. We then have $R(N') \le R(N)$ and $R(N'') \le R(N)$. If $R(N) < R(M)$, then by the induction hypothesis, $\{N\}$, $\{N'\}$, and $\{N''\}$ are all well defined.

If $Am \cap Am' \ne 0$, then $\mathfrak{p} = \mathfrak{q}$. Take any nonzero element y in the intersection. The following exact sequence exists since $y \in Am$:

$$0 \to Am/Ay \to M/Ay \to M/Am \to 0.$$

Observe that $Am/Ay \cong (A/\mathfrak{p})/Ay$, so for any $\mathscr{E} \in \mathscr{S}_{Am/Ay}$, $\mathscr{C}_{\mathscr{E}}$ is rationally equivalent to zero. This implies $\{M/Ay\} = \{M/Am\}$. Similarly, we also have $\{M/Ay\} = \{M/Am'\}$; thus, $\{M/Am\} = \{M/Am'\}$ in $A_*(A)$ and $C_{\mathscr{F}} \sim C_{\mathscr{G}}$ in $Z_*(A)$.

If $Am \cap Am' = 0$, we have the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & Am' \arrow[r, "\alpha"] \arrow[d] & Am' \cong A/q \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & Am \arrow[r] \arrow[d] & M \arrow[r, "\beta"] \arrow[d] & M/Am \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Am \cong A/\mathfrak{p} \arrow[r] \arrow[d] & M/Am' \arrow[r] \arrow[d] & M/(Am + Am') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which α is an identity map and γ is induced by the map β. The diagram is completed by the Snake lemma such that each row and column is in exact sequence. Since both M/Am and M/Am' have rank less than $R(M)$, the two cycles now become

$$\begin{aligned}
C_{\mathscr{F}} &\sim [\operatorname{Spec} A/\mathfrak{p}] + \{M/Am\} \\
&\sim [\operatorname{Spec} A/\mathfrak{p}] + [\operatorname{Spec} A/\mathfrak{q}] + \{M/(Am + Am')\} \\
C_{\mathscr{G}} &\sim [\operatorname{Spec} A/\mathfrak{q}] + \{M/Am'\} \\
&\sim [\operatorname{Spec} A/\mathfrak{q}] + [\operatorname{Spec} A/\mathfrak{p}] + \{M/(Am + Am')\}.
\end{aligned}$$

This shows $C_{\mathscr{F}} \sim C_{\mathscr{G}}$ and completes the proof. ∎

3. THE RELATION BETWEEN THE CHOW GROUP AND THE GROTHENDIECK GROUP

Let X be an algebraic scheme over a field and let K_0X be the Grothendieck group of coherent sheaves on X. By the Riemann–Roch theorem (Fulton [2, Chap. 18]), there exists a map

$$\tau = \tau_X : K_0X \to A_*X_{\mathbb{Q}},$$

and this map induces an isomorphism (Corollary 18.3.2 in Fulton [2])

$$\tau_{\mathbb{Q}}: K_0 X_{\mathbb{Q}} \to A_* X_{\mathbb{Q}}.$$

$K_0 X_{\mathbb{Q}}$ (resp. $A_* X_{\mathbb{Q}}$) denotes the Grothendieck group (resp. the Chow group) tensoring with the rational number field and is called the rational Grothendieck group (resp. the rational Chow group).

To define the map τ, one has to go through the definitions of Chern classes and local Chern characters (see Fulton [2] and Roberts [6]), which involve a lot of machinery from intersection theory. Furthermore, it is interesting to know whether the Grothendieck group and the Chow group are isomorphic without tensoring by the rational number field.

If $X = \operatorname{Spec} A$, there is an equivalence of categories between the category of coherent $\mathscr{O}_X$-sheaves and the category of finitely generated A-modules. Thus, instead of working on $K_0 X$, we may work on $K_0 A$, the Grothendieck group of finitely generated A-modules. Each finitely generated A-module M determines an element, denoted $[M]$, in $K_0 A$. $K_0 A$ is the free Abelian group on the set of isomorphism classes of finitely generated modules, modulo the relations $[M] = [M'] + [M'']$ whenever

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of A-modules. Note that $K_0 A$ is generated by $[A/\mathfrak{p}]$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Let $G_i K_0 A$ denote the subgroup of $K_0 A$ generated by the set $\{[N] \in K_0 A : \dim N \le i\}$; equivalently, it is generated by the set $\{[A/\mathfrak{p}] : \mathfrak{p} \in \operatorname{Spec} A, \dim A/\mathfrak{p} \le i\}$. Obviously, $K_0 A = G_d K_0 A$.

In Bourbaki [1, Chap. VII, Sect. 4, No. 7], it is stated that if A is an integrally closed domain, then there exists an isomorphism between $K_0 A / G_{d-2} K_0 A$ and $\mathbb{Z} \oplus \mathrm{Cl}(A)$, which is isomorphic to $A_d(A) \oplus A_{d-1}(A)$. With Theorem 1, we are able to construct a map, for an arbitrary ring A,

$$\sigma : K_0 A \to A_d(A) \oplus A_{d-1}(A),$$

by sending $[M]$ to $\{M\}$. This map is well defined because the cycle classes associated to modules are additive.

We have the following corollaries to Theorem 1.

COROLLARY 1.

$$G_d K_0 A / G_{d-1} K_0 A \cong A_d(A)$$
$$G_{d-1} K_0 A / G_{d-2} K_0 A \cong A_{d-1}(A).$$

Proof. Assume $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal prime ideals of A with dimension d. Then $A_d(A) \cong \oplus^n \mathbb{Z}$ and is generated by $[\operatorname{Spec} A/\mathfrak{p}_i]$.

Consider $G_d K_0 A / G_{d-1} K_0 A$, which is generated by modules of dimension d. There are no relations among $\overline{[A/\mathfrak{p}_i]}$ because $\mathfrak{p}_i$ is minimal. Thus, by taking filtrations, one can easily see that $G_d K_0 A / G_{d-1} K_0 A$ is generated freely by $\overline{[A/\mathfrak{p}_i]}$; i.e., it is also isomorphic to $\oplus^n \mathbb{Z}$. Therefore, $G_d K_0 A / G_{d-1} K_0 A \cong A_d(A)$.

Define

$$\varphi_{d-1}\colon G_{d-1} K_0 A \to A_{d-1}(A)$$

by setting $\varphi_{d-1} = \sigma|_{G_{d-1}K_0 A}$. It is a well-defined map since φ_{d-1} is the restriction of σ to the subgroup $G_{d-1} K_0 A$ of $K_0 A$. From the definition of φ_{d-1}, it also induces a map

$$\overline{\varphi}_{d-1}\colon G_{d-1} K_0 A / G_{d-2} K_0 A \to A_{d-1}(A),$$

because $\sigma([N]) = \{N\} = 0$ if $\dim N \leq d-2$.

On the other hand, define a map

$$\psi_i'\colon Z_i(A) \to G_i K_0 A / G_{i-1} K_0 A$$

by sending [Spec $A/\mathfrak{p}$] to $\overline{[A/\mathfrak{p}]}$. Let $\mathfrak{q} \in \operatorname{Spec} A$ with dimension $i+1$ and let x be a nonzero element in A but not in $\mathfrak{q}$. Then $\dim((A/\mathfrak{q})/x(A/\mathfrak{q})) \leq i$ and we have

$$\operatorname{div}(\mathfrak{q}, x) = \sum_{\dim A/\mathfrak{p} = i} \operatorname{length}_{A_p}((A/\mathfrak{q})/x(A/\mathfrak{q}))_{\mathfrak{p}}[\operatorname{Spec} A/\mathfrak{p}].$$

Furthermore, $[(A/\mathfrak{q})/x(A/\mathfrak{q})] = 0$ in $G_i K_0 A$. Taking a filtration of $(A/\mathfrak{q})/x(A/\mathfrak{q})$, we have

$$[(A/\mathfrak{q})/x(A/\mathfrak{q})] - \sum_{\dim A/\mathfrak{p} = i} \operatorname{length}_{A_{\mathfrak{p}}}((A/\mathfrak{q})/x(A/\mathfrak{q}))_{\mathfrak{p}}[A/\mathfrak{p}] \in G_{i-1} K_0 A.$$

This implies $\psi_i'(\operatorname{div}(\mathfrak{q}, x)) = 0$ and thus, ψ_i' induces a map on $A_i(A)$, denoted by ψ_i.

We now have

$$\psi_{d-1}\colon A_{d-1}(A) \to G_{d-1} K_0 A / G_{d-2} K_0 A$$

and

$$\overline{\varphi}_{d-1}\colon G_{d-1} K_0 A / G_{d-2} K_0 A \to A_{d-1}(A).$$

By checking generators, it is not hard to see that ψ_{d-1} is the inverse map of $\overline{\varphi}_{d-1}$; thus, $\overline{\varphi}_{d-1}$ is an isomorphism. ∎

COROLLARY 2. *Assume* $G_{d-2}K_0A = 0$ *and* $A_0 \oplus \cdots \oplus A_{d-2} = 0$ (*in particular, these conditions hold if* $\dim A \leq 1$). *Then*

$$K_0 A \cong A_*(A).$$

This is true because

$$K_0 A \cong (G_d K_0 A / G_{d-1} K_0 A) \oplus G_{d-1} K_0 A$$

and the assumptions imply that $A_*(A) = A_d(A) \oplus A_{d-1}(A)$. The result follows Corollary 1.

Corollary 2 is a generalization of the result stated in Bourbaki [1, Chap. VII, Sect. 4, No. 7] and also a special case in which the Grothendieck group $K_0 A$ and the Chow group $A_*(A)$ are isomorphic. It is proved independently from the Riemann–Roch theorem. However, the more general cases are still open.

Recall that the cycles in Theorem 1 are obtained by summing prime ideals of dimension at least $d-1$ in a filtration. We end this section with an example showing that the theorem cannot be extended if lower dimensional prime ideals occurring in a filtration are added to the cycles. Keep the same notation. For any filtration of a module, say $\mathscr{F} \in \mathscr{S}_M$, let $\mathscr{C}'_{\mathscr{F}}$ denote the cycle in $Z_*(A)$ obtained by summing all prime ideals of dimension at least $d-2$ occurring in $\mathscr{F}$. Set $A = k[\{x_{ij} : i, j = 1, 2, 3\}]/I_2(x_{ij})$, where k is a field and I_2 denotes

$$I_2 \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

the ideal in $k[\{x_{ij} : i, j = 1, 2, 3\}]$ generated by all 2×2 minors of the matrix.

It is known that A is an integral domain and $\dim A = 5$. Consider A as a module over itself. There are two filtrations $\mathscr{F}$ and $\mathscr{G}$ in $\mathscr{S}_A$,

$$\begin{aligned} \mathscr{F} : 0 = F_0 &\subseteq Ax_{11} = F_1 \subseteq Ax_{11} + Ax_{21} = F_2 \subseteq Ax_{11} + Ax_{21} + Ax_{31} \\ &= F_3 \subseteq A = F_4 \\ \mathscr{G} : 0 = G_0 &\subseteq A = G_1, \end{aligned}$$

such that $F_i/F_{i-1} \cong A/\mathfrak{p}_i$ and $G_1/G_0 = A$;

$$\begin{aligned} \mathscr{C}'_{\mathscr{F}} &= \sum_{i=1}^{i=4} [\operatorname{Spec} A/\mathfrak{p}_i] \\ &= [\operatorname{Spec} A] + [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13})] \\ &\quad + [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})] + [\operatorname{Spec} A/(x_{11}, x_{21}, x_{31})] \\ \mathscr{C}'_{\mathscr{G}} &= [\operatorname{Spec} A]. \end{aligned}$$

We remark that $\mathscr{G}$ is a trivial filtration since A is an integral domain. Moreover,

$$\left[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13})\right] + \left[\operatorname{Spec} A/(x_{11}, x_{21}, x_{31})\right] = \operatorname{div}(0, x_{11}).$$

We then have

$$\mathscr{C}'_{\mathscr{F}} \sim [\operatorname{Spec} A] + \left[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})\right]$$

and

$$\mathscr{C}'_{\mathscr{G}} = [\operatorname{Spec} A].$$

A result of Kurano [4], which we will recall in the next section, shows that the cycle $[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})]$ is not rationally equivalent to zero, so $\mathscr{C}'_{\mathscr{F}}$ and $\mathscr{C}'_{\mathscr{G}}$ are not rationally equivalent. More generally, one cannot define a unique class in $A_*(A)$ by simply collecting the prime ideals occurring in the filtration.

Using this example, let us see what goes wrong with the proof as we try to extend Theorem 1 by adding lower dimensional prime ideals as we did above. If A is an integral domain, then $[\operatorname{Spec} A]$ is a generator of $A_*(A)$. Any nonzero element a of A is a nonzero divisor, so there is an exact sequence

$$0 \to A \xrightarrow{a} A \to A/Aa \to 0.$$

Let $\{M\}'$ represent the analogous cycle class, which we wish to have, in $A_*(A)$ associated to a module M. Suppose the theorem is true for $\mathscr{C}'_{\mathscr{F}}$ as defined above. Then $\{M\}'$ is well defined and

$$\{A\}' = \{A\}' + \{A/Aa\}'$$

would be expected; that means $\mathscr{C}'_{\mathscr{E}} \sim 0$ for all $\mathscr{E} \in \mathscr{S}_{A/Aa}$. Let A and $a = x_{11}$ be as given in the above example. Then $\mathscr{F}$ gives a filtration $\mathscr{F}'$ in $\mathscr{S}_{A/Ax_{11}}$ such that the associated cycle $\mathscr{C}'_{\mathscr{F}'} \sim [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})]$, which has dimension 3 and is not rationally equivalent to zero. This problem does not arise in the original theorem, since for any $\mathscr{E} \in \mathscr{S}_{A/Aa}$, $\mathscr{C}_{\mathscr{E}}$ collects prime ideals of dimension at least 4; thus, $\mathscr{C}_{\mathscr{E}}$ is either 0 or equal to $\operatorname{div}(0, a)$.

4. THE RELATION BETWEEN σ AND THE RIEMANN–ROCH MAP

In this section, we discuss the difference between the map τ constructed in the Riemann–Roch theorem and the map σ defined in Section 3.

Recall the map τ,

$$\tau\colon K_0 A \to A_*(A)_{\mathbb{Q}},$$

followed by the projection from $A_*(A)$ to the component $A_i(A)_{\mathbb{Q}}$. Then there is a map

$$\tau_i\colon K_0 A \to A_i(A)_{\mathbb{Q}}.$$

It is known that τ_i satisfies the top term property; i.e.,

$$\tau_i([A/\mathfrak{p}]) = [\operatorname{Spec} A/\mathfrak{p}]$$

for any $\mathfrak{p}$ an i-dimensional prime ideal. Given a finitely generated A-module M, take any of its filtrations $\mathscr{F}$ in $\mathscr{S}_M$. Assume that $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_k}$ are distinct prime ideals of dimension d occurring in $\mathscr{F}$. By the definitions of τ_i and σ, it is clear that

$$(\tau_d + \tau_{d-1})(M) = \sigma([M]) + \sum_{j=j_1}^{j_k} \operatorname{length}\left(M_{\mathfrak{q}_j}\right)\tau_{d-1}\left([A/\mathfrak{q}_j]\right).$$

The difference between σ and $\tau_d + \tau_{d-1}$ is determined by the image of prime ideals of dimension d through the map τ_{d-1}; in many cases, it is nonzero. For instance, if $R = k[\{x_{ij} : i = 1, 2,\ j = 1, 2, 3\}]/I_2(x_{ij})$, which has dimension 4, then $\tau_3([R]) = -\frac{1}{2}[\operatorname{Spec} R/\mathfrak{p}]$, where $\mathfrak{p}$ is a prime ideal generated by elements in one row of the matrix (Kurano [4]).

If we also write σ as the sum $\sigma_d + \sigma_{d-1}$ through projections, such that for $i = d, d-1$,

$$\sigma_i\colon K_0 A \to A_i(A),$$

then by definition, $\sigma_{d-1}([A/\mathfrak{p}]) = 0$ for any prime ideal $\mathfrak{p}$ of dimension $\neq d-1$. Whereas we are interested in knowing whether or not σ could be extended by a map with nontrivial image in the other components of lower dimensions in the Chow group of A, we make a further observation. For each prime ideal $\mathfrak{p}$ of dimension d, we may redefine $\sigma_{d-1}([A/\mathfrak{p}])$ to be an arbitrary cycle, say $\eta_{\mathfrak{p}}$ and still have a well-defined map, which is only different from the original σ, for each module M, by the cycle

$$\sum \operatorname{length}\left(M_{\mathfrak{p}}\right)\eta_{\mathfrak{p}},$$

where the sum is over all minimal prime ideals of dimension d in the support of M. In the following, we will explain how the idea can be extended and work on a special case for the example presented in the previous section.

Let A be as defined in Section 3,

$$A = k\left[\{x_{ij} : i = 1, 2, 3,\ j = 1, 2, 3\}\right]/I_2(x_{ij}).$$

In Kurano [4], it is shown that

$$A_*(A)_{\mathbb{Q}} \overset{\xi}{\cong} \mathbb{Q}[a]/(a^3),$$

in which [Spec A] corresponds to 1, [Spec $A/\mathfrak{p}$] corresponds to a with $\mathfrak{p}$ a prime ideal generated by elements in a row, and [Spec $A/\mathfrak{q}$] corresponds to $-a$ with $\mathfrak{q}$ generated by elements in a column. Moreover, if $\mathfrak{r}$ is a prime ideal generated by i rows (resp. i columns) in the matrix, then $\xi([\operatorname{Spec} A/\mathfrak{r}]) = a^i$ (resp. $(-a)^i$). $A_i(A)$ is cyclic and generated by [Spec $A/\mathfrak{r}$] with $\mathfrak{r}$ a prime ideal described above.

There is a ring isomorphism

$$t \colon A \to A$$

taking x_{ij} to x_{ji}. It is also easy to see that t induces an isomorphism

$$t^* \colon A_*(A) \to A_*(A).$$

Suppose there exists a well-defined map

$$\sigma' = (\sigma_5', \sigma_4', \sigma_3') \colon K_0 A \to A_5(A)_{\mathbb{Q}} \oplus A_4(A)_{\mathbb{Q}} \oplus A_3(A)_{\mathbb{Q}}$$

such that each σ_k' satisfies the top term property for $k = 5$, 4, and 3, and the following diagram commutes:

$$\begin{array}{ccc}
K_0 A & \xrightarrow{\sigma'} & A_5(A)_{\mathbb{Q}} \oplus A_4(A)_{\mathbb{Q}} \oplus A_3(A)_{\mathbb{Q}} \\
\big\downarrow{\scriptstyle t^*} & & \big\downarrow{\scriptstyle t^*_{\mathbb{Q}}} \\
K_0 A & \xrightarrow{\sigma'} & A_5(A)_{\mathbb{Q}} \oplus A_4(A)_{\mathbb{Q}} \oplus A_3(A)_{\mathbb{Q}}
\end{array}$$

Both the Chow group and the Grothendieck group are functorial for certain classes of ring homomorphisms which include isomorphisms. Therefore, by naturality, the above commutative diagram is expected. We remark that such a diagram exists for the map τ. Let ${}^tM$ denote the module M when it is considered as an A-module through the map t and call ${}^tM$ a ${}^tA$-module to avoid confusion. An element $[M]$ in $K_0 A$ goes to $[{}^tM]$ in $K_0 A$ through t^*.

According to the filtration $\mathscr{F} \in \mathscr{S}_M$ in the previous section, if the map σ' is well defined, we must have

$$\begin{aligned}\sigma'([A]) &= [\operatorname{Spec} A] + [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13})] \\ &\qquad + [\operatorname{Spec} A/(x_{11}, x_{21}, x_{31})] \\ &\qquad + \sigma_3'([A/(x_{11}, x_{12}, x_{13})]) + \sigma_3'([A/(x_{11}, x_{21}, x_{31})]) \\ &\qquad + [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})] + \eta \\ &= [\operatorname{Spec} A] + \sigma_3'([A/(x_{11}, x_{12}, x_{13})]) \\ &\qquad + \sigma_3'([A/(x_{11}, x_{21}, x_{31})]) \\ &\qquad + [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})] + \eta,\end{aligned}$$

where η is the image of $[A]$ under $\sigma_4' + \sigma_3'$. This implies that

$$\begin{aligned}0 = \sigma_3'([A/(x_{11}, x_{12}, x_{13})]) &+ \sigma_3'([A/(x_{11}, x_{21}, x_{31})]) \\ &+ [\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})].\end{aligned}$$

$A_3(A)$ is generated by $[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}x_{23})]$, so we may assume

$$\sigma_3'([\operatorname{Spec} A/(x_{11}, x_{12}, x_{13})]) = \alpha[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})]$$

and

$$\sigma_3'([\operatorname{Spec} A/(x_{11}, x_{21}, x_{31})]) = \beta[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})].$$

As a ${}^tA$-module, ${}^t(A/(x_{11}, x_{12}, x_{13}))$ is isomorphic to $A/(x_{11}, x_{21}, x_{31})$; i.e.,

$$t^*([A/(x_{11}, x_{12}, x_{13})]) = [A/(x_{11}, x_{21}, x_{31})].$$

Since $[\operatorname{Spec} A/(\{x_{ij} : i = 1,2,3,\ j = 1,2\})] = [\operatorname{Spec} A/(\{x_{ij} : i = 1,2,\ j = 1,2,3\})]$ in $A_*(A)$, from the above commutative diagram, we, therefore, conclude $\alpha = \beta = -\frac{1}{2}$.

Recall $\mathscr{F}$ and $\mathscr{G}$ from Section 3. Following the above computation, through σ', both the image of the sum of $[A/\mathfrak{p}_i]$ for all i and the image of $[A]$ are equal to $\sigma'([A]) = [\operatorname{Spec} A] + \eta$ in the rational Chow group. Thus, in this particular case and in the rational Chow group, by adding a possible nonzero term to $[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13})]$ and $[\operatorname{Spec} A/(x_{11}, x_{21}, x_{31})]$, we obtain a cycle independent from the choice of either filtration.

The above discussion also shows that if σ' exists and if $\mathfrak{p}$ is generated by elements in either one row or one column, then

$$\sigma_3'([A/\mathfrak{p}]) = -\tfrac{1}{2}[\operatorname{Spec} A/(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})]$$

is the unique choice, but the image of other prime ideals of dimension 4 through σ_3' is unknown. However, it is worth knowing whether or not, in general, a well-defined map

$$\sigma'' : K_0 A \to A_*(A)_{\mathbb{Q}}$$

exists as an extension of σ. If it does exist, then the isomorphism between the rational Grothendieck group and the rational Chow group can be proved without using the Riemann–Roch theorem.

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