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MODULE STRUCTURE OF AN INJECTIVE RESOLUTION

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Let A be the ring obtained by localizing the polynomial ring $\kappa[X, Y, Z, W]$ over a field κ at the maximal ideal (X, Y, Z, W) and modulo the ideal $(XW - YZ)$. Let $\mathfrak{p}$ be the ideal of A generated by X and Y. We study the module structure of a minimal injective resolution of $A/\mathfrak{p}$ in detail using local cohomology. Applications include the description of $\operatorname{Ext}^i_A(M, A/\mathfrak{p})$, where M is a module constructed by Dutta, Hochster and McLaughlin, and the Yoneda product of $\operatorname{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$.

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1. INTRODUCTION

In the category of modules over a commutative ring, injective and projective modules are dual notions.To study cohomology properties of a module, we may consider a minimal free resolution or a minimal injective resolution of the module. The boundary maps of the former are given by matrices in terms of given basis. The coboundary maps of the latter are discussed less extensively. In general, there are no simple descriptions for injective resolutions. The subtlety comes partly from the fact that there are no canonical ways to identify minimal injective modules (injective hulls) for a given module, even though they are all isomorphic. On works regarding concrete realizations of Grothendieck duality, one finds many natural injective hulls with different guises for a given module. The difference of these injective hulls is a part of the structure of the underlying module. From this viewpoint, injective hulls for a given module are not unique, just as there are different vector spaces of

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the same dimension (for instance, a finite dimensional vector space and its dual). Intriguing structures such as residues support this viewpoint since they arise from isomorphisms between injective hulls.

Here is a typical example: Let R be a formal power series ring of n variables over a field κ. The nth local cohomology module of R supported at the maximal ideal gives rise to an injective hull of κ. The elements of this local cohomology module have a concrete description using generalized fractions. As an R-module, κ has another injective hull consisting of the κ-linear homomorphisms from R to κ annihilated by some power of the maximal ideal. Residues appear when an explicit isomorphism between these two injective hulls is constructed. The reader is referred to Huang (1995, 2000) for more details and further developments along this direction.

The goal of our work is to develop a concrete means to study the structure of injective resolutions of modules. It consists of two steps: constructing injective modules explicitly and then describing the coboundary maps explicitly in a resolution built up from the injective modules obtained in the previous step. The goal has been achieved for modules related to residual complexes, which are of particular interests due to their central role in Grothendieck duality theory. We recall that residual complexes are build by injective hulls of the residue fields of points on a scheme and resolve canonical modules in certain Cohen–Macaulay cases. In Huang (2000), residual complexes are constructed concretely in a relatively canonical way. The construction in Huang (2000) is local. One of its globalizations gives rise to injective resolutions for the vector bundles on projective spaces (Huang, 2001). The injective resolutions obtained in Huang (2000, 2001) are for modules (*resp.* sheaves of modules) whose structures (*resp.* local structures) are determined completely by the underlying rings (*resp.* schemes). Not much is known in general about concrete constructions of injective resolutions of nonflat modules.

Studies of the homology and cohomology modules from the viewpoint of injective objects are often restricted to some subcategories of the category of modules, such as the category of graded (or multigraded) modules (see for example Goto and Watanabe, 1978; Miller, 2000; Miller and Sturmfels, 2005) or the category of squarefree modules (Yanagawa, 2002). Injective resolutions in these smaller categories drastically differ from those in the category of all modules. For instance, a multigraded injective resolution of the polynomial ring $\kappa[X, Y]$ of two variables over a field κ consists of only four indecomposable multi-graded injective modules (Miller and Sturmfels, 2005, Example 11.20). In the category of modules concerning no gradings, a minimal injective resolution of $\kappa[X, Y]$ consists of infinitely many indecomposable injective modules indexed by the prime ideals of $\kappa[X, Y]$ due to its Gorenstein property (Matsumura, 1986, Theorem 18.8). Minimal injective resolutions, especially those for modules over a local ring, are still full of mysteries and are not possible to be deduced from graded cases. At the time when more case studies of minimal injective resolutions are available, a general theory may be developed for a larger class of modules. This paper serves as a first step towards such direction by carrying out the above goal for a module related to an important example in the discussions of several homological conjectures (c.f. Dutta et al., 1985; Roberts, 1998, 13.2).

In this article, our study emphasizes the module structure of injective resolutions rather than its category structure. More precisely, we would like to construct explicitly an injective resolution of a given module and obtain its cohomological information from the resolution. Let S be the polynomial ring $\kappa[X, Y, Z, W]$ over a field κ. In this article, we consider the ring

$$A = S_{(X,Y,Z,W)}/(XW - YZ)$$

and the ideal $\mathfrak{p}$ of A generated by X, Y. For each prime ideal $\mathfrak{q}$ of $\kappa[Z, W]$ contained in $\mathfrak{m} := (Z, W)$, we construct an injective hull $E(A/(\mathfrak{q}, X, Y))$ of $A/(\mathfrak{q}, X, Y)$. In the sequel, we write $E(A/(\mathfrak{q}, X, Y))$ simply as $E(\mathfrak{q})$. In terms of *generalized fractions* (defined in Definition 2.1) of elements of $E(\mathfrak{q})$, our main result describes a minimal injective resolution

$$E(0) \to \bigoplus_{\mathfrak{q}\neq\mathfrak{m}} E(\mathfrak{q}) \to \bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \to E(\mathfrak{m})^2 \to E(\mathfrak{m})^2 \to E(\mathfrak{m})^2 \to \cdots$$

of $A/\mathfrak{p}$. According to the authors' knowledge, this is the first detailed analysis of an injective resolution for a module which does not come from duality theory.

As applications, we read explicitly:

- Local cohomology modules $\mathrm{H}^i_I(A/\mathfrak{p})$ of $A/\mathfrak{p}$ supported at an ideal I of A;
- An isomorphism $\mathrm{Hom}_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{p}) \to \mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p})$ of normal modules;
- The product of the Yoneda algebra $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$;
- $\mathrm{Ext}^i_A(M, A/\mathfrak{p})$, where M is the A-module constructed by Dutta et al. (1985).

The article is organized as follows: In Section 2, we recall the notion of generalized fractions which describe elements in certain top local cohomology modules. Technical properties are prepared for latter use. In Section 3, we construct injective hulls in terms of generalized fractions. These injective hulls are building blocks for our injective resolution. In Section 4, we define homomorphisms for these injective hulls and show that they give rise to a minimal injective resolution. In Section 5, we carry out the computations for the applications listed in the previous paragraph.

2. GENERALIZED FRACTIONS

Our description of injective modules and coboundary maps of an injective resolution is based on local cohomology modules and the representation of their elements by generalized fractions. We recall the definition and some properties of generalized fractions and refer the details to Huang (1995, Chapter 2). Let R be a Noetherian ring and I be an ideal of R generated *up to radical* by n elements $x_1, \ldots, x_n$ and another ideal J. Let N be an R-module, whose elements are annihilated by a power (depending on the element) of J. Elements of the nth local cohomology module $\mathrm{H}^n_I(N)$ of N supported at I can be described by the following exact sequence

$$\bigoplus_{i=1}^{n} N_{x_1\ldots\widehat{x_i}\ldots x_n} \xrightarrow{\alpha} N_{x_1\ldots x_n} \xrightarrow{\beta} \mathrm{H}^n_I(N) \to 0, \tag{1}$$

where α is the map given by

$$\frac{\omega}{(x_1 \ldots \widehat{x_i} \ldots x_n)^s} \mapsto \frac{(-1)^i x_i^s \omega}{(x_1 \ldots x_n)^s}$$

for $\omega \in N$ and $s \geq 0$.

Definition 2.1. A generalized fraction

$$\begin{bmatrix} \omega \\ x_1^{i_1}, \ldots, x_n^{i_n} \end{bmatrix} \in \mathrm{H}_I^n(N),$$

where $\omega \in N$ and $i_1, \ldots, i_n \in \mathbb{Z}$, is the image of $x_1^{s-i_1} \ldots x_n^{s-i_n}\omega/(x_1 \ldots x_n)^s$ under the map β in (1) for a sufficiently large s. ω is called the *numerator of the generalized fraction* and $x_1^{i_1}, \ldots, x_n^{i_n}$ are called the *denominators of the generalized fraction.*

If some i_j is less than one, then the above generalized fraction vanishes. Generalized fractions satisfy the following properties.

Linearity Law. For $\omega_1, \omega_2 \in N$ and $a_1, a_2 \in R$,

$$\begin{bmatrix} a_1\omega_1 + a_2\omega_2 \\ x_1, \ldots, x_n \end{bmatrix} = a_1 \begin{bmatrix} \omega_1 \\ x_1, \ldots, x_n \end{bmatrix} + a_2 \begin{bmatrix} \omega_2 \\ x_1, \ldots, x_n \end{bmatrix}.$$

Transformation Law. For $\omega \in N$ and elements $x'_1, \ldots, x'_n$, which together with J generate I up to radical,

$$\begin{bmatrix} \omega \\ x_1, \ldots, x_n \end{bmatrix} = \begin{bmatrix} \det(r_{ij})\omega \\ x'_1, \ldots, x'_n \end{bmatrix}$$

if $x'_i = \sum_{j=1}^n r_{ij}x_j$ for $i = 1, \ldots, n$.

Vanishing Law. For $\omega \in N$,

$$\begin{bmatrix} \omega \\ x_1, \ldots, x_n \end{bmatrix} = 0$$

if and only if $(x_1 \ldots x_n)^s\omega \in \left(x_1^{s+1}, \ldots, x_n^{s+1}\right)N$ for some $s \geq 0$.

Note that powers of $x_1, \ldots, x_n$ together with J also generate I up to radical. So the above laws apply to generalized fractions with arbitrary denominators. An easy application of these laws is that adding to one of the denominators by a linear combination of other denominators does not change the value.

Example 2.2. Look at the case $n = 2$. I is generated by $x_1, x_2 - ax_1$ and J up to radical for any $a \in R$. We have

$$\begin{bmatrix} \omega \\ x_1, x_2 - ax_1 \end{bmatrix} = \begin{bmatrix} x_2\omega \\ x_1, x_2(x_2 - ax_1) \end{bmatrix}$$

$$= \begin{bmatrix} (x_2 - ax_1)\omega \\ x_1, x_2(x_2 - ax_1) \end{bmatrix} + \begin{bmatrix} ax_1\omega \\ x_1, x_2(x_2 - ax_1) \end{bmatrix}$$
$$= \begin{bmatrix} \omega \\ x_1, x_2 \end{bmatrix}.$$

Proposition 2.3. *Let R be a Noetherian local ring and N be an R-module, whose elements are annihilated by a power (depending on the element) of the maximal ideal of R. An element of $\mathrm{H}^n_{(X_1,\dots,X_n)}(R[X_1,\dots,X_n]_{(X_1,\dots,X_n)} \otimes N)$ can be written as*

$$\Psi = \sum_{i_1,\dots,i_n>0} \begin{bmatrix} 1 \otimes \alpha_{i_1\dots i_n} \\ X_1^{i_1},\dots,X_n^{i_n} \end{bmatrix}, \tag{2}$$

where $\alpha_{i_1\dots i_n} \in N$. The expression is unique in the sense that $\Psi = 0$ if and only if $\alpha_{i_1\dots i_n} = 0$ for all $i_1,\dots,i_n > 0$.

Proof. N has a natural module structure over the completion $\widehat{R}$ of R. Elements of $\mathrm{H}^n_{(X_1,\dots,X_n)}(\hat{R}[[X_1,\dots,X_n]] \otimes N)$ can be written uniquely in the form of (2), see Huang (1995, p. 21). The proposition follows from the canonical isomorphism

$$\mathrm{H}^n_{(X_1,\dots,X_n)}(R[X_1,\dots,X_n]_{(X_1,\dots,X_n)} \otimes N) \simeq \mathrm{H}^n_{(X_1,\dots,X_n)}(\hat{R}[[X_1,\dots,X_n]] \otimes N). \qquad \square$$

Let S be the polynomial ring $\kappa[X, Y, Z, W]$ over a field κ as in Section 1.

Corollary 2.4. *Elements of $\mathrm{H}^4_{(X,Y,Z,W)}(S_{(X,Y,Z,W)})$ can be written uniquely as*

$$\sum_{i,j,k,\ell>0} \begin{bmatrix} a_{ijk\ell} \\ Z^i, W^j, X^k, Y^\ell \end{bmatrix}, \tag{3}$$

where $a_{ijk\ell} \in \kappa$.

We call $a_{ijk\ell}$ the coefficient of $\left[\begin{smallmatrix} 1 \\ Z^i, W^j, X^k, Y^\ell \end{smallmatrix}\right]$ for the element (3).

Corollary 2.5. *Elements of $\mathrm{H}^2_{(X,Y)}(S_{(X,Y)})$ can be written uniquely as*

$$\sum_{i,j>0} \begin{bmatrix} \varphi_{ij} \\ (XW)^i, (YZ)^j \end{bmatrix},$$

where $\varphi_{ij} \in \kappa(Z, W)$.

Proof. $S_{(X,Y)} \simeq \kappa(Z, W)[XW, YZ]_{(XW,YZ)}$. $\square$

Corollary 2.6. *Elements of $\mathrm{H}^3_{(X,Y,Z)}(S_{(X,Y,Z)})$ can be written uniquely as*

$$\sum_{i,j,k>0} \begin{bmatrix} \varphi_{ijk} \\ Z^i, (XW)^j, Y^k \end{bmatrix},$$

where $\varphi_{ijk} \in \kappa(W)$.

Proof. $S_{(X,Y,Z)} \simeq \kappa(W)[XW, Y, Z]_{(XW,Y,Z)}$. □

Corollary 2.7. *Elements of* $\mathrm{H}^3_{(X,Y,W)}(S_{(X,Y,W)})$ *can be written uniquely as*

$$\sum_{i,j,k>0} \begin{bmatrix} \varphi_{ijk} \\ W^i, X^j, (YZ)^k \end{bmatrix},$$

where $\varphi_{ijk} \in \kappa(Z)$.

Proof. $S_{(X,Y,W)} \simeq \kappa(Z)[X, YZ, W]_{(X,YZ,W)}$. □

Corollary 2.8. *Let* (f) *be a nonzero prime ideal of* $\kappa[Z, W]$ *contained in* (Z, W) *but not containing* Z *or* W. *An element* Ψ *in* $\mathrm{H}^3_{(X,Y,f)}(S_{(X,Y,f)})$ *can be written as*

$$\Psi = \sum_{i,j>0} \begin{bmatrix} g_{ij} \\ h_{ij}, (XW)^i, (YZ)^j \end{bmatrix},$$

where $g_{ij} \in \kappa[Z, W]$ *and* $0 \neq h_{ij} \in \kappa[Z, W]$. $\Psi = 0$ *if and only if* $g_{ij} \in h_{ij}\kappa[Z, W]_{(f)}$ *for all* i, j.

Proof. Since $S_{(X,Y,f)} \simeq \kappa[Z, W]_{(f)}[XW, YZ]_{(XW,YZ)}$, there is an isomorphism

$$\mathrm{H}^2_{(XW,YZ,f)}(S_{(X,Y,f)} \otimes_{\kappa[Z,W]_{(f)}} \mathrm{H}^1_{(f)}(\kappa[Z, W]_{(f)})) \simeq \mathrm{H}^3_{(X,Y,f)}(S_{(X,Y,f)})$$

(Huang, 1995, (2.5)) given by

$$\sum_{i,j>0} \begin{bmatrix} 1 \otimes \left[\begin{smallmatrix} g_{ij} \\ h_{ij} \end{smallmatrix} \right] \\ (XW)^i, (YZ)^j \end{bmatrix} \mapsto \sum_{i,j>0} \begin{bmatrix} g_{ij} \\ h_{ij}, (XW)^i, (YZ)^j \end{bmatrix}.$$

The result follows from Proposition 2.3. Moreover, $\Psi = 0$ if and only if $\left[\begin{smallmatrix} g_{ij} \\ h_{ij} \end{smallmatrix} \right] = 0$, equivalently $g_{ij} \in h_{ij}\kappa[Z, W]_{(f)}$, for all i, j. □

The following lemma will be used in Section 4.

Lemma 2.9. *Let* $f \in \kappa[Z, W]$ *be an irreducible polynomial in* (Z, W). *For any* $s, t > 0$, *there exist* $\ell > 0$, $g \in \kappa[Z, W]$ *and* $h \in \kappa[Z, W] \backslash (f)$ *such that*

$$\begin{bmatrix} g \\ h, f^\ell \end{bmatrix} = \begin{bmatrix} 1 \\ W^t, Z^s \end{bmatrix}$$

in $\mathrm{H}^2_{(Z,W)}(\kappa[Z, W]_{(Z,W)})$.

Proof. We may assume $(f) \neq (W)$ to avoid the trivial case. Write

$$f = f_0 Z^u + f_1 W^v$$

for some $f_0 \in \kappa[Z]\backslash(Z)$, $f_1 \in \kappa[Z, W]\backslash(W)$ and $u, v > 0$. Divide s by u:

$$s = uq + r \qquad (0 \leq q \text{ and } 0 \leq r < u).$$

We choose h to be W^t and prove the lemma by induction on $\lceil t/v \rceil$, the smallest integer greater than or equal to t/v. In the case where $\lceil t/v \rceil = 1$ (i.e., $t \leq v$),

$$f^{q+1} = (f_0 Z^u + f_1 W^v)^{q+1} = f_0^{q+1} Z^{u(q+1)} + wW^t,$$

for some $w \in \kappa[Z, W]$. The following can be computed using Example 2.2:

$$\begin{bmatrix} f_0^{q+1} Z^{u-r} \\ W^t, f^{q+1} \end{bmatrix} = \begin{bmatrix} f_0^{q+1} Z^{u-r} \\ W^t, f_0^{q+1} Z^{uq+u} \end{bmatrix} = \begin{bmatrix} 1 \\ W^t, Z^s \end{bmatrix}.$$

Assume the lemma holds for $\lceil t/v \rceil = n$. For the case $\lceil t/v \rceil = n + 1$, let

$$F = \sum_{i=0}^{n-1} (f_0 Z^u)^i (-f_1 W^v)^{n-i-1},$$
$$G = \sum_{j=0}^{q} \binom{q+1}{j} (f_0 Z^u)^{nj} (-f_1 W^v F)^{q-j}.$$

Then

$$f((f_0 Z^u)^n - f_1 W^v F) = (f_0 Z^u)^{n+1} - (-f_1 W^v)^{n+1},$$
$$((f_0 Z^u)^n - f_1 W^v F)^{q+1} = (f_0 Z^u)^{n(q+1)} - f_1 W^v F G,$$

and

$$\begin{aligned}
\begin{bmatrix} f_0^{q+1} Z^{u-r} \\ W^t, f^{q+1} \end{bmatrix} &= \begin{bmatrix} f_0^{q+1} Z^{u-r} ((f_0 Z^u)^n - f_1 W^v F)^{q+1} \\ W^t, ((f_0 Z^u)^{n+1} - (-f_1 W^v)^{n+1})^{q+1} \end{bmatrix} \\
&= \begin{bmatrix} f_0^{q+1} Z^{u-r} ((f_0 Z^u)^n - f_1 W^v F)^{q+1} \\ W^t, (f_0 Z^u)^{(n+1)(q+1)} \end{bmatrix} \\
&= \begin{bmatrix} f_0^{q+1} Z^{u-r} (f_0 Z^u)^{n(q+1)} \\ W^t, (f_0 Z^u)^{(n+1)(q+1)} \end{bmatrix} - \begin{bmatrix} f_0^{q+1} Z^{u-r} f_1 W^v F G \\ W^t, (f_0 Z^u)^{(n+1)(q+1)} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ W^t, Z^s \end{bmatrix} - \begin{bmatrix} f_0^{-n(q+1)} Z^{u-r} f_1 F G \\ W^{t-v}, Z^{u(n+1)(q+1)} \end{bmatrix}.
\end{aligned}$$

Since $\lceil (t-v)/v \rceil = n$, there exist $\ell_0 > 0$ and $g_0 \in \kappa[Z, W]$ such that

$$\begin{bmatrix} g_0 \\ W^{t-v}, f^{\ell_0} \end{bmatrix} = \begin{bmatrix} 1 \\ W^{t-v}, Z^{u(n+1)(q+1)} \end{bmatrix}.$$

We get the required elements:

$$\begin{aligned}\begin{bmatrix} f^{\ell_0} f_0^{q+1} Z^{u-r} + f^{q+1} g_0 f_0^{-n(q+1)} Z^{u-r} W^v f_1 FG \\ W^t, f^{\ell_0+q+1} \end{bmatrix} &= \begin{bmatrix} f_0^{q+1} Z^{u-r} \\ W^t, f^{q+1} \end{bmatrix} + \begin{bmatrix} g_0 f_0^{-n(q+1)} Z^{u-r} f_1 FG \\ W^{t-v}, f^{\ell_0} \end{bmatrix} \\ &= \begin{bmatrix} f_0^{q+1} Z^{u-r} \\ W^t, f^{q+1} \end{bmatrix} + \begin{bmatrix} f_0^{-n(q+1)} Z^{u-r} f_1 FG \\ W^{t-v}, Z^{u(n+1)(q+1)} \end{bmatrix} = \begin{bmatrix} 1 \\ W^t, Z^s \end{bmatrix}. \qquad \square\end{aligned}$$

3. INJECTIVE HULLS

In this section, we study the module structure of an injective hull $E(\mathfrak{q})$ of $A/\mathfrak{Q}$ for each prime ideal $\mathfrak{Q}$ of A generated by X, Y and a prime ideal $\mathfrak{q}$ of $\kappa[Z, W]$ contained in (Z, W). We use the following two well-known constructions for injective hulls.

Lemma 3.1. *Let R be a Noetherian ring, $\mathfrak{Q} \subset \mathfrak{P}$ be prime ideals of R and $E(R/\mathfrak{Q})$ be an injective hull of $R/\mathfrak{Q}$. Then $E(R/\mathfrak{Q})$ is an $R_{\mathfrak{P}}$-module and it is an injective hull of $(R/\mathfrak{Q})_{\mathfrak{P}}$ over $R_{\mathfrak{P}}$.*

Lemma 3.2. *Let R be a Noetherian ring, I be an ideal of R, $\mathfrak{P}$ be a prime ideal of R containing I and $E(R/\mathfrak{P})$ be an injective hull of $R/\mathfrak{P}$. Then, as an R/I-module, $\mathrm{Hom}_R(R/I, E(R/\mathfrak{P}))$ is an injective hull of $R/\mathfrak{P}$.*

Let $\mathfrak{q} = (f_1, \ldots, f_m)$ be a prime ideal of $\kappa[Z, W]$ contained in (Z, W) and $\mathfrak{Q}$ be the prime ideal of S generated by $X, Y, f_1, \ldots, f_m$. We denote by $\mathfrak{Q}$ also the element of $\operatorname{Spec} A$, $\operatorname{Spec} S_{\mathfrak{Q}}$ and $\operatorname{Spec} S_{(X,Y,Z,W)}$, which canonically embed into $\operatorname{Spec} S$. Recall that $\mathrm{H}^{\mathrm{ht}\mathfrak{Q}}_{\mathfrak{Q}}(S_{\mathfrak{Q}})$ is an injective hull of $S/\mathfrak{Q}$, as S is a Gorenstein ring.

Definition 3.3.

$$E(f_1, \ldots, f_m) := E(\mathfrak{q}) := \{\omega \in \mathrm{H}^{\mathrm{ht}\mathfrak{Q}}_{\mathfrak{Q}}(S_{\mathfrak{Q}}) \mid XW\omega = YZ\omega\}$$

By Lemma 3.1, $\mathrm{H}^{\mathrm{ht}\mathfrak{Q}}_{\mathfrak{Q}}(S_{\mathfrak{Q}})$ as an $S_{(X,Y,Z,W)}$-module is also an injective hull of $S_{(X,Y,Z,W)}/\mathfrak{Q}$. By Lemma 3.2, with the A-module structure via the bijection

$$E(\mathfrak{q}) \simeq \mathrm{Hom}_{S_{(X,Y,Z,W)}}(A, \mathrm{H}^{\mathrm{ht}\mathfrak{Q}}_{\mathfrak{Q}}(S_{\mathfrak{Q}})),$$

$E(\mathfrak{q})$ is an injective hull of $A/\mathfrak{Q}$. Next, we describe elements in $E(\mathfrak{q})$ using certain maps $\Omega^n_{\mathfrak{q}}$. If $\mathfrak{q}$ is principal,

$$\Omega^n_{\mathfrak{q}} \colon \kappa(Z, W) \to \mathrm{H}^{\mathrm{ht}\mathfrak{Q}}_{\mathfrak{Q}}(S_{\mathfrak{Q}})$$

is a $\kappa[Z, W]_{\mathfrak{q}}$-linear map. If $\mathfrak{q} = (Z, W)$,

$$\Omega^n_{\mathfrak{q}} \colon \mathrm{H}^2_{(Z,W)}(\kappa[Z, W]_{(Z,W)}) \to \mathrm{H}^4_{(X,Y,Z,W)}(S_{(X,Y,Z,W)})$$

is a κ-linear map. $\Omega^n_{\mathfrak{q}}$ is defined to be zero for $n < 0$ and is defined below for $n \geq 0$.

Definition 3.4. Let $n \geq 0$ and $\mathfrak{q} = (f)$ be a prime ideal of $\kappa[Z, W]$ contained in (Z, W). Given $s \in \mathbb{Z}$, $g \in \kappa[Z, W]$ and $h \in \kappa[Z, W] \backslash \mathfrak{q}$, we define

$$\Omega^n_{\mathfrak{q}}\left(\frac{g}{h}\right) := \frac{g}{h} \sum_{i=0}^{n} \left[\frac{1}{(XW)^{i+1}, (YZ)^{n+1-i}}\right], \qquad \text{if } (f) = (0);$$

and

$$\Omega^n_{\mathfrak{q}}\left(\frac{g}{hf^s}\right) := \begin{cases} \dfrac{g}{h} \displaystyle\sum_{i=0}^{n} \left[\dfrac{1}{f^s Z^{n+1-i}, (XW)^{i+1}, Y^{n+1-i}}\right], & \text{if } (f) = (Z); \\ \dfrac{g}{h} \displaystyle\sum_{i=0}^{n} \left[\dfrac{1}{f^s W^{i+1}, X^{i+1}, (YZ)^{n+1-i}}\right], & \text{if } (f) = (W); \\ \dfrac{g}{h} \displaystyle\sum_{i=0}^{n} \left[\dfrac{1}{f^s, (XW)^{i+1}, (YZ)^{n+1-i}}\right], & \text{if } (f) \neq (0), (Z) \text{ or } (W). \end{cases}$$

The κ-linear map $\Omega^n_{(Z,W)}$ is defined by

$$\Omega^n_{(Z,W)}\left[\frac{1}{Z^u, W^v}\right] := \sum_{i=0}^{n} \left[\frac{1}{Z^{n+1-i+u}, W^{i+1+v}, X^{i+1}, Y^{n+1-i}}\right],$$

where $u, v > 0$.

If $\mathfrak{q}$ is principal, $\Omega^n_{\mathfrak{q}}$ is independent of the choice of a generator f. We use also the notation $\Omega^n_f := \Omega^n_{\mathfrak{q}}$. If $\mathfrak{q} = (Z, W)$, we use also the notation $\Omega^n_{Z,W} := \Omega^n_{\mathfrak{q}}$. The following facts are not hard to check. Details are left to the reader.

Proposition 3.5. *Let $n \geq 0$.*

(1) *For $\varphi \in \kappa(Z, W)$, $\Omega^n_0(\varphi) \neq 0$ if and only if $\varphi \neq 0$.*
(2) *$\Omega^n_Z(Z^s) \neq 0$ (resp. $\Omega^n_W(W^s) \neq 0$) if and only if $s \leq n$. In the κ-vector space $E(Z)$ (resp. $E(W)$), elements of the form $\Omega^n_Z(Z^s W^t)$ (resp. $\Omega^n_W(Z^t W^s)$), where $s \leq n$ and $t \in \mathbb{Z}$, are linearly independent.*
(3) *For nonzero f with $Z, W \notin (f)$, $\Omega^n_f(f^s) \neq 0$ if and only if $s < 0$.*

For any $\varphi \in \kappa(Z, W)$,

$$XW\Omega^n_f \varphi = YZ\Omega^n_f \varphi = \Omega^{n-1}_f \varphi.$$

Therefore Ω^n_f has image in $E(f)$.

Example 3.6. For any nonzero φ in $\kappa(Z, W)$,

$$(XW)^n \Omega^n_0(\varphi) = (YZ)^n \Omega^n_0(\varphi) = \Omega^0_0(\varphi)$$

and it is nonzero in $E(f)$ by Proposition 3.5(1).

Next, we explain the structure of $E(Z, W)$. For any $\varphi \in \mathrm{H}^2_{(Z,W)}(\kappa[Z, W]_{(Z,W)})$,

$$XW\Omega^n_{Z,W}\varphi = YZ\Omega^n_{Z,W}\varphi = \Omega^{n-1}_{Z,W}\varphi.$$

Therefore $\Omega^n_{Z,W}$ has image in $E(Z, W)$. Note that $\Omega^n_{Z,W}$ is not $\kappa[Z, W]$-linear for $n \geq 0$. For instance, $\left[\begin{smallmatrix} 1 \\ Z, W \end{smallmatrix}\right]$ is annihilated by W^{n+1} but

$$W^{n+1}\Omega^n_{Z,W}\begin{bmatrix} 1 \\ Z, W \end{bmatrix} = \begin{bmatrix} 1 \\ Z^2, W, X^{n+1}, Y \end{bmatrix} \neq 0.$$

Definition 3.7. For $s, t \in \mathbb{Z}$, we choose $u, v > 0$ with $u + s, v + t \geq 0$ and define the notation

$$\Omega^n(Z^s W^t) := Z^{u+s} W^{v+t} \Omega^n_{Z,W} \begin{bmatrix} 1 \\ Z^u, W^v \end{bmatrix}.$$

This definition is independent of the choice of u and v, indeed,

$$\Omega^n(Z^s W^t) = \sum_{i=0}^{n} \begin{bmatrix} 1 \\ Z^{n+1-i-s}, W^{i+1-t}, X^{i+1}, Y^{n+1-i} \end{bmatrix}. \tag{4}$$

In general, $\Omega^n(Z^s W^t)$ does not equal to $\Omega^n_{Z,W}\left[\begin{smallmatrix} 1 \\ Z^{-s}, W^{-t} \end{smallmatrix}\right]$. For instance,

$$\Omega^1_{Z,W}\begin{bmatrix} 1 \\ Z, W^{-1} \end{bmatrix} = 0,$$

but

$$\Omega^1(Z^{-1}W) = W^2\Omega^1_{Z,W}\begin{bmatrix} 1 \\ Z, W \end{bmatrix} \neq 0.$$

Proposition 3.8. *$\Omega^n(Z^s W^t) \neq 0$ if and only if $n \geq \max\{0, s, t, s + t\}$. The nontrivial $\Omega^n(Z^s W^t)$ form a basis for the κ-vector space $E(Z, W)$.*

Proof. Let $n \geq 0$. It is clear from the definition that $\Omega^n(Z^s W^t) = 0$ if one of s, t, and $t + s$ is greater than n. We show first that the elements of the form $\Omega^n(Z^s W^t)$ generate $E(Z, W)$. Let

$$\Psi = \sum_{i,j,k,\ell \geq 1} \begin{bmatrix} a_{ijk\ell} \\ Z^i, W^j, X^k, Y^\ell \end{bmatrix} \in \mathrm{H}^4_{(X,Y,Z,W)}(S_{(X,Y,Z,W)}),$$

where $a_{ijk\ell} \in \kappa$. Assume that $\Psi \in E(Z, W)$, that is, $XW\Psi = YZ\Psi$ or

$$\sum_{i,j,k,\ell \geq 1} \begin{bmatrix} a_{ijk\ell} \\ Z^i, W^{j-1}, X^{k-1}, Y^\ell \end{bmatrix} = \sum_{i,j,k,\ell \geq 1} \begin{bmatrix} a_{ijk\ell} \\ Z^{i-1}, W^j, X^k, Y^{\ell-1} \end{bmatrix}.$$

Comparing coefficients, we get

$$a_{i(j+1)(k+1)\ell} = a_{(i+1)jk(\ell+1)}$$

for $i, j, k, \ell \geq 1$. For i', j', k', or ℓ' less than 1, if there exist $i, j, k, \ell \geq 1$ such that $i + j = i' + j'$, $k + \ell = k' + \ell'$ and $j - k = j' - k'$, we define

$$a_{i'j'k'\ell'} := a_{ijk\ell};$$

otherwise we define $a_{i'j'k'\ell'} := 0$. Then

$$\begin{aligned}\Psi &= \sum_{\substack{\ell, m \in \mathbb{Z}\\ n \geq 0}} a_{\ell m 1(n+1)} \sum_{i=0}^{n} \left[\genfrac{}{}{0pt}{}{1}{Z^{\ell-i},\, W^{m+i},\, X^{1+i},\, Y^{n+1-i}}\right]\\ &= \sum_{\substack{\ell, m \in \mathbb{Z}\\ n \geq 0}} a_{\ell m 1(n+1)} \Omega^n(Z^{n+1-\ell} W^{1-m}).\end{aligned}$$

Now we show that those $\Omega^n(Z^s W^t)$ with $n \geq \max\{0, s, t, s+t\}$ are linearly independent over κ. We study a linear combination of $\Omega^n(Z^s W^t)$:

$$\sum_{n \geq \max\{0, s, t, s+t\}} a_{nst} \Omega^n(Z^s W^t), \tag{5}$$

where $a_{nst} \in \kappa$. Setting i equal to n and $n - s$ respectively for the expression (4) of $\Omega^n(Z^s W^t)$, we have

$$\left[\genfrac{}{}{0pt}{}{1}{Z^{n+1-i-s},\, W^{i+1-t},\, X^{i+1},\, Y^{n+1-i}}\right] = \begin{cases} \left[\genfrac{}{}{0pt}{}{1}{Z^{1-s},\, W^{n-t+1},\, X^{n+1},\, Y}\right], & \text{if } i = n;\\ \left[\genfrac{}{}{0pt}{}{1}{Z,\, W^{n-s-t+1},\, X^{n-s+1},\, Y^{s+1}}\right], & \text{if } i = n - s.\end{cases}$$

We note that

$$\left[\genfrac{}{}{0pt}{}{1}{Z^{1-s},\, W^{n-t+1},\, X^{n+1},\, Y}\right] = 0, \qquad \text{if } s > 0$$

and

$$\left[\genfrac{}{}{0pt}{}{1}{Z,\, W^{n-s-t+1},\, X^{n-s+1},\, Y^{s+1}}\right] = 0, \qquad \text{if } s < 0.$$

For given n, s, t in the summation in (5), $\left[\genfrac{}{}{0pt}{}{1}{Z^{1-s}, W^{n-t+1}, X^{n+1}, Y}\right]$ occurs if $s < 0$. By Corollary 2.4, $\left[\genfrac{}{}{0pt}{}{1}{Z^i, W^j, X^k, Y^\ell}\right]$ are linearly independent for all $i, j, k, \ell > 0$ in $\mathrm{H}^4_{(X,Y,Z,W)}(S_{(X,Y,Z,W)})$. This implies that if there exist n', s', t' such that

$$\left[\genfrac{}{}{0pt}{}{1}{Z^{n'+1-i'-s'},\, W^{i'+1-t'},\, X^{i'+1},\, Y^{n'+1-i'}}\right] = \left[\genfrac{}{}{0pt}{}{1}{Z^{1-s},\, W^{n-t+1},\, X^{n+1},\, Y}\right]$$

for some $i' \in \{0, \ldots, n'\}$, then $i' = n' = n$, $s' = s$ and $t' = t$. Thus, for fixed n, s, t with $s < 0$, the generalized fraction $\left[{1 \atop Z^{1-s}, W^{n-t+1}, X^{n+1}, Y} \right]$ occurs in (5) exactly once with the coefficient a_{nst}. Similarly for $\left[{1 \atop Z, W^{n-s-t+1}, X^{n-s+1}, Y^{s+1}} \right]$ with $s \geq 0$. Therefore, if

$$\sum_{n \geq \max\{0, s, t, s+t\}} a_{nst} \Omega^n(Z^s W^t) = 0,$$

then $a_{nst} = 0$ for all n, s, t by Corollary 2.4 again. Hence, $\Omega^n(Z^s W^t)$ are linearly independent. □

The A-module structure of $E(Z, W)$ is clear: For $s_1, t_1 \geq 0$ and $s_2, t_2 \in \mathbb{Z}$, we have

$$Z^{s_1} W^{t_1} \Omega^n(Z^{s_2} W^{t_2}) = \Omega^n(Z^{s_1+s_2} W^{t_1+t_2}), \tag{6}$$

$$X^{t_1} Y^{s_1} \Omega^n(Z^{s_2} W^{t_2}) = \Omega^{n-t_1-s_1}(Z^{s_2-s_1} W^{t_2-t_1}). \tag{7}$$

For arbitrary $\varphi \in A$, $n \geq 0$ and $s_2 + t_2 \leq n$, we choose $f \in \kappa[X, Y, Z, W]$ such that $\varphi - f \in (X, Y)^{n+1} + (Z, W)^{n-s_2-t_2+1}$, then

$$\varphi \Omega^n(Z^{s_2} W^{t_2}) = f \Omega^n(Z^{s_2} W^{t_2}).$$

Replaced φ by f, we can use the equalities (6), (7), and κ-linearity to multiply $\Omega^n(Z^{s_2} W^{t_2})$ by φ.

Example 3.9. By Proposition 3.8, for any nonzero $\Omega^n(Z^s W^t)$, there exist n_1 and n_2 with $n_1 + n_2 = n$ such that

$$X^{n_1} Y^{n_2} \Omega^n(Z^s W^t) = \Omega^0(Z^{s-n_2} W^{t-n_1}) \neq 0.$$

Note that $\Omega^0(Z^s W^t)$ obtained by multiplying $\Omega^n(Z^s W^t)$ by $(XW)^n$ may be zero if s or t is positive.

The computations in Examples 3.6 and 3.9 will be used in proving our main result Theorem 4.12.

Divisions by X, Y, Z, and W can be defined as well. In general, for $i, j, k, l \in \mathbb{Z}$, let $X^i Y^j Z^k W^l$ be the κ-linear operator on $E(Z, W)$ satisfying

$$X^i Y^j Z^k W^l \Omega^n(Z^s W^t) = \Omega^{n-i-j}(Z^{s+k-j} W^{t+l-i}).$$

Using the above description of the A-module structure of $E(Z, W)$, one can check that this operator is A-linear.

Definition 3.10. Let $\mathfrak{q} = (f_1, \ldots, f_m)$ be a prime ideal of $\kappa[Z, W]$ contained in (Z, W). We define $E_n(\mathfrak{q})$, denoted also by $E_n(f_1, \ldots, f_m)$, to be the $\kappa[Z, W]_{(Z,W)}$-submodule of $E(\mathfrak{q})$ generated by the image of $\Omega^n_{\mathfrak{q}}$.

$E_n(f)$ consists of elements of the form $\Omega_f^n\varphi$. $E_n(Z, W)$ consists of elements of the form $Z^sW^t\Omega_{Z,W}^n\varphi$. Note that powers of Z and W are necessary to represent elements of $E_n(Z, W)$. For instance,

$$\begin{bmatrix} 1 \\ Z, W, X^2, Y \end{bmatrix} = ZW^2\Omega_{Z,W}^1\begin{bmatrix} 1 \\ Z, W \end{bmatrix}$$

does not equal to $\Omega_{Z,W}^n\varphi$ for any $\varphi \in \mathrm{H}_{(Z,W)}^2(\kappa[Z, W]_{(Z,W)})$. The multiplication by XW (equals YZ in A) takes elements of $E_n(f)$ into $E_{n-1}(f)$. The multiplication by X or Y takes elemens of $E_n(Z, W)$ into $E_{n-1}(Z, W)$.

Proposition 3.11. *$E(\mathfrak{q}) = \bigoplus_n E_n(\mathfrak{q})$ as $\kappa[Z, W]_{(Z,W)}$-modules.*

Proof. We prove the proposition in five cases.

Case 1. $\mathfrak{q} = (Z, W)$. Already shown in Proposition 3.8.

Case 2. $\mathfrak{q} = (0)$. Let

$$\Psi = \sum_{i,j\geq 1}\begin{bmatrix} \varphi_{ij} \\ (XW)^i, (YZ)^j \end{bmatrix} \in \mathrm{H}_{(X,Y)}^2(S_{(X,Y)}),$$

where $\varphi_{ij} \in \kappa(Z, W)$, be an element of $E(0)$. From the identity $XW\Psi = YZ\Psi$, we get

$$\varphi_{i(j+1)} = \varphi_{(i+1)j}$$

for $i, j \geq 1$. We have the expression

$$\Psi = \sum_{n\geq 0}\left(\sum_{i+j=n+2}\begin{bmatrix} \varphi_{ij} \\ (XW)^i, (YZ)^j \end{bmatrix}\right) = \sum_{n\geq 0}\Omega_0^n(\varphi_{1(n+1)}),$$

since for any i, j with $i + j = n + 2$, it is clear that $\varphi_{ij} = \varphi_{i(n+2-i)} = \varphi_{1(n+1)}$. This shows $E(0) = \sum_n E_n(0)$.

An element

$$\Psi_n = \sum\begin{bmatrix} \varphi_{ij} \\ (XW)^i, (YZ)^j \end{bmatrix} \in E_n(0),$$

where $\varphi_{ij} \in \kappa(Z, W)$, satisfies $i + j = n + 2$. If $\sum\Psi_n = 0$, by Corollary 2.5, $\Psi_n = 0$ for all n. This shows $E(0) = \bigoplus_n E_n(0)$.

Case 3. $\mathfrak{q} = (Z)$. Let

$$\Psi = \sum_{i,j,k\geq 1}\begin{bmatrix} \varphi_{ijk} \\ Z^i, (XW)^j, Y^k \end{bmatrix} \in \mathrm{H}_{(X,Y,Z)}^3(S_{(X,Y,Z)})$$

where $\varphi_{ijk} \in \kappa(W)$, be an element of $E(Z)$. From the identity $XW\Psi = YZ\Psi$, we get

$$\varphi_{i(j+1)k} = \varphi_{(i+1)j(k+1)}$$

for $i, j, k \geq 1$. For i', j', or k' less than 1, if there exist $i, j, k \geq 1$ such that $i + j = i' + j'$ and $j + k = j' + k'$, we define

$$\varphi_{i'j'k'} := \varphi_{ijk};$$

otherwise we define $\varphi_{i'j'k'} := 0$. Then

$$\Psi = \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \varphi_{m1(n+1)} \sum_{i=0}^{n} \begin{bmatrix} 1 \\ Z^{m-i}, (XW)^{i+1}, Y^{n+1-i} \end{bmatrix} = \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \Omega_Z^n\left(\frac{\varphi_{m1(n+1)}}{Z^{m-n-1}}\right).$$

This shows $E(Z) = \sum_n E_n(Z)$.

An element

$$\Psi_n = \sum_{i=0}^{n} \begin{bmatrix} \varphi_{ijk} \\ Z^i, (XW)^j, Y^k \end{bmatrix} \in E_n(Z),$$

where $\varphi_{ijk} \in \kappa(W)$, satisfies $j + k = n + 2$. If $\sum \Psi_n = 0$, by Corollary 2.6, $\Psi_n = 0$ for all n. This shows $E(Z) = \bigoplus_n E_n(Z)$.

Case 4. $\mathfrak{q} = (W)$. Similar to Case 3.

Case 5. $\mathfrak{q} = (f)$ not equal to (Z) or (W). Let

$$\Psi = \sum_{i,j\geq 1} \begin{bmatrix} g_{ij} \\ h_{ij}, (XW)^i, (YZ)^j \end{bmatrix} \in \mathrm{H}^3_{(X,Y,f)}(S_{(X,Y,f)}),$$

where $g_{ij} \in \kappa[Z, W]$ and $0 \neq h_{ij} \in \kappa[Z, W]$, be an element in $E(f)$. Multiplying the numerators and denominators by h_{ij}'s, we may assume that all h_{ij} equals a fixed $h \in \kappa[Z, W]$. From the identity $XW\Psi = YZ\Psi$, we get

$$g_{i(j+1)} - g_{(i+1)j} \in h\kappa[Z, W]_{(f)}$$

for $i, j \geq 1$. Hence

$$\Psi = \sum_{n\geq 0}\left(\sum_{i+j=n+2} \begin{bmatrix} g_{1(n+1)} \\ h, (XW)^i, (YZ)^j \end{bmatrix}\right) = \sum_{n\geq 0} \Omega_f^n\left(\frac{g_{1(n+1)}}{h}\right).$$

This shows $E(f) = \sum_n E_n(f)$.

An element

$$\Psi_n = \sum_{i,j\geq 1} \begin{bmatrix} g_{ij} \\ h, (XW)^i, (YZ)^j \end{bmatrix} \in E_n(f),$$

where $g_{ij} \in \kappa[Z, W]$ and $0 \neq h \in \kappa[Z, W]$, satisfies $i + j = n + 2$. If $\sum \Psi_n = 0$, by Corollary 2.8, $\Psi_n = 0$ for all n. This shows $E(f) = \bigoplus_n E_n(f)$. □

$E_n(\mathfrak{q})$ is not a $\kappa[X, Y, Z, W]$-module. In fact, $XWE_n(\mathfrak{q}) = E_{n-1}(\mathfrak{q})$.

Proposition 3.12. *Let $\mathfrak{q}$ be a prime ideal of $\kappa[Z, W]$ contained in (Z, W). Then $E_0(\mathfrak{q}) = 0:_{E(\mathfrak{q})}(X, Y)$.*

It is clear that $E_0(\mathfrak{q})$ is annihilated by (X, Y) for all $\mathfrak{q}$. To prove the proposition, it remains to show that $E_0(\mathfrak{q})$ contains all the elements annihilated by (X, Y). For a prime ideal (f) of $\kappa[Z, W]$ contained in (Z, W), we denote

$$f^{\triangle} := \begin{cases} X, & \text{if } (f) = (W); \\ Y, & \text{if } (f) = (Z); \\ XW, & \text{otherwise.} \end{cases}$$

The multiplication on $E(f)$ by Z (resp. W) is an isomorphism if $Z \notin (f)$ (resp. $W \notin (f)$). Since elements of $E(f)$ are annihilated by $XW - YZ$, an element of $E(f)$ is annihilated by $f^{\triangle}$ if and only if it is annihilated by X and Y. For instance, let $f = Z + W$ and $\Psi \in E(f)$. Then $f^{\triangle} = XW$ and $(XW - YZ)\Psi = 0$. If $X\Psi = Y\Psi = 0$, then clearly $f^{\triangle}\Psi = 0$. Conversely if $f^{\triangle}\Psi = 0$, then $X\Psi = 0$, since the multiplication by W is an isomorphism. $YZ\Psi = 0$ as well, since $YZ\Psi = f^{\triangle}\Psi$. Now $Y\Psi = 0$, because the multiplication by Z is an isomorphism.

For Proposition 3.12, what we need to prove is the following.

Proposition 3.13. *$E_0(f)$ contains all the elements of $E(f)$ annihilated by $f^{\triangle}$. $E_0(Z, W)$ contains all the elements of $E(Z, W)$ annihilated by (X, Y).*

Proof. Let

$$\Psi = \sum \Omega_0^n(\varphi_n)$$

be an element of $E(0)$ annihilated by XW, where $\varphi_n \in \kappa(Z, W)$. Then

$$\sum \Omega_0^{n-1}(\varphi_n) = XW\Psi = 0.$$

By Proposition 3.11, $\Omega_0^{n-1}(\varphi_n) = 0$ for all n, which implies $\varphi_n = 0$ for $n \geq 1$ by Proposition 3.5. Therefore,

$$\Psi = \Omega_0^0(\varphi_0) \in E_0(0).$$

Now assume that $(f) \neq 0$, (Z) or (W). Let

$$\Psi = \sum \Omega_f^n\left(\frac{g_n}{h_n} f^{s_n}\right)$$

be an element of $E(f)$ annihilated by XW, where $g_n, h_n \in \kappa[Z, W] \setminus (f)$. Then

$$\sum \Omega_f^{n-1}\left(\frac{g_n}{h_n} f^{s_n}\right) = XW\Psi = 0.$$

By Propositions 3.11 and 3.5, $\Omega_f^{n-1}(\frac{g_n}{h_n} f^{s_n}) = 0$ for all n and this implies $s_n \geq 0$ for $n \geq 1$. Therefore $\Omega_f^n(\frac{g_n}{h_n} f^{s_n}) = 0$ for all $n \geq 1$ and

$$\Psi = \Omega_f^0\left(\frac{g_0}{h_0} f^{s_0}\right) \in E_0(f).$$

Let

$$\Psi = \sum \Omega_Z^n\left(\frac{g_n}{h_n} Z^{s_n}\right)$$

be an element of $E(Z)$ annihilated by Y, where $g_n, h_n \in \kappa[Z, W]\backslash(Z)$. Then

$$\sum_{n\geq 1} \Omega_Z^{n-1}\left(\frac{g_n}{h_n} Z^{s_n - 1}\right) = Y\Psi = 0.$$

Using Propositions 3.11 and 3.5 again, a similar argument as in the previous cases shows $\Omega_Z^{n-1}(\frac{g_n}{h_n} Z^{s_n-1}) = 0$ for all n and $s_n - 1 > n - 1$ for $n \geq 1$. Therefore $\Omega_Z^n(\frac{g_n}{h_n} Z^{s_n}) = 0$ for all $n \geq 1$ and

$$\Psi = \Omega_Z^0\left(\frac{g_0}{h_0} Z^{s_0}\right) \in E_0(Z).$$

Similarly, we see that $E_0(W)$ contains all the elements of $E(W)$ annihilated by X.

Let

$$\Psi = \sum_{n \geq \max\{0, s, t, s+t\}} a_{nst} \Omega^n(Z^s W^t)$$

be an element of $E(Z, W)$ annihilated by X and Y, where $a_{nst} \in \kappa$. Then

$$\sum_{n \geq \max\{1, s+1, t, s+t\}} a_{nst} \Omega^{n-1}(Z^s W^{t-1}) = X\Psi = 0.$$

By Proposition 3.8, the coefficient $a_{nst} = 0$, if $n \geq \max\{1, s+1, t, s+t\}$. For $n \geq 1$, possible nontrivial coefficients are those a_{nnt} with $t \leq 0$. Similarly, $Y\Psi = 0$ implies that possible nontrivial coefficients are those a_{nsn} with $s \leq 0$. Therefore $a_{nst} = 0$ for $n \geq 1$ and

$$\Psi = \sum_{0 \geq \max\{s, t, s+t\}} a_{0st} \Omega^0(Z^s W^t) \in E_0(Z, W). \qquad \square$$

4. AN INJECTIVE RESOLUTION

In this section, we construct explicitly an injective resolution of $A/\mathfrak{p}$ using the injective modules given in Section 3. The coboundary maps of the injective resolution involve multiplications and divisions by elements of A and certain maps d_f^0, d_f^1 appeared in a residual complex.

A prime ideal (f) is also generated by gf for any invertible element g. For convenience, we use the notation $\bigoplus_{f\neq 0} E(f)$ for the direct sum of modules $E(f)$ indexed by the ideals generated by the irreducible polynomial $f \in \mathfrak{m} = (Z, W)$; that is,

$$\bigoplus_{f\neq 0} E(f) := \bigoplus_{\mathfrak{q}\neq (0),\mathfrak{m}} E(\mathfrak{q}),$$

where f ranges over irreducible polynomials contained in (Z, W). We use the notation $\sum_f$ for representing elements in $\bigoplus_{f\neq 0} E(f)$.

Recalling the notation $f^{\triangle}$ defined in Section 3, we have the following exact sequence by Propositions 3.12 and 3.13.

$$0 \to \bigoplus_{f\neq 0} E_0(f) \to \bigoplus_{f\neq 0} E(f) \xrightarrow{\oplus f^{\triangle}} \bigoplus_{f\neq 0} E(f) \to 0. \tag{8}$$

Now we define d^0_f using Corollary 2.5.

Definition 4.1. For an irreducible polynomial $f \in (Z, W)$, we define

$$d^0_f\colon \mathrm{H}^2_{(X,Y)}(S_{(X,Y)}) \to \mathrm{H}^3_{(X,Y,f)}(S_{(X,Y,f)})$$

to be the map

$$\sum_{i,j>0} \begin{bmatrix} g_{ij}/h_{ij} \\ (XW)^i, (YZ)^j \end{bmatrix} \mapsto \sum_{i,j>0} \begin{bmatrix} g_{ij} \\ h_{ij} W^i Z^j, X^i, Y^j \end{bmatrix},$$

where $g_{ij} \in \kappa[Z, W]$ and $0 \neq h_{ij} \in \kappa[Z, W]$.

We note that d^0_f induces a restriction (by abusing the notation)

$$d^0_f\colon E(0) \to E(f)$$

since $d^0_f(\Omega^n_0(g/h)) = \Omega^n_f(g/h)$ and $d^0_f(E_n(0)) \subset E_n(f)$. The product

$$\prod_{f\neq 0} d^0_f\colon E(0) \to \prod_{f\neq 0} E(f)$$

has image in $\bigoplus_{f\neq 0} E(f)$.

Definition 4.2. We define

$$d^0\colon E(0) \to \bigoplus_{f\neq 0} E(f),$$

where f ranges over irreducible polynomials in $(Z, W)\kappa[Z, W]$, to be the A-linear map

$$\Omega_0^n\left(\frac{g}{h}\right) \mapsto \sum_{f\neq 0} \Omega_f^n\left(\frac{g}{h}\right),$$

where $g \in \kappa[Z, W]$ and $0 \neq h \in \kappa[Z, W]$.

Now we define d_f^1 using Corollaries 2.6–2.8.

Definition 4.3. For an irreducible polynomial $f \in (Z, W)$, we define

$$d_f^1\colon \mathrm{H}^3_{(X,Y,f)}(S_{(X,Y,f)}) \to \mathrm{H}^4_{(X,Y,Z,W)}(S_{(X,Y,Z,W)})$$

to be the map

$$\sum_{i,j,k>0} \begin{bmatrix} g_{ijk}/h_{ijk} \\ f^i, X^j, Y^k \end{bmatrix} \mapsto \sum_{i,j,k>0} \begin{bmatrix} g_{ijk} \\ h_{ijk}, f^i, X^j, Y^k \end{bmatrix},$$

where g_{ijk} and h_{ijk} are in $\kappa[Z, W]$ and h_{ijk} has no factor f.

For instance,

$$\begin{aligned} d_W^1 \Omega_W^n(Z^s W^t) &= \Omega^n(Z^s W^t) \\ d_Z^1 \Omega_Z^n(Z^s W^t) &= -\Omega^n(Z^s W^t). \end{aligned} \tag{9}$$

Lemma 4.4. $d_f^1(E_n(f)) \subset E_n(Z, W)$

Proof. Let $g, h \in \kappa[Z, W]\backslash(f)$.

Case 1. $(f) \neq (Z)$ or (W). For $j > 0$, the elements $hW^{n+1}Z^{n+1}$ and f^j form a system of parameters for $k[Z, W]_{(Z,W)}$, so there exist $\alpha_{ij} \in \kappa[Z, W]_{(Z,W)}$ and $s, t > n+1$ such that

$$\begin{cases} Z^s = \alpha_{11} h W^{n+1} Z^{n+1} + \alpha_{12} f^j, \\ W^t = \alpha_{21} h W^{n+1} Z^{n+1} + \alpha_{22} f^j. \end{cases}$$

Then

$$\begin{aligned} d_f^1\left(\Omega_f^n\left(\frac{g}{hf^j}\right)\right) &= \sum_{i=0}^n d_f^1\left(\begin{bmatrix} g/h \\ f^j, (XW)^{i+1}, (YZ)^{n+1-i} \end{bmatrix}\right) \\ &= \sum_{i=0}^n \begin{bmatrix} g \\ hW^{i+1}Z^{n+1-i}, f^j, X^{i+1}, Y^{n+1-i} \end{bmatrix} \qquad (10) \\ &= \sum_{i=0}^n \begin{bmatrix} g(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) \\ Z^{s-i}, W^{t-n+i}, X^{i+1}, Y^{n+1-i} \end{bmatrix} \\ &= g(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\Omega^n(Z^{n+1-s}W^{n+1-t}) \in E_n(Z, W). \qquad (11) \end{aligned}$$

The equality (10) holds because W and Z are invertible and (11) is due to the transformation law in Section 2.

Case 2. $f = Z$. For $s > 0$, $\Omega_Z^n(\frac{g}{h}Z^s) = 0$. For $s \leq n$, the elements h and Z^{n-s+1} form a system of parameters, so one may choose $t > 0$ and $\alpha_{ij} \in \kappa[Z, W]_{(Z,W)}$ such that

$$W^t = \alpha_{21}h + \alpha_{22}Z^{n-s+1}.$$

Then, a similar computation as in the previous case shows

$$\begin{aligned} d_Z^1\left(\Omega_Z^n\left(\frac{g}{h}Z^s\right)\right) &= \sum_{i=0}^n d_f^1\left(\begin{bmatrix} g/h \\ Z^{n+1-i-s}, (XW)^{i+1}, Y^{n+1-i} \end{bmatrix}\right) \\ &= \sum_{i=0}^n \begin{bmatrix} g \\ hW^{i+1}, Z^{n+1-i-s}, X^{i+1}, Y^{n+1-i} \end{bmatrix} \\ &= \sum_{i=0}^n \begin{bmatrix} g\alpha_{21} \\ W^{t+i+1}, Z^{n+1-i-s}, X^{i+1}, Y^{n+1-i} \end{bmatrix} \\ &= -g\alpha_{21}\Omega^n(Z^s W^{-t}) \in E_n(Z, W). \end{aligned}$$

Case 3. $f = W$. Similar to Case 2. □

So we have a restriction (by abusing the notation)

$$d_f^1 \colon E(f) \to E(Z, W).$$

Definition 4.5. We define

$$d^1 \colon \bigoplus_{f \neq 0} E(f) \to E(Z, W)$$

to be $d^1 = \bigoplus_{f \neq 0} d_f^1$, where f ranges over irreducible polynomials in $(Z, W)\kappa[Z, W]$.

Proposition 4.6. $d^1 \circ d^0 = 0$.

Proof. We apply the argument in the proof of Huang (1997, Prop. 1). Recall that an arbitrary element in $E(0)$ can be written as a sum of elements in the form of $\left[\begin{smallmatrix} g/h \\ X^j, Y^k \end{smallmatrix}\right]$. It is enough to show the image of such an element under $d^1 \circ d^0$ is zero in $E(Z, W)$. We write $h = f_1 \dots f_n$ where $f_1, \dots, f_n$ are powers of distinct irreducible polynomials. It suffices to show

$$\sum_{i=1}^n \begin{bmatrix} g \\ f_1 \dots \hat{f}_i \dots f_n, f_i, X^j, Y^k \end{bmatrix} = 0. \tag{12}$$

We induct on n to prove that (12) holds for a more general case where $f_1, \dots, f_n$ are assumed to be products of powers of irreducible polynomials but each irreducible factor appears in only one f_i. The case $n = 2$ is trivial. Assume that $n = 3$. If some $f_i \notin (Z, W)$, (12) clearly holds. So we assume all $f_i \in (Z, W)$. That f_1 and f_2 are a system of parameters for $\kappa[Z, W]_{(Z,W)}$ implies f_3^{ℓ} is in (f_1, f_2) for some $\ell \gg 0$. By multiplying g and f_3 by $f_3^{\ell-1}$ and replacing them by the latter elements, we may assume that $f_3 = g_1f_1 + g_2f_2$ for some $g_1, g_2 \in \kappa[Z, W]_{(Z,W)}$. Then

$$\begin{aligned}
\left[\begin{matrix} g \\ f_1f_2, f_3, X^j, Y^k \end{matrix}\right] &= \left[\begin{matrix} g(g_1f_1 + g_2f_2) \\ f_1f_2, (g_1f_1 + g_2f_2)^2, X^j, Y^k \end{matrix}\right] \\
&= \left[\begin{matrix} gg_1 \\ f_2, (g_1f_1 + g_2f_2)^2, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} gg_2 \\ f_1, (g_1f_1 + g_2f_2)^2, X^j, Y^k \end{matrix}\right] \\
&= \left[\begin{matrix} g \\ f_2, g_1f_1^2, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1, g_2f_2^2, X^j, Y^k \end{matrix}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left[\begin{matrix} g \\ f_2f_3, f_1, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1f_3, f_2, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1f_2, f_3, X^j, Y^k \end{matrix}\right] \\
&\quad = \left[\begin{matrix} g \\ f_2^2g_2, f_1, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1^2g_1, f_2, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1f_2, f_3, X^j, Y^k \end{matrix}\right] = 0.
\end{aligned}$$

Now assume that $n > 3$ and (12) holds for numbers of f_i's less than n of the general case stated above. Then

$$\begin{aligned}
&\left[\begin{matrix} g \\ f_4 \dots f_n, (f_1f_2f_3), X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ (f_1f_2f_3)f_5 \dots f_n, f_4, X^j, Y^k \end{matrix}\right] \\
&\quad + \dots + \left[\begin{matrix} g \\ (f_1f_2f_3)f_4 \dots f_{n-1}, f_n, X^j, Y^k \end{matrix}\right] = 0, \qquad (13) \\
&\left[\begin{matrix} g \\ f_3(f_4 \dots f_n), (f_1f_2), X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ (f_1f_2)(f_4 \dots f_n), f_3, X^j, Y^k \end{matrix}\right] \\
&\quad + \left[\begin{matrix} g \\ (f_1f_2)f_3, (f_4 \dots f_n), X^j, Y^k \end{matrix}\right] = 0, \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\begin{matrix} g \\ f_2(f_3 \dots f_n), f_1, X^j, Y^k \end{matrix}\right] + \left[\begin{matrix} g \\ f_1(f_3 \dots f_n), f_2, X^j, Y^k \end{matrix}\right] \\
&\quad + \left[\begin{matrix} g \\ f_1f_2, (f_3 \dots f_n), X^j, Y^k \end{matrix}\right] = 0. \qquad (15)
\end{aligned}$$

Add identities (13), (14), and (15), we get identity (12). $\square$

Definition 4.7. Let $E^\bullet$ be the complex

$$E(0) \xrightarrow{d^0} \bigoplus_{f \neq 0} E(f) \xrightarrow{d^1} E(Z, W) \to 0 \to \cdots$$

and $E_n^\bullet$

$$E_n(0) \to \bigoplus_{f\neq 0} E_n(f) \to E_n(Z, W) \to 0 \to \cdots$$

be its restriction.

Lemma 4.8. *d^1 is surjective. Let $f \in \kappa[Z, W]$ be an irreducible polynomial in (Z, W). Then $d_f^1 E_0(f) = E_0(Z, W)$.*

Proof. d^1 is surjective, since the generators $\Omega^n(Z^s W^t)$ of $E(Z, W)$ are in the image of d^1 as seen in (9).

To prove the second assertion, we assume that $(f) \neq (W)$ to avoid the trivial case. For any $s, t \leq 0$, we choose $\ell > 0$ and $g \in \kappa[Z, W]$ as in the proof of Lemma 2.9 such that

$$\begin{bmatrix} g \\ W^{1-t}, f^\ell \end{bmatrix} = \begin{bmatrix} 1 \\ W^{1-t}, Z^{1-s} \end{bmatrix}. \tag{16}$$

Since $(f) \neq (W)$, there exists $n \geq 1 - s$ such that Z^n is a combination of W^{1-t} and f^ℓ over $\kappa[Z, W]_{(Z,W)}$, that is,

$$Z^n = \varphi W^{1-t} + \eta f^\ell \tag{17}$$

for some $\varphi, \eta \in \kappa[Z, W]_{(Z,W)}$. Using (16) and (17), we observe the following:

$$\begin{aligned} \begin{bmatrix} \eta g \\ W^{1-t}, Z^n \end{bmatrix} &= \begin{bmatrix} \eta g \\ W^{1-t}, Z^n - \varphi W^{1-t} \end{bmatrix} \\ &= \begin{bmatrix} g \\ W^{1-t}, f^\ell \end{bmatrix} = \begin{bmatrix} 1 \\ W^{1-t}, Z^{1-s} \end{bmatrix} = \begin{bmatrix} Z^{n+s-1} \\ W^{1-t}, Z^n \end{bmatrix}. \end{aligned}$$

This implies

$$\eta g - Z^{n+s-1} \in (W^{1-t}, Z^n) \tag{18}$$

in $\kappa[Z, W]_{(Z,W)}$. The relations in (17) and (18) can be extended to $S_{(X,Y,Z,W)}$. Therefore, the second assertion follows from the computation

$$d_f^1\left(\Omega_f^0\left(\frac{-gZW^t}{f^\ell}\right)\right) = \begin{bmatrix} -g \\ W^{1-t}, f^\ell, X, Y \end{bmatrix} = \begin{bmatrix} \eta g \\ Z^n, W^{1-t}, X, Y \end{bmatrix} = \Omega^0(Z^s W^t). \qquad \square$$

Lemma 4.9. *$E_0^\bullet$ is exact.*

Proof. We only need to prove that an element of $\bigoplus_{f\neq 0} E_0(f)$ is in the image of d^0 if it is in the kernel of d^1.Working on the polynomial ring $\kappa(Z)[W]$ and using Gauss lemma, one sees that elements in $\kappa(Z, W)$ can be written as a partial fraction

$$\frac{g_0}{h_0} + \frac{g_1}{h_1 f_1} + \cdots + \frac{g_s}{h_s f_s},$$

where $g_i \in \kappa[Z, W]$, $h_i \in \kappa[Z]$, and $f_i \in \kappa[Z, W]$ is a power of irreducible polynomial. This implies that, if $Z \notin (f)$, elements of $E_0(f)$ can be written as

$$\Omega_f^0(\varphi Z^s f^t), \tag{19}$$

where $\varphi \in \kappa[Z, W]_{(Z,W)}$, $s \in \mathbb{Z}$ and $t \leq -1$. Since

$$d^0\Omega_0^0(\varphi Z^s f^t) = \Omega_f^0(\varphi Z^s f^t) + \Omega_Z^0(\varphi Z^s f^t),$$

and $\bigoplus_{f\neq 0} E_0(f)$ is generated by the image of Ω_f^0 (Definition 3.10), to prove the kernel of d^1 contained in the image of d^0, we may reduce it to the case that an element $\Psi \in E_0(Z)$ with $d^1\Psi = 0$ is in the image of d^0. Working on $\kappa(W)[Z]$ instead of $\kappa(Z)[W]$, we may replace Z by W and choose f to be Z in (19). Multiplying Ψ by an element in $\kappa[Z, W]\backslash(Z, W)$, we may assume

$$\Psi = \sum_{\substack{s\leq 0\\ t\in\mathbb{Z}}} a_{st}\Omega_Z^0(Z^s W^t)$$

for some $a_{st} \in \kappa$ and write the map explicitly

$$\begin{aligned} d^0\left(\sum_{\substack{s\leq 0\\ t>0}} a_{st}\Omega_0^0(Z^s W^t)\right) &= \sum_{\substack{s\leq 0\\ t>0}} a_{st}\Omega_Z^0(Z^s W^t) + \sum_{s\leq 0<t} a_{st}\Omega_W^0(Z^s W^t)\\ &= \sum_{\substack{s\leq 0\\ t>0}} a_{st}\Omega_Z^0(Z^s W^t) = \Psi \end{aligned}$$

since $\Omega_W^0(Z^s W^t) = 0$ for all $t > 0$. □

Note that $E_n^\bullet$ is not exact for $n \geq 1$. For instance,

$$d^1\Omega_Z^n(Z^n W) = 0,$$

but $\Omega_Z^n(Z^n W)$ is not in the image of d^0.

The maps XW and $\bigoplus_{f\neq 0} f^\triangle$ are surjective by (8). Moreover, $E_0(0)$ is in the kernel of the composition $(\bigoplus_{f\neq 0} f^\triangle) \circ d^0$ and $\bigoplus_{f\neq 0} E_0(f)$ is in the kernel of the composition $(X \oplus Y) \circ d^1$. We make the following definition.

Definition 4.10. We define

$$\pi^0\colon E(0) \to \bigoplus_{f\neq 0} E(f)$$

and

$$\pi^{11} \oplus \pi^{12}\colon \bigoplus_{f\neq 0} E(f) \to E(Z, W)^2$$

to be the maps making the following diagram commutative

$$\begin{array}{ccccc} E(0) & \xrightarrow{\pi^0} & \bigoplus_{f\neq 0} E(f) & \xrightarrow{\pi^{11}\oplus\pi^{12}} & E(Z,W)^2 \\ {\scriptstyle XW}\uparrow & & \uparrow{\scriptstyle \oplus f^{\triangle}} & & \uparrow{\scriptstyle X\oplus Y} \\ E(0) & \xrightarrow{d^0} & \bigoplus_{f\neq 0} E(f) & \xrightarrow{d^1} & E(Z,W). \end{array} \tag{20}$$

π^0 and $\pi^{11} \oplus \pi^{12}$ can be described using the maps

$$\pi^0_f : E(0) \to E(f),$$
$$\pi^{11}_f : E(f) \to E(Z, W),$$
$$\pi^{12}_f : E(f) \to E(Z, W),$$

where

$$\begin{cases} \pi^0_Z = d^0_Z \circ \dfrac{1}{Z}, \\ \pi^0_W = d^0_W \circ \dfrac{1}{W}, \\ \pi^0_f = d^0_f, \end{cases} \quad \begin{cases} \pi^{11}_Z = d^1_Z \circ \dfrac{Z}{W}, \\ \pi^{11}_W = d^1_W, \\ \pi^{11}_f = d^1_f \circ \dfrac{1}{W}, \end{cases} \quad \begin{cases} \pi^{12}_Z = d^1_Z, \\ \pi^{12}_W = d^1_W \circ \dfrac{W}{Z}, \\ \pi^{12}_f = d^1_f \circ \dfrac{1}{Z}, \end{cases}$$

for $Z, W \notin (f)$. We have

$$\pi^{11} = \bigoplus_{f\neq 0} \pi^{11}_f \qquad \text{and} \qquad \pi^{12} = \bigoplus_{f\neq 0} \pi^{12}_f.$$

The product $\prod_{f\neq 0} \pi^0_f$: $E(0) \to \prod_{f\neq 0} E(f)$ has image in $\bigoplus_{f\neq 0} E(f)$ and equals π^0.

Definition 4.11. We define

$$E(0) \overset{\delta^0}{\to} \bigoplus_{\mathfrak{q}\neq(Z,W)} E(\mathfrak{q}) \overset{\delta^1}{\to} \bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \overset{\delta^2}{\to} E(Z, W)^2 \tag{21}$$

to be the total complex associated to the double complex (20) with a negative sign on $\bigoplus f^{\triangle}$. For $n \geq 3$, we define

$$\delta^n : E(Z, W)^2 \to E(Z, W)^2$$

to be the map

$$\delta^n(\Psi_1 \oplus \Psi_2) = \begin{cases} (W\Psi_1 - Z\Psi_2) \oplus (-Y\Psi_1 + X\Psi_2), & \text{if } n \text{ is odd;} \\ (X\Psi_1 + Z\Psi_2) \oplus (Y\Psi_1 + W\Psi_2), & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4.12.

$$E(0) \overset{\delta^0}{\to} \bigoplus_{\mathfrak{q}\neq\mathfrak{m}} E(\mathfrak{q}) \overset{\delta^1}{\to} \bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \overset{\delta^2}{\to} E(Z,W)^2 \overset{\delta^3}{\to} E(Z,W)^2 \overset{\delta^4}{\to} E(Z,W)^2 \cdots \tag{22}$$

is a minimal injective resolution of $A/\mathfrak{p}$.

Proof. $\pi^0 \circ (XW) - (\bigoplus f^{\triangle}) \circ d^0 = 0$, by the definition of π^0. This identity, together with Proposition 4.6 (that is, $d^1 \circ d^0 = 0$), implies that $\delta^1 \circ \delta^0 = 0$. The multiplication by XW on $E(0)$ is surjective. Therefore to show $\delta^2 \circ \delta^1 = 0$, we only need to check whether the image of an element of $\bigoplus_{f\neq 0} E(f)$ vanishes. This is easy, since $(X \oplus Y) \circ d^1 - (\pi^{11} \oplus \pi^{12}) \circ (\bigoplus f^{\triangle}) = 0$ by the definition of $\pi^{11} \oplus \pi^{12}$. The map $\bigoplus f^{\triangle}$ is surjective. Therefore to show $\delta^3 \circ \delta^2 = 0$, we only need to compute the image of an element of $E(Z,W)$. This is also easy, since elements of $E(Z,W)$ are annihilated by $XW - YZ$. It is straightforward to show that $\delta^{n+1} \circ \delta^n = 0$ for $n \geq 3$. We conclude that (22) is a complex.

The maps XW and $-(\bigoplus f^{\triangle})$ are surjective. Chasing diagram (20), it is easy to see that (21) is exact because

$$E_0(0) \to \bigoplus_{f\neq 0} E_0(f) \to E_0(Z,W) \to 0$$

is exact. The complex

$$\cdots A^2 \xrightarrow{\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)} A^2 \xrightarrow{\left(\begin{smallmatrix} W & -Y \\ -Z & X \end{smallmatrix}\right)} A^2 \xrightarrow{\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)} A^2 \xrightarrow{\left(\begin{smallmatrix} W & -Y \\ -Z & X \end{smallmatrix}\right)} A^2 \xrightarrow{(X\ \ Y)} A$$

is exact. Apply the functor $\operatorname{Hom}_A(-, E(Z,W))$, we get the exact sequence

$$E(Z,W) \overset{X\oplus Y}{\to} E(Z,W)^2 \overset{\delta^3}{\to} E(Z,W)^2 \overset{\delta^4}{\to} E(Z,W)^2 \overset{\delta^5}{\to} E(Z,W)^2 \to \cdots.$$

Hence

$$\bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \overset{\delta^2}{\to} E(Z,W)^2 \overset{\delta^3}{\to} E(Z,W)^2 \overset{\delta^4}{\to} E(Z,W)^2 \overset{\delta^5}{\to} E(Z,W)^2 \to \cdots \tag{23}$$

is also exact. Combining (21) and (23), we conclude that (22) is exact.

The kernel of δ^0 equals the kernel of d^0 restricting to $E_0(0)$. Let

$$\Psi = \begin{bmatrix} g/h \\ XW, YZ \end{bmatrix}$$

be an element of $E_0(0)$ in the kernel of d^0, where $g \in \kappa[Z,W]$ and $0 \neq h \in \kappa[Z,W]$. Then $g \in hZW\kappa[Z,W]_{(f)}$ for any irreducible polynomial $f \in (Z,W)$. Therefore $g = \varphi hZW$ for some $\varphi \in \kappa[Z,W]_{(Z,W)}$ and

$$\Psi = \begin{bmatrix} \varphi \\ X, Y \end{bmatrix}. \tag{24}$$

All elements of the above form is in the kernel of d^0. These elements form a module isomorphic to $A/\mathfrak{p}$. Therefore (22) is an injective resolution of $A/\mathfrak{p}$.

Let $\Psi = \sum_{i=0}^{n} \Omega_0^i(\varphi_i)$ be a nonzero element of $E(0)$ with $\varphi_n \neq 0$ and $\varphi_n = g/h$ for some $g, h \in \kappa[Z, W]$. By the structure of $E(f)$ discussed in Example 3.6,

$$hWZ(XW)^n\Psi = hWZ\Omega_0^0(\varphi_n) = \begin{bmatrix} g \\ X, Y \end{bmatrix}$$

which is a nonzero element in $E_0(0)$ of the form (24), so it is in the image of $A/\mathfrak{p}$. Hence $E(0)$ is an injective hull of $A/\mathfrak{p}$.

Every nonzero element of $\bigoplus_{\mathfrak{q}\neq(Z,W)} E(\mathfrak{q})$ multiplied by a suitable element of $\kappa[Z, W]$ becomes a nonzero element in the summand $E(0)$ of the form $\sum \Omega_0^i(g_i)$, $g_i \in \kappa[Z, W]$. This element is in the image of δ^0.

Every nonzero element of $\bigoplus_{f\neq 0} E(f)$ multiplied by suitable powers of irreducible polynomials becomes a nonzero element $\Psi_1 = \sum \Omega_f^i(h_i/f^{r_i}) \in E(f)$ for some nonzero $f, h_i \in \kappa[Z, W]$, and $n, r_i \in \mathbb{N}$. By definition,

$$\sum \Omega_f^i\left(\frac{h_i}{f^{r_i}}\right) = d_f^0 \sum \Omega_0^i\left(\frac{h_i}{f^{r_i}}\right) = \pi^0 \sum \Omega_0^i\left(\frac{h_i}{f^{r_i}}\right)$$

if $Z, W \notin (f)$ and

$$\sum \Omega_f^i\left(\frac{h_i}{f^{r_i}}\right) = d_f^0 \circ \frac{1}{f}\left(f \sum \Omega_0^i\left(\frac{h_i}{f^{r_i}}\right)\right) = \pi^0 \sum f\Omega_0^i\left(\frac{h_i}{f^{r_i}}\right)$$

if $f = Z$ or W. So $(\Psi_1, 0)$ is in the image of δ^1. Every nonzero element of $E(Z, W)$ multiplied by suitable powers of X and Y becomes a nonzero element $\Psi_2 \in E_0(Z, W)$ (see Example 3.9). By Lemma 4.8 and the definition of $f^{\triangle}$, $(0, \Psi_2)$ is in the image of δ^1. Now for a general case, multiplied by suitable powers of irreducible polynomials in $\kappa[Z, W]$, and those of X and Y, a nonzero element of $(\bigoplus_{f\neq 0} E(f)) \oplus E(Z, W)$ becomes a nonzero element (Ψ_1, Ψ_2) with Ψ_1, Ψ_2 as described in the above and therefore, it is in the image of δ^1.

Every nonzero element of $E(Z, W)^2$ multiplied by suitable powers of X and Y becomes a nonzero element of $E_0(Z, W)^2$. Multiplied again by suitable powers of Z and W, this element becomes a nonzero element of the form

$$\alpha\Omega^0(1) \oplus \beta\Omega^0(1) \qquad (\alpha, \beta \in \kappa),$$

which is in the image of δ^n $(n \geq 2)$, since

$$\begin{cases} \pi^{11}(\alpha\Omega_W^0(1) \oplus -\beta\Omega_Z^0(1)) = \alpha\Omega^0(1) \\ \pi^{12}(\alpha\Omega_W^0(1) \oplus -\beta\Omega_Z^0(1)) = \beta\Omega^0(1) \end{cases}$$

$$\begin{cases} \alpha X\Omega^1(W) + \beta Z\Omega^0(W^{-1}) = \alpha\Omega^0(1) \\ \alpha Y\Omega^1(W) + \beta W\Omega^0(W^{-1}) = \beta\Omega^0(1) \end{cases}$$

$$\begin{cases} \alpha W\Omega^0(W^{-1}) - \beta Z\Omega^1(W) = \alpha\Omega^0(1) \\ -\alpha Y\Omega^0(W^{-1}) + \beta X\Omega^1(W) = \beta\Omega^0(1). \end{cases}$$

Therefore the resolution is minimal. □

Corollary 4.13. *The Bass numbers of $A/\mathfrak{p}$ are as follows. Let f be an irreducible polynomial contained in* (Z, W).

$$\mu_i((X, Y, Z, W), A/\mathfrak{p}) = \begin{cases} 0, & \text{if } i < 2; \\ 1, & \text{if } i = 2; \\ 2, & \text{if } i > 2. \end{cases}$$

$$\mu_i((X, Y), A/\mathfrak{p}) = \begin{cases} 1, & \text{if } i < 2; \\ 0, & \text{if } i \geq 2. \end{cases}$$

$$\mu_i((X, Y, f), A/\mathfrak{p}) = \begin{cases} 0, & \text{if } i = 0; \\ 1, & \text{if } i = 1 \text{ or } 2; \\ 0, & \text{if } i > 2. \end{cases}$$

All other Bass numbers of $A/\mathfrak{p}$ are zero.

Minimal injective resolutions of the above $A/\mathfrak{p}$ are eventually periodic of period 2. In a private communication with K. Yanagawa, we learn that this is true in a general setting: Over a local ring R which is a hypersurface with an isolated singularity, minimal injective resolutions of any finitely generated modules are eventually periodic. His proof uses Matlis duality and a result of Eisenbud (1980). We sketch the proof as follows. Denote the maximal ideal and the residue field of R by $\mathfrak{m}$ and κ respectively. Let M be a finitely generated R-module and E an injective hull of κ. Since R has only isolated singularity, $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) = 0$ for any prime ideal $\mathfrak{p}$ of R not equal to $\mathfrak{m}$ and all i less than n, the dimension of R. Therefore, M has a minimal resolution in form of

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to E^{\mu_n(\mathfrak{m},M)} \xrightarrow{d^n} E^{\mu_{n+1}(\mathfrak{m},M)} \to \cdots. \tag{25}$$

Let K be the kernel of d^n. The dual of K, $\mathrm{Hom}_{\hat{R}}(K, E)$, is a Noetherian $\hat{R}$-module. Its minimal free resolution

$$\cdots \to \hat{R}^{\mu_n(\mathfrak{m},M)} \to \hat{R}^{\mu_n(\mathfrak{m},M)} \to \mathrm{Hom}_{\hat{R}}(K, E) \to 0$$

is eventually periodic by Theorem 6.1 in Eisenbud (1980) and so is the injective resolution (25) of M.

5. APPLICATIONS

5.1. Local Cohomology

We compute the local cohomology module $\mathrm{H}^i_I(A/\mathfrak{p})$ of $A/\mathfrak{p}$ supported at an ideal I of A. Recall that the injective resolution (22) of $A/\mathfrak{p}$ is built up by injective hulls $E(\mathfrak{q})$ of modules $A/(\mathfrak{q}, X, Y)$, where $\mathfrak{q}$ is a prime ideal of $\kappa[Z, W]$. Since elements in $E(\mathfrak{q})$ are annihilated by powers of X and Y, the functors Γ_I and $\Gamma_{I+(X,Y)}$ have the same effect on the complex (22). Hence $\mathrm{H}^i_I(A/\mathfrak{p}) = \mathrm{H}^i_{I+(X,Y)}(A/\mathfrak{p})$ for all i and, to compute the local cohomology modules, we may assume that $I = (I_0, X, Y)$ for some ideal I_0 of $\kappa[Z, W]$. If $I_0 \subset \mathfrak{q}$, then $E(\mathfrak{q})$ being $\mathfrak{q}$-torsion is also I-torsion. If $I_0 \not\subset \mathfrak{q}$, there is an element $a \in I_0 \backslash \mathfrak{q}$. The only element of $E(\mathfrak{q})$ annihilated by powers of a is zero, so $E(\mathfrak{q})$ is I-torsion free in this case. Therefore, applying the I-torsion functor $\Gamma_I(-)$ simply means taking away those $E(\mathfrak{q})$ with $I_0 \not\subset \mathfrak{q}$ from the complex (22).

If $\mathrm{ht}(\kappa[Z, W] \cap I) = 2$, (X, Y, Z, W) is the only prime containing I. Apply the functor $\Gamma_I(-)$ to the injective resolution (22) of $A/\mathfrak{p}$, we get the complex

$$0 \to 0 \to E(Z, W) \xrightarrow{X \oplus Y} E(Z, W)^2 \xrightarrow{\delta^3} E(Z, W)^2 \xrightarrow{\delta^4} E(Z, W)^2 \to \cdots$$

whose only nontrivial cohomology is $E_0(Z, W)$. Therefore

$$\mathrm{H}^i_I\left(\frac{A}{\mathfrak{p}}\right) = \begin{cases} E_0(Z, W), & \text{if } i = 2; \\ 0, & \text{if } i \neq 2. \end{cases} \tag{26}$$

As a κ-vector space, $\mathrm{H}^2_I(A/\mathfrak{p})$ has a basis consisting of $\Omega^0(Z^s W^t)$, where $s, t \leq 0$.

A local cohomology module is a direct limit of extension modules. Using the injective resolution (22), we can see clearly the behavior of the limit

$$\lim_{n\to\infty} \mathrm{Ext}^2_A\left(\frac{A}{(X, Y, Z, W)^n}, \frac{A}{\mathfrak{p}}\right) = \mathrm{H}^2_{(X,Y,Z,W)}\left(\frac{A}{\mathfrak{p}}\right).$$

Apply the functor $\mathrm{Hom}_A(A/(X, Y, Z, W)^n, -)$ to (22), we get

$$0 \to 0 \to \mathrm{Hom}_A\left(\frac{A}{(X, Y, Z, W)^n}, E(Z, W)\right) \xrightarrow{\binom{X}{Y}} \mathrm{Hom}_A\left(\frac{A}{(X, Y, Z, W)^n}, E(Z, W)^2\right) \to \cdots.$$

$\mathrm{Ext}^2_A(A/(X, Y, Z, W)^n, A/\mathfrak{p})$ is isomorphic to the submodule of $E(Z, W)$ consisting of those elements annihilated by X, Y and $(X, Y, Z, W)^n$. As a κ-vector space, $\mathrm{Ext}^2_A(A/(X, Y, Z, W)^n, A/\mathfrak{p})$ has a basis consisting of $\Omega^0(Z^s W^t)$, where $s, t \leq 0$ satisfy $s + t + n > 0$. In particular,

$$\dim_\kappa \mathrm{Ext}^2\left(\frac{A}{(X, Y, Z, W)^n}, \frac{A}{\mathfrak{p}}\right) = \frac{n(n+1)}{2}.$$

As n increasing, the set $\{\Omega^0(Z^s W^t) | s, t \leq 0, s + t + n > 0\}$ becomes larger and closer to the basis $\{\Omega^0(Z^s W^t) | s, t \leq 0\}$ of $\mathrm{H}^2_{(X,Y,Z,W)}(A/\mathfrak{p})$.

If $\mathrm{ht}(\kappa[Z, W] \cap I) = 0$, then $I = (X, Y)$. The functor $\Gamma_I(-)$ does not change the complex (22). Therefore

$$\mathrm{H}^i_{(X,Y)}\left(\frac{A}{\mathfrak{p}}\right) = \begin{cases} A/\mathfrak{p}, & \text{if } i = 0; \\ 0, & \text{if } i \neq 0. \end{cases} \tag{27}$$

Now we look at the case $\mathrm{ht}(\kappa[Z, W] \cap I) = 1$. Applying $\Gamma_I(-)$ to (22), we get a complex quasi-isomorphic to

$$0 \to \bigoplus_{I \subset (f, X, Y)} E_0(f) \to E_0(Z, W) \to 0 \to \cdots .$$

By Lemma 4.8, the nontrivial map in the above complex is surjective. Therefore

$$\mathrm{H}^i_I\left(\frac{A}{\mathfrak{p}}\right) = \begin{cases} \text{kernel of } \bigoplus_{I \subset (f,X,Y)} E_0(f) \to E_0(Z, W), & \text{if } i = 1; \\ 0, & \text{if } i \neq 1. \end{cases} \tag{28}$$

For instance, if $I = (Z, X, Y)$, the above complex becomes

$$\cdots \to 0 \to E_0(Z) \to E_0(Z, W) \to 0 \to \cdots .$$

As a $\kappa[W]_{(W)}$-module, $\mathrm{H}^1_{(Z,X,Y)}(A/\mathfrak{p})$ is generated freely by $\Omega^0_Z(Z^s W)$, where $s \leq 0$.

5.2. Normal Module

We would like to make explicit the canonical isomorphism

$$\mathrm{Hom}_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{p}) \to \mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p}) \tag{29}$$

in terms of the injective resolution (22) of $A/\mathfrak{p}$. Note that the canonical map

$$\mathrm{Hom}_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{p}) \to \mathrm{Hom}_A(\mathfrak{p}, A/\mathfrak{p})$$

is an isomorphism. We describe an isomorphism between $\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p})$ and $\mathrm{Hom}_A(\mathfrak{p}, A/\mathfrak{p})$ to establish (29).

We compute $\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p})$ by applying the functor $\mathrm{Hom}_A(A/\mathfrak{p}, -)$ to (22). By Proposition 3.12, $\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p})$ is the cohomology of the complex

$$E_0(0) \xrightarrow{\delta^0} \bigoplus_{\mathfrak{q} \neq (Z,W)} E_0(\mathfrak{q}) \xrightarrow{\delta^1} \bigoplus_{\mathfrak{q} \neq (0)} E_0(\mathfrak{q}).$$

By Lemma 4.9, it is also the kernel of

$$E_0(0) \xrightarrow{\pi^0} \bigoplus_{f \neq 0} E_0(f).$$

Explicitly,

$$\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p}) \simeq \{\Omega^0_0(gZ^2W^2) | g \in \kappa[Z, W]_{(Z,W)}\} \subset E_0(0).$$

Consider the diagram

$$
\begin{array}{ccccc}
 & & & & \mathrm{Hom}(\mathfrak{p}, A/\mathfrak{p}) \\
 & & & & \downarrow \\
E_0(0) & \longrightarrow & E(0) & \longrightarrow & \mathrm{Hom}(\mathfrak{p}, E(0)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{\mathfrak{q}\neq(Z,W)} E_0(\mathfrak{q}) & \longrightarrow & \bigoplus_{\mathfrak{q}\neq(Z,W)} E(\mathfrak{q}) & \longrightarrow & \mathrm{Hom}(\mathfrak{p}, \bigoplus_{\mathfrak{q}\neq(Z,W)} E(\mathfrak{q})) \\
\downarrow & & \vdots & & \vdots \\
\bigoplus_{\mathfrak{q}\neq(0)} E_0(\mathfrak{q}) & & & & \\
\vdots & & & &
\end{array}
$$

obtained by applying the Hom functors on the short exact sequence

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$$

and using (22) to establish the vertical maps. Chasing the above diagram, we get an isomorphism

$$\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p}) \to \mathrm{Hom}_A(\mathfrak{p}, A/\mathfrak{p}),$$

which maps $\Omega^0_0(gZ^2W^2)$ to the A-linear map $\mathfrak{p} \to A/\mathfrak{p}$ determined by

$$X \mapsto gZ \qquad \text{and} \qquad Y \mapsto gW$$

for $g \in \kappa[Z, W]_{(Z,W)}$.

5.3. Yoneda Algebra

First we compute $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p}) = \sum_{i=0}^{\infty} \mathrm{Ext}^i_A(A/\mathfrak{p}, A/\mathfrak{p})$. In Subsection 5.2, we have seen that

$$\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p}) \simeq \left\{\Omega^0_0\big(gZ^2W^2\big) \mid g \in \kappa[Z, W]_{(Z,W)}\right\}.$$

Let

$$
\begin{aligned}
e_0 &= \Omega^0_0(ZW) \in E(0),\\
e_1 &= \Omega^0_0(Z^2W^2) \in \bigoplus_{\mathfrak{q}\neq\mathfrak{m}} E(\mathfrak{q}),\\
e_2 &= \Omega^0_W(Z) \in \bigoplus_{\mathfrak{q}\neq 0} E(\mathfrak{q}),\\
e_{2n} &= 0 \oplus \Omega^0(1) \text{ in the } 2n\text{th term } E(Z, W)^2 \text{ of (22)},
\end{aligned}
$$

where $n > 1$, be (the equivalence classes of) the cycles of the complex (22). It should be pointed out that all the above e_i represent nontrivial cohomology classes in $\mathrm{Ext}^i_A(A/\mathfrak{p}, A/\mathfrak{p})$. For each e_j, we define

$$\iota_j \colon A/\mathfrak{p} \to \text{ the } j\text{th term of (22)}$$

to be the map sending 1 to e_j, in which ι_0 is the embedding making (22) in Theorem 4.12 an injective resolution of $A/\mathfrak{p}$.

The following Lemma 5.1 describes $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$ as an A-module using independent generators e_i. Later in Proposition 5.3, we will present $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$ as an A-algebra.

Lemma 5.1. *As an A-module, the Yoneda algebra* $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$ *is generated by* $e_0, e_1, e_2, e_4, e_6, \cdots$. *The annihilators of* e_0 *and* e_1 *are* $\mathfrak{p}$; *for* $i > 0$, *the annihilator of* e_{2i} *is* $\mathfrak{p} + AZ + AW$.

Proof. It is clear that $\mathrm{Ext}^0_A(A/\mathfrak{p}, A/\mathfrak{p})$ is generated by e_0, whose annihilator is $\mathfrak{p}$. The module $\mathrm{Ext}^1_A(A/\mathfrak{p}, A/\mathfrak{p})$ has been treated in Subsection 5.2.

As seen in Subsection 5.2, for all $i \geq 2$, $\mathrm{Ext}^i_A(A/\mathfrak{p}, A/\mathfrak{p})$ is a cohomology module of the complex

$$E_0(0) \xrightarrow{\pi^0} \bigoplus_{f \neq 0} E_0(f) \xrightarrow{\pi^{11} \oplus \pi^{12}} E_0(Z, W)^2 \xrightarrow{\delta^3} E_0(Z, W)^2 \to \cdots .$$

For $n \geq 2$, $\mathrm{Ext}^{2n+1}_A(A/\mathfrak{p}, A/\mathfrak{p}) = 0$, since the complex

$$E_0(Z, W)^2 \xrightarrow{\left(\begin{smallmatrix} 0 & Z \\ 0 & W \end{smallmatrix}\right)} E_0(Z, W)^2 \xrightarrow{\left(\begin{smallmatrix} W & -Z \\ 0 & 0 \end{smallmatrix}\right)} E_0(Z, W)^2$$

is exact. The exactness of the above sequence means that the kernel of $\left(\begin{smallmatrix} W & -Z \\ 0 & 0 \end{smallmatrix}\right)$ consists of elements of the form $Z\Psi \oplus W\Psi$, where $\Psi \in E_0(Z, W)$. By Lemma 4.8, these elements are in the image of $W\left(\pi^{11}_Z \oplus \pi^{12}_Z\right) = (Zd^1_Z) \oplus (Wd^1_Z)$. Therefore,

$$E_0(Z) \xrightarrow{\pi^{11}_Z \oplus \pi^{12}_Z} E_0(Z, W)^2 \xrightarrow{\left(\begin{smallmatrix} W & -Z \\ 0 & 0 \end{smallmatrix}\right)} E_0(Z, W)^2$$

is exact and $\mathrm{Ext}^3_A(A/\mathfrak{p}, A/\mathfrak{p}) = 0$ as well.

The cohomology of the complex

$$E_0(Z, W)^2 \xrightarrow{\left(\begin{smallmatrix} W & -Z \\ 0 & 0 \end{smallmatrix}\right)} E_0(Z, W)^2 \xrightarrow{\left(\begin{smallmatrix} 0 & Z \\ 0 & W \end{smallmatrix}\right)} E_0(Z, W)^2$$

is generated by the element in $\mathrm{Ext}^{2n}_A(A/\mathfrak{p}, A/\mathfrak{p})$ represented by $0 \oplus \Omega^0(1)$. Therefore $\mathrm{Ext}^{2n}_A(A/\mathfrak{p}, A/\mathfrak{p}) = Ae_{2n}$ for $n \geq 2$. The element e_{2n} is nonzero and annihilated by Z, W and $\mathfrak{p}$. Its annihilator is hence $\mathfrak{p} + AZ + AW$.

As seen in the proof of Lemma 4.9, if $Z, W \notin (f)$, elements of $E_0(f)$ can be written as

$$\Omega^0_f(\varphi Z^s f^t),$$

where $\varphi \in \kappa[Z, W]_{(Z,W)}$, $s \in \mathbb{Z}$ and $t \leq -1$. Since

$$\Omega^0_f(\varphi Z^s f^t) = \pi^0\Omega^0_0(\varphi Z^s f^t) - \Omega^0_Z(\varphi Z^{s-1} f^t) - \Omega^0_W(\varphi Z^s f^t W^{-1}),$$

to compute $\mathrm{Ext}^2_A(A/\mathfrak{p}, A/\mathfrak{p})$, we may restrict $\pi^{11} \oplus \pi^{12}$ to $E_0(Z) \oplus E_0(W)$. Multiplied by an element in $\kappa[Z, W]\backslash(Z, W)$, an element in $E_0(Z) \oplus E_0(W)$ can be written as the form

$$\sum_{\substack{s\leq 0\\ t\in\mathbb{Z}}} a_{st}\Omega^0_Z(Z^s W^t) + \sum_{\substack{s\in\mathbb{Z}\\ t\leq 0}} b_{st}\Omega^0_W(Z^s W^t).$$

Since

$$\Omega^0_Z(Z^s W^t) = \pi^0\Omega^0_0(Z^{s+1} W^t) - \Omega^0_W(Z^{s+1} W^{t-1})$$

and

$$\Omega^0_W(Z^s W^t) = \pi^0\Omega^0_0(Z^s W^{t+1}) \qquad \text{for } s > 1,$$

to compute $\mathrm{Ext}^2_A(A/\mathfrak{p}, A/\mathfrak{p})$, we may work on elements of the form

$$\sum_{\substack{s\leq 1\\ t\leq 0}} b_{st}\Omega^0_W(Z^s W^t)$$

and assume that it is in the kernel of $\pi^{11} \oplus \pi^{12}$.

$$\pi^{11}\left(\sum_{\substack{s\leq 1\\ t\leq 0}} b_{st}\Omega^0_W(Z^s W^t)\right) = 0$$

implies $b_{st} = 0$ for all $s, t \leq 0$. Furthermore,

$$\pi^{12}\left(\sum_{t\leq 0} b_{1t}\Omega^0_W(Z W^t)\right) = 0$$

implies $b_{1t} = 0$ for all $t \leq -1$. Therefore, b_{10} is the only possible nonzero coefficient and $\mathrm{Ext}^2_A(A/\mathfrak{p}, A/\mathfrak{p}) = Ae_2$. Clearly, e_2 is annihilated by W and $\mathfrak{p}$. Since $Ze_2 = \pi^0\Omega^0_0(Z^2 W)$, it is also annihilated by Z. Finally, e_2 is nonzero, so its annihilator is $\mathfrak{p} + AW + AZ$. $\square$

Now we compute the Yoneda pairing

$$\mathrm{Ext}^i_A(A/\mathfrak{p}, A/\mathfrak{p}) \times \mathrm{Ext}^j_A(A/\mathfrak{p}, A/\mathfrak{p}) \to \mathrm{Ext}^{i+j}_A(A/\mathfrak{p}, A/\mathfrak{p}).$$

Since the pairing is A-bilinear, we only need to compute $e_i \times e_j$.

Lemma 5.2.

$$e_i \times e_j = \begin{cases} e_i, & \textit{if } j = 0; \\ e_j, & \textit{if } i = 0; \\ 0, & \textit{if } i = 1 \textit{ or } j = 1, \textit{ but } ij \neq 0; \\ -e_{i+j}, & \textit{if } ij \neq 0 \textit{ and } i, j \textit{ are both even.} \end{cases}$$

Proof. To compute $e_2 \times e_2$, we need to construct a commutative diagram

$$\begin{tikzcd} A/\mathfrak{p} \arrow[r, "\iota_0"] \arrow[dr, "\iota_2"'] & E(0) \arrow[r, "\delta^0"] \arrow[d] & \bigoplus_{\mathfrak{q}\neq\mathfrak{m}} E(\mathfrak{q}) \arrow[r, "\delta^1"] \arrow[d] & \bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \arrow[d] \\ & \bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \arrow[r, "\delta^2"'] & E(Z,W)^2 \arrow[r, "\delta^3"'] & E(Z,W)^2. \end{tikzcd} \tag{30}$$

It is straightforward to check that the diagrams

$$\begin{tikzcd} A/\mathfrak{p} \arrow[r, "\iota_0"] \arrow[dr, "\iota_2"'] & E(0) \arrow[r, "d^0"] \arrow[d, "d^0_W \frac{1}{W}"] & (\bigoplus_{f\neq 0,Z,W} E(f)) \oplus E(Z) \oplus E(W) \arrow[d, "\mathcal{M}_{23}"] \\ & E(W) \arrow[r, "\begin{pmatrix} d^1_W \\ d^1_W \frac{W}{Z} \end{pmatrix}"'] & E(Z,W)^2 \end{tikzcd}$$

and

$$\begin{tikzcd} \left(\bigoplus_{f\neq 0,Z,W} E(f)\right) \oplus E(Z) \oplus E(W) \arrow[r, "\mathscr{M}_{43}"] \arrow[d, "\mathscr{M}_{23}"'] & E(Z,W) \oplus \left(\bigoplus_{f\neq 0,Z,W} E(f)\right) \oplus E(Z) \oplus E(W) \arrow[d, "\mathscr{M}_{24}"] \\ E(Z,W)^2 \arrow[r, "\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}"'] & E(Z,W)^2 \end{tikzcd}$$

are commutative, where

$$\mathscr{M}_{23} = \begin{pmatrix} -\bigoplus_{f\neq 0,Z,W} d^1_f \dfrac{1}{W} & -d^1_Z \dfrac{1}{W} & 0 \\ 0 & 0 & d^1_W \dfrac{1}{Z} \end{pmatrix},$$

$$\mathscr{M}_{24} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\bigoplus_{f\neq 0,Z,W} d^1_f \dfrac{1}{ZW} & -d^1_Z \dfrac{1}{W} & -d^1_W \dfrac{1}{Z} \end{pmatrix},$$

$$\mathscr{M}_{43} = \begin{pmatrix} \bigoplus_{f\neq 0,Z,W} d^1_f & d^1_Z & d^1_W \\ -\bigoplus_{f\neq 0,Z,W} f^\Delta & 0 & 0 \\ 0 & -Z^\Delta & 0 \\ 0 & 0 & -W^\Delta \end{pmatrix}.$$

We define the vertical maps in (30) to be those in the above and zero maps if a certain component is not included above. The product $e_2 \times e_2$ is the image of e_2 under the map $\mathscr{M}_{24}$, which equals $-e_4$.

We use the same method to compute other $e_i \times e_j$. For $i > 1$, the diagram

$$\begin{tikzcd}
A/\mathfrak{p} \arrow[r, "\iota_0"] \arrow[dr, "\iota_{2i}"'] & E(0) \arrow[r, "d^0"] \arrow[d, "\begin{pmatrix} 0 \\ d^1_W d^0_W \frac{1}{ZW} \end{pmatrix}"] & \bigoplus_{f\neq 0} E(f) \arrow[r, "\mathcal{M}_{43}"] \arrow[d, "\mathcal{M}_{23}"] & \oplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \arrow[d, "\mathcal{M}_{24}"] \\
& E(Z,W)^2 \arrow[r, "\begin{pmatrix} X & Z \\ Y & W \end{pmatrix}"'] & E(Z,W)^2 \arrow[r, "\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}"'] & E(Z,W)^2
\end{tikzcd}$$

commutes. Therefore $e_2 \times e_{2i} = -e_{2i+2}$ for $i > 1$. The diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak{q}\neq(0)} E(\mathfrak{q}) \arrow[r, "\delta^2"] \arrow[d, "\mathscr{M}_{24}"'] & E(Z,W)^2 \arrow[r, "\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}"] & E(Z,W)^2 \cdots \arrow[d, "\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}"] \\
E(Z,W)^2 \arrow[r, "\begin{pmatrix} X & Z \\ Y & W \end{pmatrix}"'] & E(Z,W)^2 \arrow[r, "\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}"'] & E(Z,W)^2 \cdots
\end{tikzcd}$$

commutes. Therefore $e_{2j} \times e_{2i} = -e_{2i+2j}$ for $i \geq 1$ and $j > 1$. The diagram

$$\begin{tikzcd}
A/\mathfrak{p} \arrow[r, "\iota_0"] \arrow[dr, "\iota_1"'] & E(0) \arrow[r, "d^0"] \arrow[d, "ZW"] & \left(\bigoplus_{f\neq 0,Z,W} E(f)\right) \oplus E(Z) \oplus E(W) \arrow[d, "\begin{pmatrix} ZW & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & Z \end{pmatrix}"] \\
& E(0) \arrow[r, "\pi^0"'] & \left(\bigoplus_{f\neq 0,Z,W} E(f)\right) \oplus E(Z) \oplus E(W)
\end{tikzcd}$$

commutes and ι_1 has image in $E(0)$. Therefore $e_1 \times e_1 = 0$.

$\mathrm{Ext}^{2i+1}_A(A/\mathfrak{p}, A/\mathfrak{p}) = 0$ for $i \geq 1$. Therefore $e_1 \times e_{2i} = e_{2i} \times e_1 = 0$ for $i \geq 1$. It is easy to see that $e_0 \times e_i = e_i \times e_0 = e_i$. □

Proposition 5.3. *The Yoneda algebra* $\mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$ *is isomorphic to the polynomial ring* $A/\mathfrak{p}[U, V]$ *modulo the ideal generated by* ZV, WV, U^2, *and* UV.

Proof. All e_i are annihilated by $\mathfrak{p}$. So there is an A-algebra homomorphism

$$A/\mathfrak{p}[U, V] \to \mathrm{Ext}^*_A(A/\mathfrak{p}, A/\mathfrak{p})$$

given by $1 \mapsto e_0$, $U \mapsto e_1$ and $V \mapsto e_2$. Since $\mathrm{Ext}_A^*(A/\mathfrak{p}, A/\mathfrak{p})$ is generated by e_i and $(-1)^{n+1}e_{2n} = e_2^n$ (the Yoneda product of n copies of e_2), the homomorphism is surjective. By Lemmas 5.1 and 5.2, the kernel of the homomorphism is generated by ZV, WV, U^2, and UV. □

Corollary 5.4. *The Yoneda algebra* $\mathrm{Ext}_A^*(A/\mathfrak{p}, A/\mathfrak{p})$ *is commutative and finitely generated.*

5.4. Dutta, Hochster, and McLaughlin's Module

We recall the definition of the module M given by Dutta et al. (1985) (see also Roberts, 1998, 13.2). As a κ-vector space, it is 15-dimensional:

$$M = (\kappa u_1 + \cdots + \kappa u_5) + (\kappa v_1 + \cdots + \kappa v_4) + (\kappa w_1 + \cdots + \kappa w_6).$$

Its module structure is given by

$$\begin{array}{llll}
Xu_i = Yu_i = Zu_i = Wu_i = 0 & (i = 1, \ldots, 5) & & \\
Xv_1 = u_1 & Yv_1 = 0 & Zv_1 = 0 & Wv_1 = 0 \\
Xv_2 = u_2 & Yv_2 = 0 & Zv_2 = 0 & Wv_2 = 0 \\
Xv_3 = 0 & Yv_3 = 0 & Zv_3 = u_1 & Wv_3 = 0 \\
Xv_4 = 0 & Yv_4 = 0 & Zv_4 = u_2 & Wv_4 = 0 \\
Xw_1 = v_1 & Yw_1 = u_3 & Zw_1 = 0 & Ww_1 = u_1 \\
Xw_2 = v_2 & Yw_2 = u_4 & Zw_2 = 0 & Ww_2 = u_2 \\
Xw_3 = v_3 & Yw_3 = u_5 & Zw_3 = v_1 & Ww_3 = 0 \\
Xw_4 = v_4 & Yw_4 = 0 & Zw_4 = v_2 & Ww_4 = u_3 \\
Xw_5 = u_4 & Yw_5 = 0 & Zw_5 = v_3 & Ww_5 = u_4 \\
Xw_6 = u_5 & Yw_6 = 0 & Zw_6 = u_3 + v_4 & Ww_6 = u_5.
\end{array}$$

Note that all monomials of degree greater than one act on the basis u_i, v_j, w_k trivially except the following cases:

$$\begin{array}{lll}
X^2w_1 = u_1 & XZw_3 = u_1 & Z^2w_5 = u_1 \\
X^2w_2 = u_2 & XZw_4 = u_2 & Z^2w_6 = u_2
\end{array}$$

An A-linear homomorphism Φ from M to an A-module N is determined by its values at $w_1, \ldots, w_6$ and satisfies the conditions

$$\begin{aligned}
&Z\Phi(w_1) = Z\Phi(w_2) = W\Phi(w_3) = Y\Phi(w_4) = Y\Phi(w_5) = Y\Phi(w_6) = 0 \\
&X\Phi(w_1) = Z\Phi(w_3) \\
&X\Phi(w_2) = Z\Phi(w_4) \\
&X\Phi(w_3) = Z\Phi(w_5)
\end{aligned}$$

$$
\begin{aligned}
Y\Phi(w_1) &= W\Phi(w_4)\\
X\Phi(w_5) &= Y\Phi(w_2) = W\Phi(w_5)\\
X\Phi(w_6) &= Y\Phi(w_3) = W\Phi(w_6)\\
Z\Phi(w_6) &= Y\Phi(w_1) + X\Phi(w_4)\\
W\Phi(w_1) &= X^2\Phi(w_1) = XZ\Phi(w_3) = Z^2\Phi(w_5)\\
W\Phi(w_2) &= X^2\Phi(w_2) = XZ\Phi(w_4) = Z^2\Phi(w_6)
\end{aligned}
$$

and the condition that all monomials of degree greater than one act trivially on $\Phi(w_i)$ except $X^2\Phi(w_1)$, $XZ\Phi(w_3)$, $Z^2\Phi(w_5)$, $X^2\Phi(w_2)$, $XZ\Phi(w_4)$, $Z^2\Phi(w_6)$. Any six elements $\Phi(w_1), \ldots, \Phi(w_6) \in N$ satisfying the above conditions extend uniquely to an A-linear map Φ: $M \to N$. Note that some of these conditions are redundant.

Lemma 5.5. $\mathrm{Hom}_A(M, E(f)) = 0$.

Proof. Let $\Phi \in \mathrm{Hom}_A(M, E(f))$. If $Z \notin (f)$, multiplication by Z is bijective. Thus

$$
\begin{aligned}
Z\Phi(\omega_1) = Z\Phi(\omega_2) = 0 &\implies \Phi(\omega_1) = \Phi(\omega_2) = 0\\
Z\Phi(\omega_3) = X\Phi(\omega_1) = 0 &\implies \Phi(\omega_3) = 0\\
Z\Phi(\omega_4) = X\Phi(\omega_2) = 0 &\implies \Phi(\omega_4) = 0\\
Z\Phi(\omega_5) = X\Phi(\omega_3) = 0 &\implies \Phi(\omega_5) = 0\\
Z^2\Phi(\omega_6) = W\Phi(\omega_2) = 0 &\implies \Phi(\omega_6) = 0
\end{aligned}
$$

If $f = Z$, multiplication by W is bijective. Thus

$$
\begin{aligned}
W\Phi(\omega_3) = 0 &\implies \Phi(\omega_3) = 0\\
W\Phi(\omega_6) = Y\Phi(\omega_3) = 0 &\implies \Phi(\omega_6) = 0\\
W\Phi(\omega_2) = Z^2\Phi(\omega_6) = 0 &\implies \Phi(\omega_2) = 0\\
W\Phi(\omega_5) = Y\Phi(\omega_2) = 0 &\implies \Phi(\omega_5) = 0\\
W\Phi(\omega_1) = Z^2\Phi(\omega_5) = 0 &\implies \Phi(\omega_1) = 0\\
W\Phi(\omega_4) = Y\Phi(\omega_1) = 0 &\implies \Phi(\omega_4) = 0
\end{aligned}
$$

In either case, $\Phi = 0$. □

Now we compute $M' := \mathrm{Hom}_A(M, E(Z, W))$. For $1 \le i, j \le 6$, let

$$\Phi_i(w_j) := \delta_{ij}\Omega^0(1).$$

Furthermore, for $1 \le j \le 6$, let

$$
\begin{aligned}
\Phi_{13}(\omega_j) &:= X^{-1}\Phi_1(\omega_j) + Z^{-1}\Phi_3(\omega_j),\\
\Phi_{24}(\omega_j) &:= X^{-1}\Phi_2(\omega_j) + Z^{-1}\Phi_4(\omega_j),
\end{aligned}
$$

$$\Phi_{35}(\omega_j) := X^{-1}\Phi_3(\omega_j) + Z^{-1}\Phi_5(\omega_j),$$
$$\Phi_{46}(\omega_j) := X^{-1}\Phi_4(\omega_j) + Z^{-1}\Phi_6(\omega_j),$$
$$\Phi_{14}(\omega_j) := Y^{-1}\Phi_1(\omega_j) + (W^{-1} - X^{-1})\Phi_4(\omega_j),$$
$$\Phi_{25}(\omega_j) := Y^{-1}\Phi_2(\omega_j) + (W^{-1} + X^{-1})\Phi_5(\omega_j),$$
$$\Phi_{36}(\omega_j) := Y^{-1}\Phi_3(\omega_j) + (W^{-1} + X^{-1})\Phi_6(\omega_j),$$
$$\Phi_{135}(\omega_j) := (W^{-1} + X^{-2})\Phi_1(\omega_j) + Z^{-1}X^{-1}\Phi_3(\omega_j) + Z^{-2}\Phi_5(\omega_j),$$
$$\Phi_{246}(\omega_j) := (W^{-1} + X^{-2})\Phi_2(\omega_j) + Z^{-1}X^{-1}\Phi_4(\omega_j) + Z^{-2}\Phi_6(\omega_j).$$

For $1 \leq i \leq 6$, it is straightforward to check that $\Phi_i(w_j)$ satisfy the conditions prior to Lemma 5.5. For $i \in \{13, 24, 35, 46, 14, 25, 36, 135, 246\}$, we use the following facts to check these conditions for $\Phi_i(w_j)$:

- Divisions or multiplications by powers of different variables on $E(Z, W)$ are commutative. For example, $X^i Y^j \Psi = Y^j X^i \Psi$ for $i, j \in \mathbb{Z}$;
- The divisions by a power of a single variable satisfy the properties: $X^i X^{-j}\Psi = X^{i-j}\Psi$ for $i, j > 0$ and the same for Y, Z and W. However, $X^{-1}X\Psi \neq \Psi$ in general.

All these Φ_i extend to well-defined elements in M'.

Lemma 5.6.

$$\{\Phi_i \mid i \in \{1, 2, 3, 4, 5, 6, 13, 24, 35, 46, 14, 25, 36, 135, 246\}\}$$

is a basis for the κ-vector space M'.

Proof. Clearly, Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 are linearly independent. Assume that

$$\Phi = \sum_i a_i \Phi_i = 0 \qquad (a_i \in \kappa).$$

Since $Z^2\Phi = W\Phi = \Phi = 0$, we evaluate the left-hand side of the above equality at ω_3, ω_4, ω_5, ω_6 and obtain

$$a_{135} = a_{246} = 0,$$
$$a_{14} = a_{25} = a_{36} = 0,$$
$$a_{13} = a_{24} = a_{35} = a_{46} = 0.$$

Hence all $a_i = 0$ and Φ_i are linearly independent. Since the Matlis dual M' of M has length 15, Φ_i generate M'. □

Proposition 5.7. *As an A-module, the minimal number of generators for M' is 5.*

Proof. $(X, Y, Z, W)M'$ as a vector space is generated by Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6, Φ_{13}, Φ_{24}, Φ_{35}, Φ_{46}. Therefore $\{\Phi_{14}, \Phi_{25}, \Phi_{36}, \Phi_{135}, \Phi_{246}\}$ is a minimal generating set for M'. □

Now we compute $\mathrm{Ext}^i_A(M, A/\mathfrak{p})$.

Proposition 5.8.

$$\dim_k \mathrm{Ext}^i_A(M, A/\mathfrak{p}) = \begin{cases} 6, & \textit{if } i = 2; \\ 7, & \textit{if } i = 3; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. For $n \geq 2$, $\mathrm{Ext}^{2n}_A(M, A/\mathfrak{p})$ is the cohomology of

$$M'^2 \xrightarrow{\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}} M'^2 \xrightarrow{\begin{pmatrix} X & Z \\ Y & W \end{pmatrix}} M'^2.$$

To simplify the notation, we write (i, j) for the element $(\Phi_i, \Phi_j) \in M'^2$. The kernel of $\left(\begin{smallmatrix} X & Z \\ Y & W \end{smallmatrix}\right)$ is generated by $(35, -13)$, $(46, -24)$, and $(i, 0)$, $(0, i)$, where $1 \leq i \leq 6$. The image of $\left(\begin{smallmatrix} W & -Z \\ -Y & X \end{smallmatrix}\right)$ is generated by $(4, -1)$, $(5, -2)$, $(6, -3)$, $(1, 0)$, $(2, 0)$, $(-3, 1)$, $(-4, 2)$, $(-5, 3)$, $(-6, 4)$, $(0, -4)$, $(0, 5)$, $(0, 6)$, $(-35, 13)$, $(-46, 24)$. Since

$$\begin{aligned}
(0, -3) &= (6, -3) + (-6, 4) + (0, -4), \\
(0, -2) &= (5, -2) + (-5, 3) + (0, -3), \\
(0, -1) &= (4, -1) + (-4, 2) + (0, -2), \\
(-3, 0) &= (-3, 1) + (0, -1), \\
(-4, 0) &= (-4, 2) + (0, -2), \\
(-5, 0) &= (-5, 3) + (0, -3), \\
(-6, 0) &= (-6, 4) + (0, -4),
\end{aligned}$$

all $(0, i)$ and $(i, 0)$ are contained in the image of $\left(\begin{smallmatrix} W & -Z \\ -Y & X \end{smallmatrix}\right)$. Therefore, $\mathrm{Ext}^{2n}_A(M, A/\mathfrak{p}) = 0$ for $n \geq 2$.

For $n \geq 2$, $\mathrm{Ext}^{2n+1}_A(M, A/\mathfrak{p})$ is the cohomology of

$$M'^2 \xrightarrow{\begin{pmatrix} X & Z \\ Y & W \end{pmatrix}} M'^2 \xrightarrow{\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}} M'^2.$$

The kernel of $\left(\begin{smallmatrix} W & -Z \\ -Y & X \end{smallmatrix}\right)$ is generated by $(13, 0)$, $(24, 0)$, $(35, 0)$, $(46, 0)$, and $(i, 0)$, $(0, i)$, where $1 \leq i \leq 6$. The image of $\left(\begin{smallmatrix} X & Z \\ Y & W \end{smallmatrix}\right)$ is generated by $(1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, $(-4, 1)$, $(5, 2)$, $(6, 3)$, $(13, 0)$, $(24, 0)$, $(5, 0)$, $(6, 0)$, $(0, 4)$, $(0, 5)$, $(0, 6)$, $(35, 1)$, $(46, 2)$. Clearly, $\mathrm{Ext}^{2n+1}_A(M, A/\mathfrak{p}) = 0$ for $n \geq 2$.

$\mathrm{Ext}^3_A(M, A/\mathfrak{p})$ is the cohomology of

$$M' \xrightarrow{\begin{pmatrix} X \\ Y \end{pmatrix}} M'^2 \xrightarrow{\begin{pmatrix} W & -Z \\ -Y & X \end{pmatrix}} M'^2.$$

The image of $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ is generated by $(1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, $(-4, 1)$, $(5, 2)$, $(6, 3)$, $(13, 0)$, $(24, 0)$. $\mathrm{Ext}^3_A(M, A/\mathfrak{p})$ is generated by the classes represented by $(0, 2)$, $(0, 3)$, $(0, 4)$, $(0, 5)$, $(0, 6)$, $(35, 0)$, $(46, 0)$. In particular, $\dim_\kappa \mathrm{Ext}^3_A(M, A/\mathfrak{p}) = 7$.

$\operatorname{Ext}_A^2(M, A/\mathfrak{p})$ has a basis $\Phi_1, \ldots, \Phi_6$, so $\dim_k \operatorname{Ext}_A^2(M, A/\mathfrak{p}) = 6$. It is easy to see that $\operatorname{Ext}_A^0(M, A/\mathfrak{p}) = \operatorname{Ext}_A^1(M, A/\mathfrak{p}) = 0$. This completes the proof of the proposition. □

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