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A Riemann–Roch formula for the blow-up of a nonsingular affine scheme

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Abstract

The main purpose of this paper is to obtain the Hilbert–Samuel polynomial of a module via blowing up and applying intersection theory rather than employing associated graded objects. The result comes in the form of a concrete Riemann–Roch formula for the blow-up of a nonsingular affine scheme at its closed point. To achieve this goal, we note that the blow-up sits naturally between two projective spaces, one over a field and one a regular local ring, and then apply the Grothendieck–Riemann–Roch Theorem to each containment.

Keywords: Riemann–Roch theorem; Hilbert–Samuel polynomial; Blow-up

Introduction

The main purpose of this paper is to elicit multiplicity data on a nonsingular affine scheme by means of a Riemann–Roch formula developed on the blow-up.

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The multiplicity, and indeed all the Hilbert–Samuel coefficients, of an ideal or module over a local ring are traditionally detected by looking modulo the powers of the maximal ideal, that is, by going to an associated graded object and examining its Hilbert function. In such a way one is placed in the setting of projective varieties over a field, where the algebraic definitions were developed by Samuel and others from the original geometric definitions. For example, it was well known (see [9, p. 65], [12, p. 277], and [15, p. 578]) that the coefficients of the Hilbert polynomial of a subvariety are the Euler–Poincaré characteristics of a sequence of intersections with generic hyperplanes. In this setting there are also many related interpretations available, such as the Hirzebruch–Riemann–Roch formula (see [5, Example 15.1.4] or [7, Lemma 1.7.1]). In the case of a projective space over a field, there is an explicit relationship between the Hilbert polynomial and the total Chern class of a coherent sheaf; see [4, Exercise 19.18] and [3].

In this paper we investigate a question raised by V. Srinivas in conversation: Can one obtain the Hilbert–Samuel polynomial, and so in particular the multiplicity, of a module over a regular local ring by blowing up the maximal ideal instead of modding out by its powers? That is, can this polynomial be interpreted via intersection theory on the blow-up of the affine scheme at its closed point, say via a Riemann–Roch type formula? Indeed, this is the case. Combining Theorems 3.1 and 2.1, we obtain the following result.

We point out first that, although the Riemann–Roch theorem exists in various forms depending on the context under consideration, it is often stated in a rather abstract way. This is due to the fact that the Todd class involved usually has no precise expression. Here, in the case of the blow-up of a nonsingular affine scheme, our result provides an explicit formulation for the Riemann–Roch theorem.

We recall some notation briefly (for precise definitions, see Section 1): Let A be a regular local ring and X the blow-up of $\operatorname{Spec} A$ at its closed point. Recall that the Chow group of X is a free abelian group on the powers of the exceptional divisor E. For a hyperplane section on (the Chow group of) X, we write h_X instead of $h_X[X]$ for simplicity. Finally, for any A-module M, let $\mathcal{R}(M)$ denote the Rees module of M.

Main Theorem. *Let A be a regular local ring*[2] *of dimension $d+1$ and M an A-module. On the blow-up X of* $\operatorname{Spec} A$ *at its closed point, one has*

$$\operatorname{ch}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1} = a_d h_X^0 + a_{d-1} h_X^1 + \cdots + a_0 h_X^d$$

where $a_0, \ldots, a_d$ are such that

$$\Delta \mathfrak{p}_M(t) = \frac{a_d}{d!} t^d + \frac{a_{d-1}}{(d-1)!} t^{d-1} + \cdots + a_0$$

is the discrete derivative of the Hilbert–Samuel polynomial of M.

This is indeed a Riemann–Roch formula on the blow-up. In fact, its similarity to the classical Hirzebruch–Riemann–Roch formula

[2] We note that one may generally assume that the ring is regular when computing the Hilbert–Samuel polynomial of a module M since the polynomial is unchanged when M is viewed over a regular cover of the ring, say from a Cohen presentation. (Similarly, the Hilbert function of a variety is often given a geometric interpretation after a suitable embedding in a projective space.)

$$\chi(P,\mathcal{F}) = \int \mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_P)$$

for a coherent sheaf $\mathcal{F}$ on a nonsingular projective scheme P over a field becomes more apparent in the following reworking of the Hirzebruch formula above on $\mathbb{P}^d_k$, which we derive in Proposition 2.5. Recall that the Chow group of $\mathbb{P}^d_k$ is a free abelian group on the linear subspaces $[\mathbb{P}^i_k]$.

Proposition. *Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}^d_k$ and suppose*

$$\tau\big([\mathcal{F}]\big) \stackrel{\text{def}}{=} \mathrm{ch}(\mathcal{F})\left(\frac{h}{1-e^{-h}}\right)^d = a_d\big[\mathbb{P}^d_k\big] + a_{d-1}\big[\mathbb{P}^{d-1}_k\big] + \cdots + a_1\big[\mathbb{P}^1_k\big] + a_0\big[\mathbb{P}^0_k\big]$$

in the Chow group of $\mathbb{P}^d_k$ where h denotes a hyperplane section on $\mathbb{P}^d_k$. Then the Hilbert polynomial of $\mathcal{F}$ is

$$\mathcal{P}_{\mathcal{F}}(t) = \frac{a_d}{d!}t^d + \frac{a_{d-1}}{(d-1)!}t^{d-1} + \cdots + a_0$$

This is derived by applying the Hirzebruch–Riemann–Roch formula to the twists of the sheaf $\mathcal{F}$ and analyzing their Euler characteristics.

However, for our main theorem above, it was not possible to apply nonsingular Riemann–Roch theory directly, since, although the blow-up is nonsingular, it is not smooth and consequently has no tangent bundle; see 1.1. Note further that in our case the natural modules of global sections to consider do not have finite length over A and so the Euler characteristics are not even defined.

Instead the strategy of the proof of our main theorem above is to squeeze the blow-up X naturally between two projective spaces:

$$\mathbb{P}^d_k = E \stackrel{f}{\hookrightarrow} X \stackrel{g}{\hookrightarrow} Z = \mathbb{P}^d_A$$

The exchange of information is attained by means of the local Riemann–Roch theorem applied to each embedding:

$$\begin{array}{ccc} K_0(Z) & \xrightarrow{\tau_{Z/A}} & \mathrm{A}_*(Z)_{\mathbb{Q}} \\ \downarrow g^* & & \downarrow \mathrm{ch}^Z_X \\ K_0(X) & \xrightarrow{\tau_{X/A}} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ \downarrow f^* & & \downarrow \mathrm{ch}^X_E \\ K_0(E) & \xrightarrow{\tau_{E/A}} & \mathrm{A}_*(E)_{\mathbb{Q}} \end{array} \qquad (0.0.1)$$

Here is a brief overview of the proof. In Section 2, the image of the Rees module sheaf $\widetilde{\mathcal{R}(M)}$ under the Riemann–Roch map $\tau_{X/A}$ is computed by a careful transfer via the lower commutative square to the setting of $E \cong \mathbb{P}^d_k$, where the proposition above is applied. This yields Theorem 2.1. Then, in Section 3, the image of $\widetilde{\mathcal{R}(M)}$ under $\tau_{X/A}$ (which is technically defined on Z rather than on X anyway) is recomputed via the top square to yield a formula involving Chern characters defined on X itself, rather than on Z, yielding Theorem 3.1. Combining these two results gives our main theorem above; we give a proof at the end of Section 3 since it is in fact simpler to derive directly from the proofs of those two theorems instead of from the statements themselves.

As for the organization of the remaining parts of the paper, Section 1 contains the necessary background and basic set-up and Section 4 addresses the issue of determining the constant term of the Hilbert–Samuel polynomial, clearly missing from our approach above. For this we simply extend a result of Johnston and Verma on ideals to the setting of modules. Although we include the proofs here for completeness, they are modeled directly on theirs. Of course, it would be nice to know whether there is a single approach that yields the entire Hilbert–Samuel polynomial, but the authors do not have further insight into this question.

We end with two remarks. First, again, since the blow-up of Spec A is not smooth over Spec A, the machinery of nonsingular Riemann–Roch theory, which concerns itself with schemes nonsingular over a field (or smooth over a regular base), could not be applied directly on the blow-up to deduce our results. In fact, that brings us to the second remark: The Riemann–Roch map itself is not well understood for nonsmooth schemes. But the proof of our main theorem proceeds by computing the image of the sheaf $\widetilde{\mathcal{R}(M)}$ under the Riemann–Roch map $\tau_{X/A}$, in two different ways. So, since the twists of these sheaves actually generate the Chow group of the blow-up X (see Proposition 1.12), in some sense this gives an explicit computation of the Riemann–Roch map for a nonsmooth scheme.

We wish to thank Vasudevan Srinivas for introducing us to the problem and Kazuhiko Kurano and Paul Roberts for their generous help along the way. We also thank Orlando Villamayor for pointing out Proposition 1.12 to us. The referee's comments are very much appreciated for they helped us make the statements in the paper more accurate.

1. Preliminaries

We first describe the basic setting of the paper. Then we review background on the parts of intersection theory used in this paper and on Hilbert polynomials.

Basic setting

Let A be a regular local ring of dimension $d+1$ with the maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Consider the blow-up

$$\begin{array}{c} X \\ \pi \big\downarrow \\ \operatorname{Spec} A \end{array}$$

of Spec A at its closed point $\mathfrak{m}$, and let E denote the exceptional divisor of the blow-up. Let f denote the embedding $E \hookrightarrow X$. We assume that the Chow group of Spec A is isomorphic to $\mathbb{Z}$; this is known if the regular local ring A contains a field and unknown otherwise.

Recall that $X = \operatorname{Proj}(\mathcal{R}(A))$, where $\mathcal{R}(A)$ is the Rees ring of A:

$$\mathcal{R}(A) = A[\mathfrak{m}t] = \bigoplus_{i \geqslant 0} \mathfrak{m}^i$$

and that $E = \operatorname{Proj}(\mathfrak{gr}(A))$, where $\mathfrak{gr}(A)$ is the associated graded ring of A:

$$\mathfrak{gr}(A) = \mathcal{R}(A)/\mathfrak{m}\mathcal{R}(A) = \bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

Furthermore, since A is regular local, $\mathfrak{gr}(A)$ is a polynomial ring in $d+1$ variables and so there is an isomorphism

$$E \cong \mathbb{P}^d_k$$

Indeed, E is the exceptional divisor of the blow-up of the nonsingular scheme $\operatorname{Spec} A$ at its closed point.

For any A-module M define the *Rees module* as

$$\mathcal{R}(M) = \bigoplus_{i \geqslant 0} \mathfrak{m}^i M$$

and the associated graded module as

$$\mathfrak{gr}(M) = \mathcal{R}(M)/\mathfrak{m}\mathcal{R}(M) = \bigoplus_{i \geqslant 0} \mathfrak{m}^i M/\mathfrak{m}^{i+1}M$$

If M is finitely generated, $\widetilde{\mathcal{R}(M)}$ and $\widetilde{\mathfrak{gr}(M)}$ are coherent sheaves on X and E, respectively.

Properties of the blow-up

Note that X is nonsingular since it is the blow-up of the nonsingular scheme $\operatorname{Spec} A$ at the closed point $\mathfrak{m}$. However, it is *not* smooth over $\operatorname{Spec} A$.

Proposition 1.1. *X is nonsingular but not smooth over A.*

Proof. To see that X is not smooth, it suffices to show that the sheaf of differentials is not locally free. Using the description of the affine patches of X in (1.3.1) below, one computes easily that the sheaf of differentials is in fact $\mathfrak{m}$-torsion. □

We record below some well-known facts about the blow-up.

1.2. We describe here equations that define the blow-up as a projective variety.

Let $a_0, a_1, \dots, a_d$ be a minimal set of generators for $\mathfrak{m}$. Since A is regular local, the Rees ring has the well-known presentation

$$\mathcal{R}(A) \cong A[X_0, X_1, \dots, X_d]/I \tag{1.2.1}$$

where I is the ideal generated by the set of (2×2)-minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_d \\ a_0 & a_1 & \cdots & a_d \end{pmatrix}$$

Under this isomorphism the image of X_i in the quotient corresponds to the image of the minimal generator a_i in the degree 1 piece of the Rees ring.

This presentation allows us to consider $X = \operatorname{Proj}(\mathcal{R}(A))$ as a closed subscheme of a projective space. We denote the embedding by

$$g : X \hookrightarrow Z = \mathbb{P}^d_A \tag{1.2.2}$$

Note further that the map g is a regular embedding, and so also local complete intersection (henceforth l.c.i), because both X and Z are nonsingular and g is an embedding. (Indeed, the maps of local rings are surjections of regular rings and thus must be given as a quotient by a regular sequence. In fact, this sequence necessarily consists of part of a set of minimal generators for the maximal ideal.)

Note that the composition

$$\mathbb{P}^d_k = E \overset{f}{\hookrightarrow} X \overset{g}{\hookrightarrow} Z = \mathbb{P}^d_A \tag{1.2.3}$$

is simply the natural map $\mathbb{P}^d_k \hookrightarrow \mathbb{P}^d_A$ defined on the ring level by going "mod $\mathfrak{m}$".

1.3. We describe here the affine patches of the blow-up.

Using (1.2.1) to consider $X = \operatorname{Proj}(\mathcal{R}(A))$ as a closed subscheme of $Z = \mathbb{P}^d_A$, we examine the affine patch where $X_i \neq 0$. Upon inverting X_i, the ideal I is generated by the d elements $a_j X_i - a_i X_j$ for $j \neq i$, equivalently, by

$$a_j - a_i \frac{X_j}{X_i} \qquad j \neq i$$

(the other generators $a_j X_k - a_k X_j$ for $j, k \neq i$ can be obtained from these). Therefore, letting $T_j = \frac{X_j}{X_i}$ for each $j \neq i$, the ith affine patch is simply

$$\operatorname{Spec}\bigl(\mathcal{R}(A)_{(X_i)}\bigr) = \operatorname{Spec}\left(\frac{A[T_0, \ldots, \widehat{T_i}, \ldots, T_d]}{(\{a_j - a_i T_j\}_{j \neq i})}\right) \tag{1.3.1}$$

Chow groups/rings

In the next few parts of this section, we review the results from intersection theory used in this paper. For basic definitions and constructions, see the books by Fulton [5] and Roberts [11].

For any projective scheme Y, let $\mathrm{A}_*(Y)$ denote the Chow group of Y, graded by dimension rather than codimension. We will work mostly with the rational Chow group $\mathrm{A}_*(Y)_{\mathbb{Q}} \overset{\text{def}}{=} \mathrm{A}_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. For any l.c.i. morphism $f : W \to Y$ of projective schemes, there is a *Gysin homomorphism*

$$g^* : \mathrm{A}_*(Y)_{\mathbb{Q}} \to \mathrm{A}_*(W)_{\mathbb{Q}}$$

as described in [5, Chapter 6], to be thought of as giving "intersection products" of elements of $\mathrm{A}_*(Y)_{\mathbb{Q}}$ with W.

The Chow ring of the blow-up X

Returning to the setting of this paper, let X be the blow-up and E its exceptional divisor. The main goal in this part is to give an explicit description of the Chow ring of X. It is derived from corresponding descriptions of the Chow ring of E.

Since E is a Cartier divisor on X, the product $[E]\cdot\alpha$, for any α in $A_*(X)$, is defined as $c_1(L)\cap\alpha$ where L is the line bundle defined by E and $c_1(L)$ is the first Chern class of L. Whether the result is thought of as an element of $\mathrm{A}_*(X)$ or of $\mathrm{A}_*(E)$ will always be obvious from the context. Considering $[E]$ as a cycle class in $\mathrm{A}_{d-1}(X)$, we also denote by $[E^i]$ the iterated self-product $[E]\cdot[E]\cdots[E]$ of i copies of $[E]$.

Recall that E is in fact the projective space $\mathbb{P}^d_k$, and thus its Chow ring has a well-known description in terms of hyperplane sections. Let h_Y denote a hyperplane section on any projective scheme Y. To simplify the appearance of the formulas in this paper, we will write simply h^i_Y for the class of $h^i_Y([Y])$ in either $\mathrm{A}_*(Y)$ or $\mathrm{A}_*(Y)_{\mathbb{Q}}$. Whether or not the Chow ring is tensored by $\mathbb{Q}$ will always be obvious from the context.

Proposition 1.4. *Let B be a regular local ring. Let W denote the projective space $\mathbb{P}^d_B$ and h_W a hyperplane section on W. The Chow ring of $\mathbb{P}^d_B$ is a free abelian group on the basis*

$$h^0_W=\big[\mathbb{P}^d_B\big],\qquad h^1_W=\big[\mathbb{P}^{d-1}_B\big],\qquad \ldots,\qquad h^i_W=\big[\mathbb{P}^{d-i}_B\big],\qquad \ldots,\qquad h^d_W=\big[\mathbb{P}^0_B\big]$$

where $\mathbb{P}^{d-i}_B$ is considered as a linear subvariety of $\mathbb{P}^d_B$. Furthermore, $h^{d+1}_W=0$.

For E, this basis can be compared with the list of powers $[E^i]\in\mathrm{A}_*(E)$ as follows, see [5, Example 3.3.4 and B.6.3]:

Proposition 1.5. *In the setting of this paper, there is an equality in $\mathrm{A}_*(E)$*

$$h^i_E=(-1)^i\big[E^{i+1}\big]$$

In particular, the powers

$$[E],\qquad \big[E^2\big],\qquad \ldots,\qquad \big[E^{d+1}\big]$$

also form a $\mathbb{Z}$-basis for the Chow group $\mathrm{A}_*(E)$. Furthermore, $[E^{d+2}]=0$.

The description of the Chow ring of X can now be derived from the one of E above using a split exact sequence involving the Chow group of the blow-up.

Proposition 1.6. *The Chow ring of X is a free abelian group on the basis*

$$\big[E^0\big]=[X],\qquad \big[E^1\big],\qquad \ldots,\qquad \big[E^d\big]$$

Furthermore, $[E^{d+1}]=0$.

Proof. Let S denote Spec A; recall that we are assuming that $\mathrm{A}_*(S) = \mathbb{Z}[S]$. There is a split exact sequence (cf. [5, Proposition 6.7(e)])

$$0 \to \mathrm{A}_i(\operatorname{Spec} A/\mathfrak{m}) \xrightarrow{\alpha} \mathrm{A}_i(E) \oplus \mathrm{A}_i(S) \xrightarrow{\beta} \mathrm{A}_i(X) \to 0$$

for all $i = 0, \dots, d+1$ such that

(a) α is an isomorphism when $i = 0$, and
(b) $\beta = (f_*, \pi^*)$,

where $f : E \to X$ and $\pi : X \to S$ are as previously defined.

First, condition (a) yields that $\mathrm{A}_0(X) = 0$ and thus also that $[E^{d+1}] = 0$.

Next, for $1 \leqslant i \leqslant d$, both $\mathrm{A}_i(\operatorname{Spec} A/\mathfrak{m})$ and $\mathrm{A}_i(S)$ vanish, and so β restricts to an isomorphism $\mathrm{A}_i(E) \xrightarrow{\cong} \mathrm{A}_i(X)$. Furthermore, by condition (b) this map is simply the pushforward f_* and so sends $[E^{d+1-i}]$ to $[E^{d+1-i}]$.

Lastly, since both $\mathrm{A}_{d+1}(\operatorname{Spec} A/\mathfrak{m})$ and $\mathrm{A}_{d+1}(E)$ vanish and $\pi^*([S]) = [X]$, we see that $\mathrm{A}_{d+1}(X) = \mathbb{Z}[X]$. □

Local Chern characters

Local Chern characters provide maps on Chow groups. We recall them only in the special case of perfect embeddings. Recall that for any closed embedding $f : W \to Y$ of schemes, f is *perfect* if $f_*\mathcal{O}_W$ can be resolved by a finite complex $E_\bullet$ of vector bundles on Y. Note that any l.c.i. morphism is perfect. In particular, both maps studied in this paper, $f : E \hookrightarrow X$ and $g : X \hookrightarrow Z$, are perfect.

For any closed perfect embedding $f : W \to Y$, there is a homomorphism

$$\mathrm{ch}_W^Y : \mathrm{A}_*(Y)_{\mathbb{Q}} \to \mathrm{A}_*(W)_{\mathbb{Q}}$$

called the *local Chern character*. For a regular embedding, this map can be described more explicitly as follows, see [5, Corollary 18.1.2] (applied to the maps $W \overset{i}{=} W \overset{j}{\hookrightarrow} Y$).

Proposition 1.7. *If $f : W \to Y$ is a regular embedding with normal bundle N, then*

$$\mathrm{ch}_W^Y = \mathrm{td}(N)^{-1} \cdot f^*$$

where f^* is the Gysin homomorphism.

The local Chern character ch_E^X

Returning to the setting of this paper, let X be the blow-up, E the exceptional divisor, $f : E \hookrightarrow X$ the inclusion, and $N = N_E^X$ the normal bundle. By Proposition 1.7,

$$\mathrm{ch}_E^X = \mathrm{td}(N)^{-1} \cdot f^*$$

We proceed to give a more explicit description of each factor above.

Lemma 1.8. *In the setting of this paper, the Gysin map* $f^*: \mathrm{A}_*(X) \to \mathrm{A}_*(E)$ *is given by multiplication by* $[E]$. *In particular, it is an isomorphism of groups.*

Proof. By [5, Corollary 8.1.1] for any $x \in \mathrm{A}_*(X)$ one has that $[E] \cdot x = f^!(x)$ in $\mathrm{A}_*(|x| \cap E)$ and so the desired formula follows upon pushing forward to $\mathrm{A}_*(E)$. The second statement follows using Propositions 1.5 and 1.6. □

Lemma 1.9. *In the setting of this paper, if* N *is the normal bundle to* E *in* X, *then*

$$\mathrm{td}(N)^{-1} = \textit{multiplication by } \frac{1 - e^{-[E]}}{[E]}$$

Proof. Note that N is a line bundle. It suffices to show that the action of the Chern class $c_1(N)$ on $\mathrm{A}_*(E)$ is the same as multiplication by $[E]$. Indeed, in view of Lemma 1.8, since E is a divisor one may apply [5, Proposition 6.1(c)] using [5, Proposition 2.6(c)] and noting that $N = f^*(\mathcal{O}_X(E))$. □

As a consequence of Lemmas 1.8 and 1.9 one obtains:

Proposition 1.10. *The local Chern character*

$$\mathrm{ch}_E^X: \mathrm{A}_*(X)_{\mathbb{Q}} \to \mathrm{A}_*(E)_{\mathbb{Q}}$$

provides an isomorphism of Chow groups with

$$\mathrm{ch}_E^X = \textit{multiplication by } 1 - e^{-[E]}$$

Grothendieck groups and Gysin maps

First we recall for Grothendieck groups the Gysin map derived from Serre's intersection multiplicity formula [13]. Let $\mathrm{K}_0(Y)$ denote the Grothendieck group of coherent sheaves on a scheme Y.

Definition 1.11. (See [5, Example 15.1.8], [1, III].) Let $f: W \to Y$ be a perfect embedding of schemes and $E_\bullet$ a finite complex of vector bundles on Y resolving $f_*\mathcal{O}_W$. There is a *Gysin homomorphism* $f^*: \mathrm{K}_0(Y) \to \mathrm{K}_0(W)$ induced by the following formula: for any coherent sheaf $\mathcal{F}$ on Y

$$f^*([\mathcal{F}]) = \sum_i (-1)^i \big[\mathrm{Tor}_i^Y(\mathcal{O}_W, \mathcal{F})\big]$$

where $\mathrm{Tor}_i^Y(\mathcal{O}_W, \mathcal{F})$ is the ith homology sheaf of the complex $E_\bullet \otimes_{\mathcal{O}_Y} \mathcal{F}$. Perfection of the map f ensures that the sum is finite.

Although we do not need such a precise description of the Grothendieck group as of the Chow group, we still present the following curious fact for the general interest of the reader: The Grothendieck group of coherent sheaves on the blow-up of the affine scheme Spec A for *any* ring A is generated by the classes defined by the Rees modules $\mathcal{R}(M)$ and their twists, as we show next. The result follows by a direct application of two well-known theorems of Serre for coherent sheaves on projective schemes.

Proposition 1.12. *Let A be a local ring, and let X be the blow-up* $\operatorname{Proj}(\mathcal{R}(A))$ *of* $\operatorname{Spec} A$ *at its closed point. For every coherent sheaf $\mathcal{F}$ on X, there exists a finitely generated A-module M and a positive integer n_0 such that $\mathcal{F} = \widetilde{\mathcal{R}(M)}(-n_0)$.*

Proof. Since X is a projective scheme over A, there exists a positive integer n_0 such that for all $n \geqslant n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections (cf. [6, Theorem II.5.17]). This is equivalent to saying that for all $n \gg 0$,

$$\Gamma\big(X, \mathcal{F}(n)\big) = \mathfrak{m}^{n-n_0}\Gamma\big(X, \mathcal{F}(n_0)\big)$$

Let M be $\Gamma(X, \mathcal{F}(n_0))$, which is a finitely generated A-module by [6, Theorem III.5.2(a)]. Since $\bigoplus_{n\geqslant 0}\Gamma(X, \mathcal{F}(n))$ and $\bigoplus_{n\geqslant n_0}\mathfrak{m}^{n-n_0}M$ coincide on all large enough homogeneous components, they define the same sheaf; equivalently, $\mathcal{F} = \widetilde{\mathcal{R}(M)}(-n_0)$. $\square$

Riemann–Roch theory

As proved in Proposition 1.1, X is not smooth over A, and so the "nonsingular Riemann–Roch theory" from Chapter 15 of [5] cannot be applied to X. Furthermore, we will require Riemann–Roch theorems for schemes not necessarily over a field; so, although references are given to specific results from Chapter 18 of [5], the versions given here are understood to include the modifications described in [5, Chapter 20] or in [11].

The Gysin maps on Grothendieck groups and the local Chern characters are connected by the Riemann–Roch Theorem [5, Theorem 18.2 or 18.3]: It states that for all algebraic schemes Y over base scheme $\operatorname{Spec} A$ there is a homomorphism

$$\tau_Y = \tau_{Y/A} : \mathrm{K}_0(Y) \to \mathrm{A}_*(Y)_{\mathbb{Q}}$$

called the Riemann–Roch map, which is defined using Todd classes after embedding Y in a smooth space, such that τ_Y satisfies various natural properties such as "covariance" (compatibility with pushforward under proper maps) and a "module" property (compatibility of the module action on Grothendieck groups with Chern characters on the Chow group). However, we will only need the local Riemann–Roch formula, a consequence of the Riemann–Roch Theorem:

Theorem 1.13 *(Local Riemann–Roch formula). (See [5, Examples 18.3.15 or 18.3.12].) If $f : W \to Y$ is a perfect morphism of schemes, then the diagram*

$$\begin{array}{ccc} K_0(Y) & \xrightarrow{\tau_Y} & \mathrm{A}_*(Y)_{\mathbb{Q}} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{ch}_W^Y} \\ K_0(W) & \xrightarrow{\tau_W} & \mathrm{A}_*(W)_{\mathbb{Q}} \end{array}$$

commutes, where f^* is the Gysin homomorphism as in Definition 1.11.

A well-known consequence of the Riemann–Roch theorem is that the Riemann–Roch map τ_Y induces an isomorphism between the rational Grothendieck and Chow groups: $K_0(Y)_{\mathbb{Q}} \cong \mathrm{A}_*(Y)_{\mathbb{Q}}$.

Hilbert polynomials

We give a brief review of the facts on Hilbert–Samuel polynomials used in this paper. Let M be a finitely generated A-module of dimension $n+1$. For integers $j \gg 0$, the length $\ell_A(M/\mathfrak{m}^j M)$ is a polynomial type function in j of degree $n+1$.

Definition 1.14. The *Hilbert–Samuel polynomial* of M is the polynomial $\mathfrak{p}_M(t)$ such that

$$\mathfrak{p}_M(j) = \ell_A\big(M/\mathfrak{m}^j M\big) \quad \text{for all } j \gg 0 \text{ in } \mathbb{Z}$$

We now connect this polynomial with the Hilbert polynomial of the associated graded module $\mathfrak{gr}(M)$ over $\mathfrak{gr}(A)$.

Remark 1.15. For any function $f : \mathbb{Z} \to \mathbb{Z}$, its discrete derivative is defined as

$$\Delta f(t) \overset{\text{def}}{=} f(t+1) - f(t)$$

It is well known that $\Delta f(t)$ is a polynomial for $t \gg 0$ if and only if $f(t)$ is a polynomial for $t \gg 0$. In this case, one can also obtain $f(t)$ up to a constant with relative ease from $\Delta f(t)$ by "discrete integration".

Notice that the discrete derivative of the Hilbert–Samuel polynomial of M is

$$\Delta \mathfrak{p}_M(t) \overset{\text{def}}{=} \ell\big(M/\mathfrak{m}^{t+1} M\big) - \ell\big(M/\mathfrak{m}^t M\big) = \ell\big(\mathfrak{m}^t M/\mathfrak{m}^{t+1} M\big) = \ell\big(\big(\mathfrak{gr}(M)\big)_t\big)$$

With this motivation, one can show that: for any finitely generated graded $\mathfrak{gr}(A)$-module $N = \bigoplus_{j \geqslant 0} N_j$, the length $\ell(N_j) = \dim_k N_j$ is a polynomial type function in j when j is large enough.

Definition 1.16. Let $N = \bigoplus_{j \geqslant 0} N_j$ be a finitely generated graded $\mathfrak{gr}(A)$-module. The *Hilbert polynomial* of N is the polynomial $\mathcal{P}_N(t)$ such that

$$\mathcal{P}_N(j) = \ell(N_j) \quad \text{for all } j \gg 0 \text{ in } \mathbb{Z}$$

Note that the Hilbert polynomial of $\mathfrak{gr}(M)$ is simply the discrete derivative of the Hilbert–Samuel polynomial of M: $\mathcal{P}_{\mathfrak{gr}(M)}(t) = \Delta \mathfrak{p}_M(t)$. In particular, it has degree n.

The analogous definition for a coherent sheaf $\mathcal{F}$ on a projective scheme Y is given by considering the vector space dimension, $\dim_k \Gamma(Y, \mathcal{F}(j))$, of the global sections of the twists of $\mathcal{F}$; this dimension is well known to be a finite number and is a polynomial type function in j whenever j is a large enough positive integer.

Definition 1.17. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme Y over a field k. The *Hilbert polynomial* of $\mathcal{F}$ is the polynomial $\mathcal{P}_{\mathcal{F}}(t)$ such that

$$\mathcal{P}_{\mathcal{F}}(j) = \dim_k \Gamma\big(Y, \mathcal{F}(j)\big) \quad \text{for all } j \gg 0 \text{ in } \mathbb{Z}$$

From [6, Exercise II.5.9] one has:

Remark 1.18. If $Y = \operatorname{Proj}(S)$ and $\mathcal{F} = \widetilde{N}$ where S is a graded ring with $S_0 = k$ and N is a finitely generated graded S-module, then $\Gamma(Y, \mathcal{F}(j)) \cong N_j$ for $j \gg 0$, see [6, Exercise II.5.9]. In particular, the Hilbert polynomials of the sheaf $\mathcal{F}$ and the module N agree:

$$\mathcal{P}_{\mathcal{F}}(t) = \mathcal{P}_N(t)$$

2. The Hilbert–Samuel polynomial

We want to express the Hilbert–Samuel polynomial $\mathfrak{p}_M(t)$ of M via intersection theory on the blow-up of $\operatorname{Spec} A$. We do this by going mod $\mathfrak{m}$ via a local Riemann–Roch formula and using classical intersection theory on the projective space $\mathbb{P}^d_k$ over the field k. Note that, although the proof itself goes mod $\mathfrak{m}$ and uses the associated graded module $\mathfrak{gr}(M)$, the statement of the theorem shows that the same multiplicity information on M over A is detected by blowing up A. Indeed, this is our main point.

It should be noted that the constant term of $\mathfrak{p}_M(t)$ is not recovered by this method; see Example 2.2. We deal separately with the constant term in Section 4.

Recall from Proposition 1.6 that the Chow group of the blow-up X is free abelian with basis $[X], [E], [E^2], \ldots, [E^d]$ where E is the exceptional divisor.

Theorem 2.1. *Let A be a regular local ring of dimension $d+1$ and M an A-module. If the discrete derivative of the Hilbert–Samuel polynomial of M is*[3]

$$\Delta\mathfrak{p}_M(t) = \frac{a_d}{d!}t^d + \frac{a_{d-1}}{(d-1)!}t^{d-1} + \cdots + a_0$$

then the image of $[\widetilde{\mathcal{R}(M)}]$ under the Riemann–Roch map $\tau_{X/A} : K_0(X) \to \mathrm{A}_*(X)_{\mathbb{Q}}$ is

$$\tau_{X/A}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \left(\frac{[E]}{1 - e^{-[E]}}\right)\big(a_d[X] - a_{d-1}[E] + \cdots + (-1)^d a_0\big[E^d\big]\big)$$

Note that since $(\frac{[E]}{1-e^{-[E]}})$ is a unit, one can read this result in reverse, namely as providing $\Delta\mathfrak{p}_M(t)$ given the expression of $(\frac{1-e^{-[E]}}{[E]})\tau_{X/A}([\widetilde{\mathcal{R}(M)}])$ in terms of the standard basis of the Chow group of X. Furthermore, one can then recover the Hilbert–Samuel polynomial of M (up to a constant term) by discrete integration; see the discussion in Remark 1.15.

The proof of the theorem requires several supporting results of independent interest which are developed first. Here we give a brief sketch of how these will eventually fit together to give the

[3] Note that $a_d = a_{d-1} = \cdots = a_{n+1} = 0$ if $n = \dim M - 1 < d$.

final proof, which can be found at the end of the section. Since the closed embedding $E \to X$ is perfect, the local Riemann–Roch formula, Theorem 1.13, yields a commutative diagram,

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau_{X/A}} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ch}_E^X} \\ K_0(E) & \xrightarrow{\tau_{E/k}} & \mathrm{A}_*(E)_{\mathbb{Q}} \end{array} \tag{2.1.1}$$

First we show that $f^*([\widetilde{\mathcal{R}(M)}]) = [\widetilde{\mathfrak{gr}(M)}]$ in Lemma 2.3. Since $E = \mathbb{P}_k^d$, we are thus placed in the classical setting of projective space over the field k. Here we apply the Hirzebruch–Riemann–Roch formula with some additional computations to obtain the Hilbert polynomial of $\mathfrak{gr}(M)$; see Proposition 2.5. By the discussion in Remark 1.15, this is exactly the discrete derivative of the Hilbert–Samuel polynomial of M. Finally, we move the result back via the isomorphism ch_E^X to get the desired expression for $\tau_{X/A}([\widetilde{\mathcal{R}(M)}])$ in the Chow group of the blow-up X.

Before we begin to prove the necessary supporting results, we give an example that shows that $\mathcal{R}(M)$ does not contain information on the constant term of the Hilbert–Samuel polynomial of M.

Example 2.2. For any nonzero A-module M of finite length, one has $\mathfrak{m}^n M = 0$ for $n \gg 0$, and so the sheaf $\widetilde{\mathcal{R}(M)}$ is the zero sheaf. But in this case, the Hilbert–Samuel polynomial of M is the *nonzero* constant $\ell(M)$, where $\ell(M)$ is the length of M.

Lemma 2.3. *Let A be a regular local ring and M a finitely generated A-module. For the Gysin homomorphism $f^* : K_0(X) \to K_0(E)$,*

$$f^*\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \big[\widetilde{\mathfrak{gr}(M)}\big]$$

Proof. Let $\mathcal{R} = \mathcal{R}(A)$. It is clear that $\mathcal{O}_E = \widetilde{\mathcal{R}/\mathfrak{m}\mathcal{R}}$. By definition,

$$f^*\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \sum_{i \geqslant 0} (-1)^i \big[\widetilde{\mathrm{Tor}_i^{\mathcal{R}}}\big(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M)\big)\big]$$

We claim that, for each $j > 0$, $\mathrm{Tor}_j^{\mathcal{R}}(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M))$ determines the zero sheaf on E and therefore the desired result follows as

$$f^*\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = [\widetilde{\mathcal{R}/\mathfrak{m}\mathcal{R} \otimes_{\mathcal{R}} \mathcal{R}(M)}] = \big[\widetilde{\mathfrak{gr}(M)}\big]$$

To prove the claim, we compute locally using the discussion from 1.2 and 1.3. In brief, the vanishing follows from the facts that E is locally principal (E is a divisor!), that X is an integral scheme, and that the high components of $\mathcal{R}(M)$ have positive depth. In more detail: Any minimal set of generators $a_0, \dots, a_d$ for $\mathfrak{m}$ induces a natural embedding of X as a subvariety of $\mathbb{P}_A^d$ whose ith affine patch has equations

$$\mathcal{R}_{(X_i)} = \frac{A[T_0, \ldots, \widehat{T_i}, \ldots, T_d]}{(\{a_\ell - a_i T_\ell\}_{\ell \neq i})}$$

in local coordinates $T_\ell = \frac{X_\ell}{X_i}$. In particular, one has that $\mathfrak{m}\mathcal{R}_{(X_i)} = a_i \mathcal{R}_{(X_i)}$ and so $\mathfrak{m}\mathcal{R}$ is locally principal on $\mathrm{Proj}(\mathcal{R})$. Therefore,

$$\mathrm{Tor}_j^{\mathcal{R}}\big(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M)\big)_{(X_i)} = 0 \quad \text{for } j \geqslant 2$$

and

$$\mathrm{Tor}_1^{\mathcal{R}}\big(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M)\big)_{(X_i)} = (0 :_{\mathcal{R}(M)} a_i)_{(X_i)} = (0 :_{\mathcal{R}(M)} \mathfrak{m})_{(X_i)}$$

However, as $\mathfrak{m}^n M$ has positive depth for $n \gg 0$, one has

$$(0 :_{\mathcal{R}(M)} \mathfrak{m})_n = (0 :_{\mathfrak{m}^n M} \mathfrak{m}) = 0 \quad \text{for } n \gg 0$$

So indeed $\mathrm{Tor}_j^{\mathcal{R}}(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M))$ determines the zero sheaf on E for each $j > 0$. □

Remark 2.4. The proof of Lemma 2.3 yields a direct relation between resolutions of the sheaf $\widetilde{\mathfrak{gr}(M)}$ on E and resolutions of the sheaf $\widetilde{\mathcal{R}(M)}$ on the blow-up X, namely that a resolution of $\widetilde{\mathfrak{gr}(M)}$ is given by the pullback of any resolution of $\widetilde{\mathcal{R}(M)}$. However, as $\mathrm{Tor}_1^{\mathcal{R}}(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{R}(M)) \neq 0$ in general, the resolutions of the $\mathcal{R}(A)$-module $\mathcal{R}(M)$ and the $\mathfrak{gr}(A)$-module $\mathfrak{gr}(M)$ are not so clearly related.

Although a crucial lemma for Theorem 2.1 on the blow-up, the next result is independently interesting for classical projective space over a field. It follows by a straightforward computation from the Hirzebruch–Riemann–Roch formula; however, the authors have not been able to find it stated in the literature.

Recall that the Chow group of $\mathbb{P}_k^d$ is a free abelian group on the linear subspaces $[\mathbb{P}_k^i]$ for $i = 0, \ldots, d$; see Proposition 1.4.

Proposition 2.5. *Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}_k^d$ and let*

$$\tau_{\mathbb{P}_k^d}\big([\mathcal{F}]\big) = a_d\big[\mathbb{P}_k^d\big] + a_{d-1}\big[\mathbb{P}_k^{d-1}\big] + \cdots + a_0\big[\mathbb{P}_k^0\big]$$

be the image of $[\mathcal{F}]$ in $\mathrm{A}_*(\mathbb{P}_k^d)_{\mathbb{Q}}$. Then the Hilbert polynomial of $\mathcal{F}$ is

$$\mathcal{P}_{\mathcal{F}}(t) = \frac{a_d}{d!}t^d + \frac{a_{d-1}}{(d-1)!}t^{d-1} + \cdots + a_0$$

Proof. By definition, the Hilbert polynomial is

$$\mathcal{P}_{\mathcal{F}}(t) = \dim_k \Gamma\big(\mathbb{P}_k^d, \mathcal{F}(t)\big) \quad \text{for } t \gg 0$$

On the other hand, the Euler–Poincaré characteristic $\chi(\mathbb{P}^d_k, \mathcal{F}(t))$ is a polynomial for *all* values of t, cf. [5, Exercise III.5.2], which must then agree with the one above by Serre's Vanishing Theorem. That is,

$$\mathcal{P}_{\mathcal{F}}(t) = \chi\big(\mathbb{P}^d_k, \mathcal{F}(t)\big) \quad \text{for all } t \in \mathbb{Z}$$

By the Hirzebruch–Riemann–Roch Theorem, [5, Corollary 15.2.1, Theorem 15.2] (applied to a finite resolution of $\mathcal{F}(t)$ by vector bundles) and the fact that the Todd class of the tangent bundle of $\mathbb{P}^d_k$ is

$$\operatorname{td}(T_{\mathbb{P}^d_k}) = \left(\frac{h}{1-e^{-h}}\right)^{d+1}$$

where h is a hyperplane section on $\mathbb{P}^d_k$, one has for all $t \in \mathbb{Z}$ that

$$\chi\big(\mathbb{P}^d_k, \mathcal{F}(t)\big) = \int \operatorname{ch}\big(\mathcal{F}(t)\big)\left(\frac{h}{1-e^{-h}}\right)^{d+1} = \int e^{th}\operatorname{ch}(\mathcal{F})\left(\frac{h}{1-e^{-h}}\right)^{d+1}$$

where integration is defined as taking the coefficient of h^d in the power series expansion of the integrand.

On the other hand, by definition,

$$\tau_{\mathbb{P}^d_k}\big([\mathcal{F}]\big) = \operatorname{ch}(\mathcal{F})\left(\frac{h}{1-e^{-h}}\right)^{d+1}$$

So, the Hilbert polynomial $\mathcal{P}_{\mathcal{F}}(t)$ is the coefficient of h^d in the following expression

$$\begin{aligned} e^{th}\cdot\tau_{\mathbb{P}^d_k}\big([\mathcal{F}]\big) &= e^{th}\big(a_d + a_{d-1}h^1 + \cdots + a_1h^{d-1} + a_0h^d\big) \\ &= \left(1 + th + \frac{t^2}{2!}h^2 + \cdots\right)\big(a_d + a_{d-1}h^1 + \cdots + a_1h^{d-1} + a_0h^d\big) \end{aligned}$$

That is,

$$\mathcal{P}_{\mathcal{F}}(t) = \frac{a_d}{d!}t^d + \frac{a_{d-1}}{(d-1)!}t^{d-1} + \cdots + a_0 \qquad \square$$

Proof of Theorem 2.1. By the discussion in Remark 1.15, the discrete derivative of the Hilbert–Samuel polynomial of M is just the Hilbert polynomial of $\mathfrak{gr}(M)$, which in turn equals the Hilbert polynomial of the sheaf $\widetilde{\mathfrak{gr}(M)}$ by Remark 1.18.

Since the closed embedding $E \to X$ is perfect (E is locally principal and X is an integral scheme), Theorem 1.13 yields a commutative diagram,

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau_{X/A}} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_E^X} \\ K_0(E) & \xrightarrow{\tau_{E/A}} & \mathrm{A}_*(E)_{\mathbb{Q}} \end{array}$$

It can be shown that $\tau_{E/A} = \tau_{E/k}$ and so the diagram yields

$$\mathrm{ch}_E^X \circ \tau_{X/A}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \tau_{E/k} \circ f^*\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \tau_{E/k}\big(\big[\widetilde{\mathfrak{gr}(M)}\big]\big) \tag{2.5.1}$$

where the last equality is from Lemma 2.3. On the one hand, since $E = \mathbb{P}_k^d$, Proposition 2.5 yields (with $[\mathbb{P}_k^i]$ replaced by h^{d-i}) that if

$$\tau_{E/k}\big(\big[\widetilde{\mathfrak{gr}(M)}\big]\big) = a_d h^0 + a_{d-1} h^1 + \cdots + a_0 h^d$$

then the Hilbert polynomial of $\widetilde{\mathfrak{gr}(M)}$ is

$$\mathcal{P}_{\widetilde{\mathfrak{gr}(M)}}(t) = \frac{a_d}{d!} t^d + \frac{a_{d-1}}{(d-1)!} t^{d-1} + \cdots + a_0 \tag{2.5.2}$$

Note that by Proposition 1.5 one has $h^i = h_E^i = (-1)^i [E^{i+1}]$. On the other hand, by Proposition 1.10, the map ch_E^X is an isomorphism given by the explicit formula there. Therefore (2.5.1) becomes

$$\begin{aligned} \tau_{X/A}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) &= \big(\mathrm{ch}_E^X\big)^{-1}\big(a_d[E] - a_{d-1}\big[E^2\big] + \cdots + (-1)^d a_0\big[E^{d+1}\big]\big) \\ &= \left(\frac{[E]}{1 - e^{-[E]}}\right)\big(a_d[X] - a_{d-1}[E] + \cdots + (-1)^d a_0\big[E^d\big]\big) \end{aligned}$$

as desired. □

Remark 2.6. Let $\mathcal{F} = \widetilde{M}$ be a coherent sheaf on $\mathbb{P}_k^d$, and set $n = \dim M - 1$. Since Hilbert polynomials are often written in terms of their Hilbert coefficients, namely in the form

$$\mathcal{P}_{\mathcal{F}}(t) = e_n \binom{t+n}{n} - e_{n-1} \binom{t+n-1}{n-1} + \cdots + (-1)^n e_0$$

and since the Hilbert polynomial of $\mathbb{P}_k^i$ is $\binom{t+i}{i}$ for $i \geqslant 0$, it is tempting to think that the two sets of coefficients, $\{a_n, \ldots, a_0\}$ in Proposition 2.5 and $\{e_n, -e_{n-1}, \ldots, (-1)^n e_0\}$ above, are equal. But they are in fact not equal. Here instead, it is surprising to us that if we display the Hilbert polynomial in t:

$$\mathcal{P}_{\mathcal{F}}(t) = b_n t^n + \cdots + b_0$$

then there is a close relationship between $\{a_n, \ldots, a_0\}$ and $\{b_n, \ldots, b_0\}$. Indeed, $b_i = \frac{a_i}{i!}$ for $i = 0, \ldots, n$ as stated in Proposition 2.5.

3. A Riemann–Roch formula

The goal of this section is to derive a Riemann–Roch formula on the blow-up scheme X. Recall that X is nonsingular, but not smooth over A, so the classical Riemann–Roch theorems are not valid on X. We embed X in a smooth scheme and use Riemann–Roch theory there. As in (1.2.2), there is a natural embedding

$$g : X \hookrightarrow Z = \mathbb{P}^n_A$$

which is a regular embedding. The following result gives the desired Riemann–Roch formula on the blow-up.

Theorem 3.1. *Let N_g be the normal bundle to X in Z and let h_X be a hyperplane section on X. For any coherent sheaf $\mathcal{F}$ on X, there is an equality*

$$\tau_{X/A}\big([\mathcal{F}]\big) = \operatorname{td}(N_g)^{-1}\left(\operatorname{ch}\big([\mathcal{F}]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1}\right)$$

Furthermore, the endomorphism $\operatorname{td}(N_g)^{-1}$ of $A_*(X)_{\mathbb{Q}}$ is determined by the values

$$\operatorname{td}(N_g)^{-1}\big(h^i_X\big) = (-1)^i\left(\frac{[E]}{1-e^{-[E]}}\right)\big[E^i\big]$$

for $i = 0, \dots, d$.

Proof. To avoid confusion in this proof, we use h_Z, h_X and h_E to denote a hyperplane section on $Z = \mathbb{P}^d_A$, on X, and on $E = \mathbb{P}^d_k$, respectively.

Since g is an l.c.i. morphism, Theorems 1.13 and 1.7 apply to give a commutative diagram

$$\begin{array}{ccc} K_0(Z) & \xrightarrow{\tau_{Z/A}} & \mathrm{A}_*(Z)_{\mathbb{Q}} \\ \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle \operatorname{td}(N_g)^{-1}\cdot g^*} \\ K_0(X) & \xrightarrow{\tau_{X/A}} & \mathrm{A}_*(X)_{\mathbb{Q}} \end{array} \tag{3.1.1}$$

We claim that $[\mathcal{F}]$ is a linear combination of twists of the structure sheaf in the rational Grothendieck group $K_0(X)_{\mathbb{Q}}$. First note that the localized Gysin map $f^*_{\mathbb{Q}} : K_0(X)_{\mathbb{Q}} \to K_0(E)_{\mathbb{Q}}$ is an isomorphism. Indeed, from the local Riemann–Roch formula (Theorem 1.13) one has that $\tau_{E/A} \circ f^* = \operatorname{ch}^X_E \circ \tau_{X/A}$, so it remains to remark that the Riemann–Roch maps $\tau_{X/A}$ and $\tau_{E/A}$ induce isomorphisms on the rational level and that ch^X_E was shown to be an isomorphism in Proposition 1.10. Next note that since $E \cong \mathbb{P}^d_k$, we may take a finite resolution of $f^*(\mathcal{F})$ of the form

$$0 \to \cdots \to \bigoplus_j \mathcal{O}_E(-n_{ij})^{b_{ij}} \to \cdots \to f^*(\mathcal{F}) \to 0$$

This yields an equality in the rational Grothendieck group of X

$$
\begin{aligned}
[\mathcal{F}] &= \left(f^*\right)^{-1}\left(\sum_{i,j}(-1)^i b_{ij}\left[\mathcal{O}_E(-n_{ij})\right]\right)\\
&= \sum_{i,j}(-1)^i b_{ij}\left[\mathcal{O}_X(-n_{ij})\right]\\
&= \sum_{i,j}(-1)^i b_{ij} g^*\left(\left[\mathcal{O}_Z(-n_{ij})\right]\right)\\
&= g^*\left(\sum_{i,j}(-1)^i b_{ij}\left[\mathcal{O}_Z(-n_{ij})\right]\right)
\end{aligned}
$$

Therefore, the commutative diagram above implies that

$$
\tau_{X/A}\left([\mathcal{F}]\right) = \mathrm{td}(N_g)^{-1}\cdot g^*\cdot \tau_{Z/A}\left(\sum_{i,j}(-1)^i b_{ij}\left[\mathcal{O}_Z(-n_{ij})\right]\right) \tag{3.1.2}
$$

We compute this in stages. First, by definition, for any n, one has

$$
\tau_{Z/A}\left(\left[\mathcal{O}_Z(-n)\right]\right) = \mathrm{ch}\left(\left[\mathcal{O}_Z(-n)\right]\right)\mathrm{td}(T_Z)
$$

since Z is smooth and where T_Z is the tangent bundle of Z. It is well known that

$$
\mathrm{ch}\left(\left[\mathcal{O}_Z(-n)\right]\right) = e^{-nh_Z} \quad \text{and} \quad \mathrm{ch}\left(\left[\mathcal{O}_X(-n)\right]\right) = e^{-nh_X}
$$

and that

$$
\mathrm{td}(T_Z) = \left(\frac{h_Z}{1-e^{-h_Z}}\right)^{d+1}
$$

Indeed, for the first two formulas apply [5, Example 3.2.3] and for the last formula apply [5, Example 3.2.4] to the exact sequence

$$
0 \to \mathcal{O}_{\mathbb{P}^d_A} \to \left(\mathcal{O}_{\mathbb{P}^d_A}(1)\right)^{d+1} \to T_{\mathbb{P}^d_A} \to 0
$$

Lastly, noting that $g^*: \mathrm{A}_*(Z)_{\mathbb{Q}} \to \mathrm{A}_*(X)_{\mathbb{Q}}$ takes h_Z to h_X by functoriality, we can now complete the computation begun in (3.1.2):

$$
\begin{aligned}
\tau_{X/A}\left([\mathcal{F}]\right) &= \mathrm{td}(N_g)^{-1}\cdot g^*\cdot \tau_{Z/A}\left(\sum_{i,j}(-1)^i b_{ij}\left[\mathcal{O}_Z(-n_{ij})\right]\right)\\
&= \mathrm{td}(N_g)^{-1}\cdot g^*\left(\sum_{i,j}(-1)^i b_{ij} e^{-n_{ij}h_Z}\left(\frac{h_Z}{1-e^{-h_Z}}\right)^{d+1}\right)
\end{aligned}
$$

$$= \mathrm{td}(N_g)^{-1}\left(\sum_{i,j}(-1)^i b_{ij} e^{-n_{ij}h_X}\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1}\right)$$

$$= \mathrm{td}(N_g)^{-1}\left(\sum(-1)^i b_{ij}\,\mathrm{ch}\big([\mathcal{O}_X(-n_{ij})]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1}\right)$$

$$= \mathrm{td}(N_g)^{-1}\left(\mathrm{ch}\big([\mathcal{F}]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1}\right)$$

It remains to verify the action of $\mathrm{td}(N_g)^{-1}$ on each individual h_X^i. Since both inclusions

$$\mathbb{P}_k^d = E \overset{f}{\hookrightarrow} X \overset{g}{\hookrightarrow} Z = \mathbb{P}_A^d$$

are perfect, they induce maps

$$\mathrm{A}_*(Z)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}_X^Z} \mathrm{A}_*(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}_E^X} \mathrm{A}_*(E)_{\mathbb{Q}}$$

Since the composition $g \circ f : E \to Z$ is the natural inclusion of projective spaces as noted in (1.2.3), we obtain

$$\mathrm{ch}_E^X \circ \mathrm{ch}_X^Z\big(h_Z^i\big) = \mathrm{ch}_E^Z\big(h_Z^i\big) = h_E^i$$

But ch_E^X is an isomorphism by Proposition 1.10 and so

$$\mathrm{ch}_X^Z\big(h_Z^i\big) = \big(\mathrm{ch}_E^X\big)^{-1}\big(h_E^i\big) = \big(\mathrm{ch}_E^X\big)^{-1}\big((-1)^i\big[E^{i+1}\big]\big) = (-1)^i\left(\frac{[E]}{1-e^{-[E]}}\right)\big[E^i\big]$$

On the other hand, by Theorem 1.13 since g is an l.c.i. morphism,

$$\mathrm{ch}_X^Z\big(h_Z^i\big) = \mathrm{td}(N_g)^{-1} \cdot g^*\big(h_Z^i\big) = \mathrm{td}(N_g)^{-1}\big(h_X^i\big)$$

So, indeed $\mathrm{td}(N_g)^{-1}(h_X^i) = (-1)^i(\frac{[E]}{1-e^{-[E]}})[E^i]$. □

The Main Theorem in the introduction now follows almost immediately from Theorems 2.1 and 3.1. In fact, the derivation is a bit simpler than that of the individual theorems if one works with the inverse image of the basis $h_E^0, h_E^1, \ldots, h_E^d$ of $\mathrm{A}_*(E)$ under the isomorphism $\mathrm{ch}_E^X : \mathrm{A}_*(X) \to \mathrm{A}_*(E)$ *without* explicitly expressing it in terms of the basis $[X], [E], \ldots, [E^d]$ of $\mathrm{A}_*(X)$.

Proof of Main Theorem. To avoid confusion, we continue to use h_X and h_E for a hyperplane section on X and E respectively.

On the one hand, from the proof of Theorem 2.1 (without converting from h_E's to $[E]$'s in $\mathrm{A}_*(E)$) one sees that

$$\tau_{X/A}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \big(\mathrm{ch}_E^X\big)^{-1}\big(a_d h_E^0 + a_{d-1}h_E^1 + \cdots + a_0 h_E^d\big)$$

On the other hand, by Theorem 3.1 we have that

$$\tau_{X/A}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big) = \mathrm{td}(N_g)^{-1}\left(\mathrm{ch}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1}\right)$$

However, from the very end of the proof of the latter, one sees that in fact

$$\big(\mathrm{ch}_E^X\big)^{-1}\big(h_E^i\big) = \mathrm{td}(N_g)^{-1}\big(h_X^i\big)$$

So indeed we have

$$\mathrm{ch}\big(\big[\widetilde{\mathcal{R}(M)}\big]\big)\left(\frac{h_X}{1-e^{-h_X}}\right)^{d+1} = a_d h_X^0 + a_{d-1}h_X^1 + \cdots + a_0 h_X^d$$

in the Chow group of X, as desired. □

4. The constant term of the Hilbert–Samuel polynomial

In this section we seek a description of the constant term of the Hilbert–Samuel polynomial, the only term not obtained in our main theorem listed in the introduction. To this end, we extend a result of Johnston and Verma [8] from the setting of ideals to that of modules. Our proof precisely mirrors the one in [8].

The result we derive describes, for any A-module M, the difference of the Hilbert–Samuel polynomial and the Hilbert–Samuel function via lengths of the graded pieces of the local cohomology of the Rees module of M. Recall that one defines the Hilbert–Samuel function of M as $\mathfrak{h}_M(n) = \ell(M/\mathfrak{m}^n M)$.

Theorem 4.1. *Let A be a local ring of dimension $d+1$ and M a finitely generated A-module. Let $\mathcal{R}$ denote the Rees ring $\mathcal{R}(A)$ and $\mathcal{R}_+$ the ideal generated by the elements of positive degree in $\mathcal{R}(A)$. Then for all $n \geqslant 0$,*

$$\mathfrak{p}_M(n) - \mathfrak{h}_M(n) = \sum_{i=0}^{d+2}(-1)^i \ell_A\big(H^i_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n\big)$$

In particular, evaluating the equation at $n=0$, one obtains a formula for the constant term (note that $\mathfrak{h}_M(0)=0$):

Corollary 4.2. *Let A be a local ring of dimension $d+1$ and M a finitely generated A-module. The constant term of the Hilbert–Samuel polynomial of M equals*

$$\mathfrak{p}_M(0) = \sum_{i=0}^{d+2}(-1)^i \ell_A\big(H^i_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_0\big)$$

Johnston and Verma's theorem and Theorem 4.1 are both consequences of the following theorem due to Serre [14, Theorem C] for Hilbert functions and polynomials of graded modules. A direct proof of Theorem 4.3 can be found in Ooishi's paper [10] that uses local and global cohomology theorem and applies Serre's Vanishing Theorem to achieve the result. It can also be proved by induction on the dimension of the module (cf. [2, Theorem 4.4.3]). Recall that one defines the Hilbert function of a graded module N as $\mathcal{H}_N(n) = \ell(N_n)$.

Theorem 4.3 *(Serre). Let B be a standard graded homogeneous Noetherian ring of dimension e over the Artinian local ring B_0. Let B_+ denote the ideal generated by the elements of positive degree. Assume that N is a finitely generated graded B-module. Then for all $n \geqslant 0$*

$$\mathcal{H}_N(n) - \mathcal{P}_N(n) = \sum_{i=0}^{e} (-1)^i \ell_{B_0}\big(H^i_{B_+}(N)_n\big)$$

Proof of Theorem 4.1. Consider the submodule $\mathcal{R}_+(M) = \mathfrak{m}M \oplus \mathfrak{m}^2 M \oplus \cdots$ of $\mathcal{R}(M)$ and the resulting exact sequence

$$0 \to \mathcal{R}_+(M) \to \mathcal{R}(M) \to {}_{\mathcal{R}}M \to 0$$

where ${}_{\mathcal{R}}M$ denotes M viewed as a module over $\mathcal{R}$. The long exact sequence of local cohomology modules implies that for $i \geqslant 2$ and for $i = 0, 1$ with $n > 0$, one has

$$H^i_{\mathcal{R}+}\big(\mathcal{R}_+(M)\big)_n \cong H^i_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n \tag{4.3.1}$$

From the short exact sequence

$$0 \to \mathcal{R}_+(M)(1) \to \mathcal{R}(M) \to \mathfrak{gr}(M) \to 0 \tag{4.3.2}$$

and the isomorphisms (4.3.1) one obtains for each $n \geqslant 0$ a long exact sequence of graded pieces of local cohomology modules

$$\begin{aligned} 0 &\to H^0_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_{n+1} \to H^0_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n \to H^0_{\mathcal{R}_+}\big(\mathfrak{gr}(M)\big)_n \\ &\to H^1_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_{n+1} \to H^1_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n \to H^1_{\mathcal{R}_+}\big(\mathfrak{gr}(M)\big)_n \\ &\vdots \\ \cdots &\to H^{d+2}_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_{n+1} \to H^{d+2}_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n \to H^{d+2}_{\mathcal{R}_+}\big(\mathfrak{gr}(M)\big)_n \to 0 \end{aligned} \tag{4.3.3}$$

Lemma 4.4. *Each local cohomology module in* (4.3.3) *has finite length over A.*

We first assume Lemma 4.4 and continue to prove Theorem 4.1. For each $n \geqslant 0$, set

$$\phi(n) = \sum_{i=0}^{d+2} (-1)^i \ell\big(H^i_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n\big)$$

Then additivity of lengths along the long exact sequence (4.3.3) yields

$$-\sum_{i=0}^{d+2} (-1)^i \ell_A\big(H^i_{\mathcal{R}_+}\big(\mathfrak{gr}(M)\big)_n\big) = \phi(n+1) - \phi(n) = \Delta\phi(n) \tag{4.4.1}$$

On the other hand, by Theorem 4.3

$$\mathcal{P}_{\mathfrak{gr}(M)}(n) - \mathcal{H}_{\mathfrak{gr}(M)}(n) = -\sum_{i=0}^{d+2}(-1)^i \ell_A\big(H^i_{\mathcal{R}_+}\big(\mathfrak{gr}(M)\big)_n\big) \tag{4.4.2}$$

Since $\mathcal{P}_{\mathfrak{gr}(M)}(n) = \Delta\mathfrak{p}_M(n)$ and $\mathcal{H}_{\mathfrak{gr}(M)}(n) = \Delta\mathfrak{h}_M(n)$, Eqs. (4.4.1) and (4.4.2) yield

$$\Delta\phi(n) = \Delta\big(\mathfrak{p}_M(n) - \mathfrak{h}_M(n)\big)$$

Since, in addition, both $\phi(n)$ and $\mathfrak{p}_M(n) - \mathfrak{h}_M(n)$ vanish for $n \gg 0$, they must be equal, completing the proof of the theorem. □

It remains to prove Lemma 4.4. We thank the referee for pointing out the following more concise proof.

Proof of Lemma 4.4. It suffices to show that $H^i_{\mathcal{R}_+}(\mathcal{R}(M))_n$ has finite length over A for each $i \geqslant 0$ and $n \geqslant 0$. By Serre's theorem (cf. [6, Theorem III.5.2(a)]), all these modules are finitely generated over A. So it suffices to show that they are supported at the maximal ideal $\mathfrak{m}$. This is equivalent to showing that $(H^i_{\mathcal{R}_+}(\mathcal{R}(M))_n)_a = 0$ for all nonzero $a \in \mathfrak{m}$, where, for an A-module N, N_a denotes the localization obtained by inverting the multiplicatively closed set $\{1, a, a^2, \ldots\}$.

Notice that $\mathcal{R}_a = \bigoplus_{i\geqslant 0}(\mathfrak{m}^i)_a t^i \cong A_a[t]$ since, for each i, $(\mathfrak{m}^i)_a = A_a$. Similarly, $\mathcal{R}(M)_a \cong M_a[t]$. Thus one obtains

$$\big(H^i_{\mathcal{R}_+}\big(\mathcal{R}(M)\big)_n\big)_a = H^i_{(t)}\big(M_a[t]\big)_n = 0$$

if $i \neq 1$, or $i = 1$ and $n \geqslant 0$, by directly computing the local cohomology module $H^i_{(t)}(M_a[t])$ using the Čech complex. □

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