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A construction of regular magic squares of odd order

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A magic square is an $n \times n$ array of numbers whose rows, columns, and the two diagonals sum to μ. A regular magic square satisfies the condition that the entries symmetrically placed with respect to the center sum to $\frac{2\mu}{n}$. Using circulant matrices we describe a construction of regular classical magic squares that are nonsingular for all odd orders. A similar construction is given that produces regular classical magic squares that are singular for odd composite orders. This paper is an extension of [3].

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1. Introduction

A *magic square* M is an $n \times n$ matrix in which entries along each row, each column, the main diagonal, and the cross diagonal add to the same value μ called the *magic sum* of M. If the entries of M are integers from 1 through n^2 where each number appears once then $\mu = \frac{n(n^2+1)}{2}$ and M is called a *classical* magic square (or *natural* magic square).

A magic square $M = [m_{i,j}]$ is said to be *regular* (also called *associated* or *symmetrical*) if the sum of the entries $m_{i,j}$ and $m_{n+1-i,n+1-j}$ that are symmetrically placed across the center of the square is equal to the number $\frac{2\mu}{n}$. In the case of classical magic square this sum is $n^2 + 1$.

Dürer's magic square

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

is an example of a regular magic square [7]. In [5] Mattingly proved that every even order regular magic square is singular (that is, determinant of the magic square is zero). In [4] Loly et al. found that not all of the 5×5 regular classical magic squares are nonsingular. In [3] an example of a 9×9 regular classical magic square that is singular is given.

As a result the question of when an odd order regular magic square is singular or nonsingular was addressed in [3]. A necessary and sufficient condition for an odd order regular magic square to be nonsingular was given. In addition a method to construct nonsingular regular classical magic squares using circulant matrices is given when the order of the magic square is an odd prime power [3].

In this paper we extend this construction method of regular classical magic squares to all odd orders. Moreover, we show that this construction method will produce a singular or nonsingular regular classical magic square based on the choice of the first row of the circulant matrix used in the construction.

2. A construction of regular magic squares

In this section we present the method of construction used in [3] to produce regular classical magic squares.

Let E denote the matrix of all 1's for its entries and e denote the column vector of all 1's. Since $Me = \mu e$ we observe that the magic sum μ is an eigenvalue of magic square M. The following theorem is found in [1].

Theorem 2.1. *If M is an $n \times n$ magic square and ρ is a complex number, then $M + \rho E$ has the same eigenvalues of M except that μ is replaced with $\mu + \rho n$.*

Definition 2.2. If M is a regular magic square we define

$$Z = M - \frac{\mu}{n}E$$

to be the corresponding zero regular magic square.

From Theorem 2.1 it follows that zero regular magic square has the same eigenvalues as M except that μ is replaced by 0.

Let J denote the permutation matrix obtained by writing 1 in each of the cross diagonal entries and 0 elsewhere. Since multiplying a matrix on the left by J reverses the order of the rows and multiplying on the right by J reverses the order of the columns we observe that an $n \times n$ matrix M is a regular magic square if and only if $M + JMJ = \frac{2\mu}{n}E$.

Definition 2.3. An $n \times n$ matrix B with real entries is said to be *centroskew* if $JBJ = -B$.

It is easy to verify that the zero regular magic square Z in Definition 2.2 is a centroskew matrix. The method of construction used in [3] uses a special type of circulant matrix which is defined below. A matrix is said to be *circulant* if each row other than the first row is obtained from the preceding row by shifting entries cyclically one column to the right.

For the rest of the paper let n denote an odd integer and S denote the set

$$S = \left\{-\frac{n-1}{2}, \dots, -1, 0, 1, \dots, \frac{n-1}{2}\right\}. \tag{1}$$

Definition 2.4. Let $\vec{a} = (a_1, a_2, \dots, a_n)$ be a list consisting of n distinct members from S in (1) and $a_1 = 0$. A circulant matrix A with its first row equal $\vec{a}$ is called an S-circulant matrix.

The following two results are from [3]:

1. Suppose A is an S-circulant matrix. Then A is a zero magic square.
2. Suppose A is an S-circulant matrix. Then A is centroskew if and only if

$$a_{j+1} + a_{n+1-j} = 0 \quad \text{for } j = 1, \dots, n-1.$$

Example 2.5. The following is an S-circulant matrix that is centroskew.

$$\begin{bmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}$$

A procedure to construct a regular classical magic square

Step 1: Let A be a centroskew S-circulant matrix of odd order n. Define $Z = nA + AJ$. Then Z is a centroskew zero magic square with n^2 distinct entries from the set

$$Q = \left\{-\frac{n^2-1}{2}, \dots, -1, 0, 1, \dots, \frac{n^2-1}{2}\right\}. \tag{2}$$

Step 2: Let $M = Z + \frac{n^2+1}{2}E$. Then M is a regular classical magic square.

Using the above procedure it is shown in [3] that

1. $\operatorname{rank}(Z) = \operatorname{rank}(A)$,
2. if n is an odd prime then $\operatorname{rank}(Z) = n-1$ and M is nonsingular,
3. if $n = p^t$ where p is an odd prime and the first row of A is $\vec{a} = (a_1, a_2, \dots, a_n)$ with $a_j = j-1$ for $j = 1, 2, \dots, \frac{n+1}{2}$, then $\operatorname{rank}(Z) = n-1$ and M is nonsingular, and
4. by using other first rows for A, examples of singular M were given for $n = 9$ and $n = 15$.

The construction method makes use of the following known facts [6, p. 243], [2, p. 33, 100] about circulant matrices whose first row is given by $\vec{a} = (a_1, a_2, \dots, a_n)$. If A is a circulant matrix then $A^*A = AA^*$, so that A is normal. Hence every circulant matrix is unitarily similar to diagonal matrix. Moreover the eigenvalues of the circulant matrix A are determined by the entries of the first row and are given by

$$\left\{\sum_{j=0}^{n-1} a_{j+1}\omega^{kj} : k = 0, 1, ..., n-1 \text{ and } \omega = e^{\frac{2\pi i}{n}}\right\}. \tag{3}$$

If there is only one zero eigenvalue in (3) the above construction method will produce a nonsingular regular classical magic square. If (3) has more than one zero eigenvalue then the construction method will produce a singular regular classical magic square.

In Section 3 we provide construction of nonsingular regular classical magic squares of *all* odd order extending the results of [3]. In Section 4 we generalize our construction to include singular regular classical magic squares of odd order. Since the construction steps are outlined above we only mention the first row $\vec{a} = (a_1, a_2, \dots, a_n)$ of the centroskew S-circulant matrix A when giving examples.

3. Nonsingular regular magic squares

We utilize the construction in previous section to create nonsingular regular magic squares for all odd n. As seen before, the designation of the first row of matrix A determines its eigenvalues by (3).

For the remainder of the paper let $\mathrm{Re}(r)$ be the real part of complex number r and let $\mathrm{Im}(r)$ be the imaginary part of r. With this notation $r = \mathrm{Re}(r) + i\,\mathrm{Im}(r)$.

Define the first row of matrix A by $a_j = j - 1$ for $j = 1, \dots, \frac{n+1}{2}$ and assign $a_{n-j+1} = -a_{j+1}$ for $1 \le j \le n-1$. Furthermore let ω be the nth root of unity, $\omega = e^{\frac{2\pi i}{n}}$.

For simplicity, we use the notation $E_n(x)$ to denote the polynomial; $E_n(x) = \sum_{j=0}^{n-1} a_{j+1}x^j$. As an example, if $n = 5$ then the associated matrix is the matrix given in Example 2.5 and the polynomial is $E_5(x) = 0 + 1x + 2x^2 - 2x^3 - 1x^4$. The polynomial notation $E_n(x)$ allows us to rewrite the set in (3) as

$$\{E_n(\omega^k) : k = 0, 1, ..., n-1\}. \tag{4}$$

With the above definition for the first row of A,

$$E_n(x) = \sum_{j=0}^{\frac{n-1}{2}} jx^j + \sum_{j=1}^{\frac{n-1}{2}} (-j)x^{n-j}. \tag{5}$$

Lemma 3.1. *If $E_n(x)$ and ω are defined as above, then $E_n(\omega) \neq 0$.*

Proof. We examine the number $E_n(\omega) = \sum_{j=0}^{\frac{n-1}{2}} j\omega^j + \sum_{j=1}^{\frac{n-1}{2}} (-j)\omega^{n-j}$. Since $\omega^n = 1$, we have $E_n(\omega) = \sum_{j=1}^{\frac{n-1}{2}} j(\omega^j - \omega^{-j})$. Note that $(\omega^j - \omega^{-j}) = 2i\,\mathrm{Im}(\omega^j)$.

Therefore, $E_n(\omega) = i\sum_{j=1}^{\frac{n-1}{2}} 2j\,\mathrm{Im}(\omega^j)$. The sum $\sum_{j=1}^{\frac{n-1}{2}} 2j\,\mathrm{Im}(\omega^j)$ must be positive since $\mathrm{Im}(\omega^j) > 0$ for $j = 1, 2, \dots, \frac{n-1}{2}$. So $E_n(\omega) \neq 0$. $\square$

Lemma 3.2. *Let k be a divisor of n where $k \neq n$. Then $E_n(\omega^k) \neq 0$.*

Proof. If $k = 1$, then we are done by Lemma 3.1. For the remainder of the proof assume $k \neq 1$. Similar computations to the proof of Lemma 3.1 show that

$$E_n(\omega^k) = i\sum_{j=0}^{\frac{n-1}{2}} 2j\big(\mathrm{Im}(\omega^{kj})\big). \tag{6}$$

Let $\frac{n}{k} = l$. Then ω^k is a primitive lth root of unity. We therefore break up $E_n(\omega^k)$ in the following way:

$$\begin{aligned} E_n(\omega^k) = {} & \sum_{m=0}^{\frac{k-1}{2}-1} i\left\{\sum_{j=ml+1}^{ml+\frac{l-1}{2}} 2j\,\mathrm{Im}(\omega^{kj}) + \sum_{j=ml+\frac{l-1}{2}+1}^{(m+1)l-1} 2j\,\mathrm{Im}(\omega^{kj})\right\} \\ & + i\sum_{j=(\frac{k-1}{2})l+1}^{(\frac{k-1}{2})l+\frac{l-1}{2}} 2j\,\mathrm{Im}(\omega^{kj}). \end{aligned} \tag{7}$$

There are $\frac{n-1}{2}+1$ terms in (6) and $\frac{n-1}{2}=\frac{k-1}{2}l+\frac{l-1}{2}$. The first summation over m in (7) gives $\frac{k-1}{2}l+1$ terms and the second summation over j accounts for the remaining $\frac{l-1}{2}$ terms. We now show that $\operatorname{Im}(E_n(\omega^k))>0$.

Notice that $-2\operatorname{Im}(\omega^{(m+1)l-z})=2\operatorname{Im}(\omega^{(m+1)l+z})$ for $z=1,2,\dots,\frac{l-1}{2}$.

Moreover, $2\operatorname{Im}(\omega^{(m+1)l+z})>0$ for $z=1,2,\dots,\frac{l-1}{2}$.

So we have that $2((m+1)l+z)\operatorname{Im}(\omega^{(m+1)l+z})+2((m+1)l-z)\operatorname{Im}(\omega^{(m+1)l-z})=2((m+1)l+z)\operatorname{Im}(\omega^{(m+1)l+z})-2((m+1)l-z)\operatorname{Im}(\omega^{(m+1)l+z})=4z\operatorname{Im}(\omega^{(m+1)l+z})>0$.

This shows that each pair of sums, $\sum_{ml+\frac{l-1}{2}}^{(m+1)l-1}2j\operatorname{Im}(\omega^{kj})+\sum_{(m+1)l+1}^{(m+1)l+\frac{l-1}{2}}2j\operatorname{Im}(\omega^{kj})$ is greater than zero for $m=0,\dots,\frac{k-1}{2}-1$. Moreover the first sum in $E_n(\omega^k)$, which is $\sum_{j=1}^{\frac{l-1}{2}}2j\operatorname{Im}(\omega^{kj})$, is also greater than zero. Therefore $\operatorname{Im}(E_n(\omega^k))>0$ so $E_n(\omega^k)\neq 0$. □

Corollary 3.3. *For any $k=1,\dots,n-1$, $E_n(\omega^k)\neq 0$.*

Proof. For any $k=1,\dots,n-1$, ω^k is a primitive lth root of unity for some $l\in\mathbb{Z}$ which divides n. Assume $\frac{n}{l}=k'$. Note that $k=ak'$ where $\gcd(a,n)=1$. There is an isomorphism ϕ from $\mathbb{Q}[\omega^{k'}]$ to $\mathbb{Q}[\omega^k]$, given by $\phi(1)=1$ and $\phi(\omega^{k'})=\omega^k$ and extended to be a ring homomorphism. If we apply this isomorphism to $E_n(\omega^{k'})$, we see that $\phi(E_n(\omega^{k'}))=E_n(\omega^k)$. From Lemma 3.2, $E_n(\omega^{k'})\neq 0$, so we must have $E_n(\omega^k)\neq 0$. □

Theorem 3.4. *Let A be the S-circulant matrix defined by $a_j=j-1$ for $1\le j\le\frac{n+1}{2}$ and $a_{n-j+1}=-a_{j+1}$ for $1\le j\le n-1$. If $Z=nA+AJ$ and E is the all ones matrix, then $M=Z+\frac{n^2+1}{2}E$ is a regular classical magic square that is nonsingular.*

Proof. By Theorem 2.1 the matrix M has the same eigenvalues as Z except the eigenvalue zero is replaced with $\frac{2\mu}{n}$. From Corollary 3.3, $\operatorname{rank}(A)=n-1$. The matrix Z has the property that $\operatorname{rank}(A)=\operatorname{rank}(Z)$. Therefore, $\operatorname{rank}(M)=n$ and M is a nonsingular classical magic square based on the construction presented in Section 2. □

For explanation, we include the following example.

Example 3.5. If $n=35$, the first row of A becomes

$$[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,-17,-16,-15,-14,-13,-12,$$
$$-11,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1].$$

This A will create a matrix M which is a nonsingular regular classical magic square using the process outlined in Section 2.

It is obvious that there are other ways to designate the first row of A that could also produce nonsingular regular magic squares. One example would be to negate the first row which we defined.

4. Singular regular magic squares of odd composite order

In [3] it was shown that any first row containing the integers from $-\frac{n-1}{2}$ through $\frac{n-1}{2}$ and having the properties that $a_1 = 0$ and $a_{j+1} = -a_{n-j+1}$ for $j = 1, 2, \ldots, n-1$ would produce a nonsingular regular magic square when n is an odd prime. However, this is not true when n is an odd composite.

Let A be an $n \times n$ centroskew S-circulant matrix with first row $\vec{b} = (b_0, b_1, \ldots, b_{n-1})$. We use A to create singular regular magic squares when n is an odd composite. To do this, we need to assign integers from $-\frac{n-1}{2}$ through $\frac{n-1}{2}$ and having the property that $b_j = -b_{n-j}$ for $j = 1, 2, \ldots, n$ such that $E_n(\omega^k) = 0$ for some $1 \le k < n$ where we recall as in Section 3 that $\omega = e^{\frac{2\pi i}{n}}$ and $E_n(x) = \sum_{j=0}^{n-1} b_j x^j$.

Example 4.1. Begin with the example of $n = 35 = 5 \cdot 7$. Denote the first row of A by $[b_0, \ldots, b_{34}]$. Examine $E_{35}(\omega^7)$. Notice that ω^7 is a primitive 5th root of unity.

$$E_{35}\left(\omega^7\right) = \sum_{j=0}^{34} \left(b_j \omega^{7j}\right) = \sum_{j=0}^{4} \left(\sum_{i=0}^{6} b_{5i+j} \right) \omega^{7j}.$$

Therefore, if each sum $\sum_{i=0}^{6} (b_{5i+j})$ is zero, then the eigenvalue is zero. Due to the fact that $b_j = -b_{n-j}$, there are restrictions on what integers you can assign the b_j's. The following two facts follow from these restrictions:

- $\sum_{i=0}^{6} (b_{5i+0})$ must be zero since $b_0 = 0$ and $-b_{5i} = b_{35-5i} = b_{5(7-i)}$.
- $\sum_{i=0}^{6} (b_{5i+j}) = -\sum_{i=0}^{6} (b_{5i+(7-j)})$ since $-b_{5i+j} = b_{35-5i-j} = b_{5(7-i)-j}$.

For $j = 0, 1, 2, 3, 4$ denote $B_j = \{b_{5i+j} : 0 \le i \le 6\}$. From the previous facts, if $b \in B_j$ then $-b \in B_{5-j}$ for $j > 0$. Therefore, if we show that $\sum_{i=0}^{6} (b_{5i+1}) = 0$ and $\sum_{i=0}^{6} (b_{5i+2}) = 0$, then we have that $E_{35}(\omega^7) = 0$.

We simultaneously assign integers into B_1 and B_2. We begin by placing integers in pairs which add to 1. Specifically, we place the numbers 17 and -16 in B_1 and 15 and -14 in B_2. Then we place pairs of integers which add to -1 in each set. To do this we place -13 and 12 into B_1 and -11 and 10 into B_2. After these elements are placed, the sum of the four elements is zero, so we must place three more numbers which add to zero in B_1 and three numbers which add to zero in B_2.

All numbers in B_1 and B_2 have distinct absolute values. So we place their opposites in B_3 and B_4. There are seven unassigned numbers from -17 to 17 not already assigned to B_1 through B_4. Among these are 0 and three pairs of opposite numbers. We put the remaining seven numbers in B_0. The complete sets B_0, B_1, B_2, B_3, and B_4 are as follows:

- $B_0 = \{0, 3, 4, 6, -3, -4, -6\}$
- $B_1 = \{17, -16, -13, 12, 9, -8, -1\}$
- $B_2 = \{15, -14, -11, 10, 7, -5, -2\}$

- $B_3 = \{2, 5, -7, -10, 11, 14, -15\}$
- $B_4 = \{1, 8, -9, -12, 13, 16, -17\}$

Therefore, the first row of matrix A is

$$[0, 17, 15, 2, 1, 3, -16, -14, 5, 8, 4, -13, -11, -7, -9, 6, 12, 10, -10, -12, -6, 9, 7, 11, 13, -4, -8, -5, 14, 16, -3, -1, -2, -15, -17].$$

If we do this, then we have $\sum_{i=0}^{6} b_{5i+j} = 0$ for each $0 \le j \le 4$ and at least two eigenvalues of A are zero. Since multiple eigenvalues of A are zero, the regular classical magic square M one gets by following the process in Section 2 would be singular.

We may apply a similar labeling to any composite odd number. This is done in Lemma 4.2.

Lemma 4.2. *Let $n = n_1n_2$ where n_1 and n_2 are odd integers greater than one. Then there exists an $n \times n$ centroskew S-circulant matrix A such that its first row $\vec{b} = (b_0, b_1, \ldots, b_{n-1})$ gives the property that $E_n(\omega^{n_2}) = 0$.*

Proof. We have that if ω is a primitive nth root of unity then ω^{n_2} is a primitive n_1th root of unity. We examine the eigenvalue $E_n(\omega^{n_2})$.

$$E_n\left(\omega^{n_2}\right) = \sum_{j=0}^{n-1} \left(b_j \omega^{jn_2}\right) = \sum_{j=0}^{n_1-1} \left(\sum_{i=0}^{n_2-1} b_{n_1 i+j}\right) \omega^{n_2 j}.$$

Let $B_j = \{b_{n_1i+j} : 0 \le i \le n_2 - 1\}$, the set of coefficients for $\omega^{n_2 j}$, for $0 \le j \le n_1 - 1$. So long as the elements of B_j add to zero for each $0 \le j \le n_1 - 1$, then $E_n(\omega^{n_2}) = 0$.

Note that since $b_0 = 0$ and $b_j = -b_{n-j}$ we must have two properties on the B_j's:

- The elements of B_0 sum to zero since $b_{n_1 i} = -b_{n-n_1 i} = -b_{n_1(n_2-i)}$.
- The elements of B_j are the opposite of the elements in B_{n_1-j} for $1 \le j \le n_1 - 1$. This is due to the fact that $b_{n_1i+j} = -b_{n-n_1i-j} = -b_{n_1(n_2-i)-j}$.

To simplify the notation let $m = \frac{n-1}{2}$. Each b_j must be an integer between m and $-m$. We construct each B_j for $1 \le j \le \frac{n_1-1}{2}$ in the following way.

Into each B_j for $1 \le j \le \frac{n_1-1}{2}$ put the elements $(-1)^i(m - i(n_1 - 1) - j + 1)$ and $(-1)^{i+1}(m - i(n_1 - 1) - j)$ for $0 \le i \le \frac{n_2-5}{2}$. When i is even each pair of numbers sum to 1 and when i is odd the pair of numbers sum to -1. This assignment of elements places a total of $n_2 - 3$ elements into each B_j. Moreover if $n_2 \equiv 1 \bmod 4$ the sum of all $n_2 - 3$ elements placed in B_j thus far is 1. If $n_2 \equiv 3 \bmod 4$ the sum of all $n_2 - 3$ elements placed in B_j thus far is 0. Therefore the placement of the remaining three elements depends on the value of n_2. We break the assignment for the last three elements based on whether $n_2 \equiv 1 \bmod 4$ or if $n_2 \equiv 3 \bmod 4$.

Case 1: $n_2 \equiv 1 \bmod 4$

Since $n_2 \equiv 1 \bmod 4$, the previous $\frac{n_2-3}{2}$ pairs of elements in B_j must sum to 1.

- Into B_1 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1))$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 2)$, and 1
- Into B_2 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 1)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 4)$, and 2
- Into B_3 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 3)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 7)$, and 3
- Into B_4 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 5)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 10)$, and 4

  $\vdots$
- Into $B_{\frac{n_1-1}{2}}$ place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - k)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - k - \frac{n_1-1}{2} - 1)$, and $\frac{n_1-1}{2}$ where k is chosen such that $|-(z - (\frac{n_2-3}{2})(n_1 - 1) - k)|$ is the highest value of any elements not previously used in a B_j.

Each triple adds to -1 so that in this case the sum of all elements in each B_j sum to zero.

Case 2: $n_2 \equiv 3 \bmod 4$

Since $n_2 \equiv 3 \bmod 4$, the previous $\frac{n_2-3}{2}$ pairs of elements in B_j must sum to 0.

- Into B_1 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1))$, $-(m - (\frac{n_2-3}{2})(n_1 - 1))$, and -1
- Into B_2 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 2)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 4)$, and -2
- Into B_3 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 3)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 6)$, and -3
- Into B_4 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 5)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 9)$ and -4

  $\vdots$
- Into $B_{\frac{n_1-1}{2}}$ place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - k)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - k - \frac{n_1-1}{2})$, and $\frac{n_1-1}{2}$ where k is chosen so that $|-(m - (\frac{n_2-3}{2})(n_1 - 1) - k)|$ is the highest value of any elements not previously used in a B_j.

Each triple adds to 0 so that in this case the sum of all elements in each B_j sum to zero.

In both cases, no integers in B_1 through $B_{\frac{n_1-1}{2}}$ have the same absolute value. Therefore we may place their opposites in B_{n_1-1} through $B_{\frac{n_1+1}{2}}$. There are n_2 remaining elements not used in any B_j for $j \geq 1$. These elements consist of zero and pairs of opposite numbers. Place these elements into B_0. Therefore, the numbers in B_0 must sum to zero.

This assignment of b_i's gives the sum of elements in each B_j is zero, namely $\sum_{i=0}^{n_2-1} b_{n_1 i+j} = 0$, therefore $E_n(\omega^{n_2}) = \sum_{j=0}^{n_1-1}(\sum_{i=0}^{n_2-1} b_{n_1 i+j})\omega^{n_2 j} = 0$. □

Theorem 4.3. *Assume that n is a composite positive odd integer and E is the all ones matrix. Let $Z = nA + AJ$ where A is an $n \times n$ matrix as obtained in Lemma 4.2. Then $M = Z + \frac{n^2+1}{2}E$ is a regular classical magic square that is singular.*

Proof. Designating the first row of the $n \times n$ matrix A as described in Lemma 4.2 creates an S-circulant matrix A which has at least 2 eigenvalues which are zero. This means that $\operatorname{rank}(A) = \operatorname{rank}(Z) \le n - 2$. If we let $M = Z + \frac{n^2+1}{2}E$ then $\operatorname{rank}(M) \le n - 1$ so M is singular. □

Remark 4.4. There are other assignments to the first row of A that will produce a singular M. For example, there are other ways one may permute the elements of the B_j's to also get zero for each summation of elements in B_j.

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