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Buchsbaum–Rim multiplicities as Hilbert–Samuel multiplicities

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Abstract

We study the Buchsbaum–Rim multiplicity $\mathrm{br}(M)$ of a finitely generated module M over a regular local ring R of dimension 2 with maximal ideal $\mathfrak{m}$. The module M under consideration is of finite colength in a free R-module F. Write $F/M \cong I/J$, where $J \subset I$ are $\mathfrak{m}$-primary ideals of R. We first investigate the colength $\ell(R/\mathfrak{a})$ of any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and its Hilbert–Samuel multiplicity $\mathrm{e}(\mathfrak{a})$ using linkage theory. As an application, we establish several multiplicity formulas that express the Buchsbaum–Rim multiplicity of the module M in terms of the Hilbert–Samuel multiplicities of ideals related to I, J and a minimal reduction of M. The motivation comes from work by E. Jones, who applied graphical computations of the Hilbert–Samuel multiplicity to the Buchsbaum–Rim multiplicity [E. Jones, Computations of Buchsbaum–Rim multiplicities, J. Pure Appl. Algebra 162 (2001) 37–52].

Keywords: Hilbert–Samuel multiplicity; Buchsbaum–Rim multiplicity; Reduction of ideals and modules; Linkage

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Let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and infinite residue field. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. In this paper, we study the connection between the *colength* of $\mathfrak{a}$, i.e., the length $\ell(R/\mathfrak{a})$ of $R/\mathfrak{a}$, and the Hilbert–Samuel multiplicity $\mathrm{e}(\mathfrak{a})$ of $\mathfrak{a}$. It is known for an $\mathfrak{m}$-primary ideal $\mathfrak{b}$ contained in $\mathfrak{a}$ that $\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\mathfrak{b})$ if and only if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ (cf. [16,17]). Furthermore, if $\mathfrak{b}$ is a minimal reduction of $\mathfrak{a}$, then

$$\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\mathfrak{b}) = \ell(R/\mathfrak{b}). \tag{1}$$

However, $\mathrm{e}(\mathfrak{a})$ and $\ell(R/\mathfrak{a})$ are not equal in general. In one of our main theorems, Theorem 2.2, we express, under certain conditions, the colength of $\mathfrak{a}$ in terms of the Hilbert–Samuel multiplicity of ideals which are in the same linkage class as $\mathfrak{a}$.

Eq. (1) can be generalized to modules using the Buchsbaum–Rim multiplicity of a module M, denoted $\mathrm{br}(M)$. Let $U \subset M$ be submodules of a free R-module F of finite rank such that $\ell(F/U) < \infty$. It is known that U and M have the same Buchsbaum–Rim multiplicity if and only if U is a reduction of M. Similar to ideals, if U is a minimal reduction of M, then

$$\mathrm{br}(M) = \mathrm{br}(U) = \ell(F/U) \tag{2}$$

(cf. [6,12–14,19]).

In the case where F has rank one, M is an $\mathfrak{m}$-primary ideal and $\mathrm{br}(M) = \mathrm{e}(M)$. Thus the Buchsbaum–Rim multiplicity is a generalization of the Hilbert–Samuel multiplicity to modules. Like the Hilbert–Samuel multiplicity, it characterizes reductions. Using the theory of reductions of modules, we reduce the problem of finding formulas for the Buchsbaum–Rim multiplicity to the task of understanding the relationship between the colength and the Hilbert–Samuel multiplicity of ideals. The latter question is answered for arbitrary licci ideals in Theorem 2.2. As an application, we obtain formulas for the Buchsbaum–Rim multiplicity of a two-dimensional module in terms of the Hilbert–Samuel multiplicities of a certain Fitting ideal and ideals linked to it, see Theorem 2.4. We also prove expressions for the Buchsbaum–Rim multiplicity that involve Bourbaki ideals associated to the module, see Theorems 3.1, 3.3, and Corollary 3.4. The last corollary contains the work of [11] as a special case.

The paper is organized in the following way: Section 1 introduces the notion of the Buchsbaum–Rim multiplicity and its basic properties. We also include some definitions and theorems that will be used in the later sections. In Section 2, we state and prove the main theorem that relates the colength and the Hilbert–Samuel multiplicity of $\mathfrak{m}$-primary ideals in regular local rings of dimension two. In Section 3, we discuss several multiplicity formulas that express the Buchsbaum–Rim multiplicity of a module in terms of the Hilbert–Samuel multiplicity of $\mathfrak{m}$-primary ideals related to the module. In Section 4 we compare the multiplicity formulas obtained in Section 3 to the results of Jones [11], who provides a method for computing the Buchsbaum–Rim multiplicity of modules of a special type.

1. Introduction to the Buchsbaum–Rim multiplicity

In 1964, Buchsbaum and Rim [6] introduced and studied the multiplicity that bears their names. It was further studied by Gaffney, Kirby, Rees and many others, including Kleiman and Thorup who investigated the geometric theory of the Buchsbaum–Rim multiplicity in [14]. In this paper, we study the connection between the Buchsbaum–Rim multiplicity and the Hilbert–Samuel multiplicity.

Throughout the paper, we assume that R is a Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. There exists a polynomial $P_{\mathfrak{a}}(n)$ of degree d such that $P_{\mathfrak{a}}(n) = \ell(R/\mathfrak{a}^n)$ for large $n \in \mathbb{N}$. This polynomial is called the Hilbert–Samuel polynomial and the coefficient of $\frac{n^d}{d!}$ is the *Hilbert–Samuel multiplicity* $\mathrm{e}(\mathfrak{a})$.

The Buchsbaum–Rim multiplicity can be viewed as a generalization of the Hilbert–Samuel multiplicity. For a submodule M of finite colength in a free R-module F of rank r, Buchsbaum and Rim [6, 3.1] proved that there exists a polynomial $\lambda(n)$ such that for all large $n \in \mathbb{N}$,

$$\lambda(n) = \ell\big(\mathcal{S}_n(F)/\mathcal{R}_n(M)\big),$$

where $\mathcal{S}(F) = \bigoplus_{n \geqslant 0} \mathcal{S}_n(F)$ is the symmetric algebra of F and $\mathcal{R}(M) = \bigoplus_{n \geqslant 0} \mathcal{R}_n(M)$ is the image of the natural map $\mathcal{S}(M) \to \mathcal{S}(F)$. Notice that $\mathcal{R}(M)$ is the R-subalgebra of $\mathcal{S}(F)$ generated by M. According to [6, 3.4], the polynomial $\lambda(n)$ has degree $d + r - 1$ unless $M = F$. The coefficient of $\frac{n^{d+r-1}}{(d+r-1)!}$ in this polynomial is defined to be the *Buchsbaum–Rim multiplicity* $\mathrm{br}(M)$. It is a positive integer whenever $M \neq F$ and only depends on F/M [6, 3.3]. Notice that if $r = 1$ and $M \neq F$, then M is an $\mathfrak{m}$-primary ideal of R, $\lambda(n) = P_M(n)$ and $\mathrm{br}(M) = \mathrm{e}(M)$.

If $\operatorname{depth} R \geqslant 2$, then any inclusion $M \subset F$ with $\ell(F/M) < \infty$ can be identified with the natural embedding of M into its double dual M^{**}. Indeed, one has $\operatorname{Ext}^i_R(F/M, R) = 0$ for $i \leqslant 1$, therefore $M \subset F$ induces the identification $F^* = M^*$ and then $M^{**} = F^{**} = F$. Hence in this case $\mathrm{br}(M)$ is independent of the embedding of M into a free module. Moreover, if R is a two-dimensional regular local ring, one can define the Buchsbaum–Rim multiplicity of *any* finitely generated R-module M: simply consider the natural map from M to M^{**} and replace M by its image. The cokernel of this map has finite length, and the module M^{**} is free by the Auslander–Buchsbaum formula because it has depth at least 2.

Let F be a free R-module of rank r, let M be a submodule of F with $\ell(F/M) < \infty$, and let U be a submodule of M. Again, we write $\mathcal{R}(U)$ and $\mathcal{R}(M)$ for the R-subalgebras of $\mathcal{S}(F)$ generated by U and M, respectively. We say that U is a *reduction* of M if $\mathcal{R}(M)$ is integral over $\mathcal{R}(U)$ as rings. A minimal reduction of M is a reduction that is minimal with respect to inclusion. A free module $M = F$ has no proper reduction. On the other hand, when $M \neq F$, $d > 0$, and the residue field of R is infinite, then a reduction U of M is minimal if and only if its minimal number of generators is $r + d - 1$ (cf. [6, 3.5], [18, 2.1 and 2.2], [7, p. 707]).

After fixing a basis for F, the submodule M of F is associated to a matrix, denoted by $\widetilde{M}$, whose columns are a finite generating set of M. Recall that the zeroth Fitting ideal $\mathrm{Fitt}_0(F/M)$ is the ideal generated by the r by r minors of $\widetilde{M}$. This ideal only depends on F/M. More generally, if N is an R-module presented by a matrix with r rows, then the ith *Fitting ideal* $\mathrm{Fitt}_i(N)$ of N is the ideal generated by all $r - i$ by $r - i$ minors of this matrix.

We recall a theorem by Rees relating reductions of ideals and modules:

Theorem 1.1. (*See Rees* [18, 1.2].) *The submodule U of M is a reduction of M if and only if the subideal* $\mathrm{Fitt}_0(F/U)$ *is a reduction of* $\mathrm{Fitt}_0(F/M)$.

Reductions of modules in turn are closely related to Buchsbaum–Rim multiplicities. If U is a reduction of M then $\mathrm{br}(U) = \mathrm{br}(M)$ [14, 6.3(i)], and the converse holds in case R is universally catenary and equidimensional with $d > 0$ (cf. [13, 4.11], [14, 6.3(ii)], [12, 2.2], [19, 5.5]). Furthermore one has:

Theorem 1.2. (*See Buchsbaum and Rim* [6, 4.5(2)], *Angéniol and Giusti* [5, 2.8 and 2.10].) *Assume that R is a Cohen–Macaulay local ring with infinite residue field. If U is a minimal reduction of M, then*

$$\operatorname{br}(M) = \operatorname{br}(U) = \ell(F/U) = \ell\big(R/\operatorname{Fitt}_0(F/U)\big).$$

We say that an R-ideal I is a *Bourbaki ideal* of an R-module N, if $I \cong N/G$ for some free submodule G of N. Now let R be a Cohen–Macaulay local ring of dimension $d \geqslant 2$ with infinite residue field and let M be a submodule of finite colength in a free R-module F of rank r. For such $M \subset F$, there exist ideals $J \subset I$ of height $\geqslant 2$ such that F/M is isomorphic to I/J. In fact one can take $I \cong F/G$ with $G \subset M$ a free submodule of rank $r-1$ and J the image of M in I (cf. [4, Chapter 7, no. 4, Theorem 6], [20, 3.2(a), (c)]). Hence I and J are Bourbaki ideals of F and M, respectively. Notice that if $r \geqslant 2$ and M is generated by 3 elements, then $M = F$ or $r = d = 2$ [6, 3.5]. In this case I and J can be chosen to be the unit ideal or $\mathfrak{m}$-primary complete intersections. Since M is its own minimal reduction, we obtain the following equalities by Theorem 1.2,

$$\operatorname{br}(M) = \ell(F/M) = \ell(R/J) - \ell(R/I) = \mathrm{e}(J) - \mathrm{e}(I). \tag{3}$$

We see that the Buchsbaum–Rim multiplicity is connected to the Hilbert–Samuel multiplicity in this special case (cf. also [11]). We are interested in such a relationship for arbitrary modules. By Theorem 1.2, $\operatorname{br}(M)$ is equal to the colength of the Fitting ideal corresponding to a minimal reduction of M. Thus, the question can be reduced to investigating the connection between the colength and the Hilbert–Samuel multiplicity of ideals.

2. Colength and Hilbert–Samuel multiplicity

In a Cohen–Macaulay local ring R, two proper ideals $\mathfrak{a}$ and $\mathfrak{a}_1$ are *linked* with respect to a complete intersection ideal $\mathfrak{c}$, denoted $\mathfrak{a} \sim \mathfrak{a}_1$, if $\mathfrak{a}_1 = \mathfrak{c} : \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{c} : \mathfrak{a}_1$. If R is Gorenstein local and $\mathfrak{a}$ is unmixed (i.e., all associated prime ideals of $\mathfrak{a}$ have the same height), it suffices to require $\mathfrak{a}_1 = \mathfrak{c} : \mathfrak{a}$ and $\mathfrak{c} \subset \mathfrak{a}$. We say an ideal $\mathfrak{a}$ is *in the linkage class of a complete intersection* (or $\mathfrak{a}$ is *licci* for simplicity) if there are ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ with $\mathfrak{a} \sim \mathfrak{a}_1 \sim \cdots \sim \mathfrak{a}_n$ and $\mathfrak{a}_n$ a complete intersection. Examples of licci ideals are $\mathfrak{m}$-primary ideals I of finite projective dimension in a local ring $(R, \mathfrak{m})$, if either R is Cohen–Macaulay of dimension 2 or else R is Gorenstein of dimension 3 and R/I is Gorenstein (cf. [1,2,8], [3, 3.2(b)], [21, proof of Theorem]).

Theorem 2.1. (*See Huneke and Ulrich* [10, proof of 2.5].) *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with infinite residue field and let $\mathfrak{a}$ be a licci $\mathfrak{m}$-primary ideal linked to a complete intersection in n steps. Then there exists a sequence of links $\mathfrak{a} = \mathfrak{a}_0 \sim \mathfrak{a}_1 \sim \cdots \sim \mathfrak{a}_n$ such that $\mathfrak{a}_n$ is a complete intersection, and $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$ are linked with respect to a minimal reduction of $\mathfrak{a}_i$.*

We are now ready to prove our result that expresses the colength of licci ideals in terms of Hilbert–Samuel multiplicities.

Theorem 2.2. *In the setting of Theorem* 2.1*, we have*

$$\ell(R/\mathfrak{a}) = \sum_{i=0}^{n} (-1)^i \, \mathrm{e}(\mathfrak{a}_i).$$

Proof. We use induction on n. If $n=0$ then $\mathfrak{a}=\mathfrak{a}_0$ is a complete intersection. Hence $\ell(R/\mathfrak{a})=\mathrm{e}(\mathfrak{a})$ and the assertion is clear. Assume $n\geqslant 1$ and let $\mathfrak{b}_0$ be a minimal reduction of $\mathfrak{a}$ such that $\mathfrak{a}_1=\mathfrak{b}_0:\mathfrak{a}$. Notice that $\mathrm{e}(\mathfrak{b}_0)=\mathrm{e}(\mathfrak{a})$. The quotient ring $R/\mathfrak{b}_0$ is Gorenstein since $\mathfrak{b}_0$ is generated by a regular sequence. Moreover,

$$(\mathfrak{b}_0:\mathfrak{a})/\mathfrak{b}_0\cong \mathrm{Hom}_{R/\mathfrak{b}_0}(R/\mathfrak{a},R/\mathfrak{b}_0)\cong \mathrm{Hom}_{R/\mathfrak{b}_0}(R/\mathfrak{a},\omega_{R/\mathfrak{b}_0})\cong D_{R/\mathfrak{b}_0}(R/\mathfrak{a}),$$

where $\omega_{R/\mathfrak{b}_0}$ is the canonical module of $R/\mathfrak{b}_0$ and D denotes the dualizing functor. Since the dualizing functor preserves length, we have

$$\ell\big((\mathfrak{b}_0:\mathfrak{a})/\mathfrak{b}_0\big)=\ell(R/\mathfrak{a}).$$

Therefore

$$\begin{aligned}\ell(R/\mathfrak{a})&=\ell(R/\mathfrak{b}_0)-\ell\big(R/(\mathfrak{b}_0:\mathfrak{a})\big)\\&=\mathrm{e}(\mathfrak{b}_0)-\ell(R/\mathfrak{a}_1)\\&=\mathrm{e}(\mathfrak{a})-\ell(R/\mathfrak{a}_1).\end{aligned}$$

Our assertion now follows from the induction hypothesis. □

Henceforth we will often use the convention that the Hilbert–Samuel multiplicity of the unit ideal be zero.

Corollary 2.3. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension 2 with infinite residue field. If $\mathfrak{a}$ is an integrally closed $\mathfrak{m}$-primary ideal, then*

$$\ell(R/\mathfrak{a})=\sum_{i=1}^{\infty}(-1)^{i+1}\mathrm{e}\big(\mathrm{Fitt}_i(\mathfrak{a})\big).$$

Proof. Since $\mathfrak{a}$ is licci we may choose $\mathfrak{a}_0,\dots,\mathfrak{a}_n$ as in Theorem 2.1. We prove the assertion by induction on n. Notice that $\mathfrak{a}=\mathrm{Fitt}_1(\mathfrak{a})$ by the Hilbert–Burch theorem. Now if $n=0$ then $\mathfrak{a}=\mathfrak{a}_0$ is a complete intersection generated by two elements. Therefore $\ell(R/\mathfrak{a})=\mathrm{e}(\mathfrak{a})=\mathrm{e}(\mathrm{Fitt}_1(\mathfrak{a}))$, whereas $\mathrm{Fitt}_i(\mathfrak{a})=R$ for every $i\geqslant 2$. Next assume $n\geqslant 1$. According to Huneke and Swanson [9, 3.1 and 3.4], $\mathfrak{a}_1$ is integrally closed and $\mathrm{Fitt}_{\mathrm{i}}(\mathfrak{a}_1)=\mathrm{Fitt}_{\mathrm{i}+1}(\mathfrak{a})$ for every $i\geqslant 1$. Now apply Theorem 2.2 and the induction hypothesis. □

Theorem 2.4. *Let R be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a proper submodule of finite colength in a free R-module F of rank r, and let U be a minimal reduction of M.*

(a) *There exists a sequence of links $\mathrm{Fitt}_0(F/U)=\mathfrak{a}_0\sim\mathfrak{a}_1\sim\cdots\sim\mathfrak{a}_{r-1}$ such that $\mathfrak{a}_{r-1}$ is a complete intersection, and $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$ are linked with respect to a minimal reduction of $\mathfrak{a}_i$.*

(b) $$\mathrm{br}(M)=\mathrm{e}\big(\mathrm{Fitt}_0(F/M)\big)+\sum_{i=1}^{r-1}(-1)^i\mathrm{e}(\mathfrak{a}_i).$$

Proof. To prove part (a), notice that $\mathfrak{a} = \mathrm{Fitt}_0(F/U)$ has height 2 and is generated by the maximal minors of an r by $r+1$ matrix. Thus $\mathfrak{a}$ can be linked to a complete intersection in $r-1$ steps $\mathfrak{a} = \mathfrak{a}_0 \sim \mathfrak{a}_1 \sim \cdots \sim \mathfrak{a}_{r-1}$ [3, 3.2(b)]. By Theorem 2.1 we may assume that $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$ are linked with respect to a minimal reduction of $\mathfrak{a}_i$.

Part (b) follows from (a), Theorems 1.1, 1.2 and 2.2. □

A different formula for $\mathrm{br}(M)$ can be obtained with the assumptions of Theorem 2.4, if in addition R is regular and $\mathrm{Fitt}_0(F/U)$ is integrally closed. In this case Theorem 1.2, Corollary 2.3 and the equalities $\mathrm{Fitt}_{i+1}(\mathrm{Fitt}_0(F/U)) = \mathrm{Fitt}_i(F/U)$ immediately show that

$$\mathrm{br}(M) = \sum_{i=0}^{r-1} (-1)^i \mathrm{e}\big(\mathrm{Fitt}_i(F/U)\big).$$

Remark 2.5. The ideals $\mathfrak{a}_i$, $0 \leqslant i \leqslant r-1$, of Theorem 2.4 can be obtained concretely in the following way: After applying general row and column operations to the matrix $\widetilde{M}$ presenting F/M, the ideal $\mathfrak{a}_i$ is generated by the maximal minors of the matrix consisting of the last $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the last $r + 1 - 2\lceil \frac{i}{2} \rceil$ columns of $\widetilde{M}$ [3, 3.2(b)]; here $\lceil k \rceil$ denotes the smallest integer greater than or equal to k.

The following remark provides another point of view on the formula of Theorem 2.4.

Remark 2.6. As in Remark 2.5 we apply general row and column operations to the matrix $\widetilde{M}$, and then obtain an exact sequence

$$R^n \xrightarrow{\widetilde{M}} F \to C_0 = F/M \to 0.$$

The Auslander transpose $\mathrm{Tr}(C_0)$ of C_0 is presented by the transpose matrix $\widetilde{M}^*$,

$$F^* \xrightarrow{\widetilde{M}^*} R^{n^*} \to \mathrm{Tr}(C_0) \to 0.$$

Let C_1 be the quotient of $\mathrm{Tr}(C_0)$ modulo the submodule generated by the image of the first $n - r + 1$ basis elements of R^{n^*}. The submatrix of $\widetilde{M}^*$ involving the last $r-1$ rows presents C_1.

Continuing this way, we obtain a sequence of modules $C_0, \ldots, C_{r-1}$, where C_i for $i \geqslant 2$ is the quotient of $\mathrm{Tr}(C_{i-1})$ modulo the submodule generated by the first two generators. Notice that C_i is presented by the matrix consisting of the last $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the last $r+1-2\lceil \frac{i}{2} \rceil$ columns of $\widetilde{M}$ if $i \geqslant 2$ is even and by the transpose of this matrix if $i \geqslant 1$ is odd. Hence $\mathrm{Fitt}_0(C_i) = \mathfrak{a}_i$ for $i \geqslant 1$ as described in Remark 2.5 and then Theorem 2.4(b) shows that

$$\mathrm{br}(M) = \sum_{i=0}^{r-1} (-1)^i \mathrm{e}\big(\mathrm{Fitt}_0(C_i)\big).$$

3. Multiplicity formulas

In this section, we discuss other connections between the Buchsbaum–Rim multiplicity of modules and the Hilbert–Samuel multiplicity of ideals. In fact, we relate the Buchsbaum–Rim multiplicity of M to the Hilbert–Samuel multiplicity of a sufficiently general Bourbaki ideal of

F with respect to M, see Theorem 3.1. However, if there is a need to fix a specific Bourbaki ideal I of F, the result in Theorem 3.1 does not apply anymore. Instead Theorem 3.3 takes care of these cases.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a proper submodule of finite colength in a free R-module F of rank r, let U be a minimal reduction of M, and let $\mathfrak{a}_i$ be ideals as in Theorem 2.4(a). Then there exists a Bourbaki ideal I of F with height $\geqslant 2$ and a subideal $J \subset I$, such that $F/M \cong I/J$ and*

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) + \mathrm{e}(\mathfrak{a}_2) + \cdots + (-1)^{r-1}\mathrm{e}(\mathfrak{a}_{r-1}).$$

In particular, if $\operatorname{rank} M = 2$, *then there exist $\mathfrak{m}$-primary ideals $J \subset I$ such that $F/M \cong I/J$ and*

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I).$$

Proof. We may assume $r \geqslant 2$. Let $\mathfrak{b}_0$ be a minimal reduction of $\mathfrak{a}_0 = \mathrm{Fitt}_0(F/U)$ defining the link $\mathfrak{a}_0 \sim \mathfrak{a}_1$. We can find generators $u_1, \ldots, u_{r+1}$ of U in F so that $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are the ideals of maximal minors of the matrices $\widetilde{U} = (u_1|\cdots|u_{r+1})$ and $\widetilde{V} = (u_1|\cdots|u_{r-1})$, and $\mathfrak{b}_0$ is generated by the determinants of $(u_1|\cdots|u_{r-1}|u_r)$ and $(u_1|\cdots|u_{r-1}|u_{r+1})$.

Let G be the submodule of U generated by $u_1, \ldots, u_{r-1}$. As $\mathfrak{a}_1 = I_{r-1}(\widetilde{V})$ has height 2, it follows that G is free and $\mathfrak{a}_1 \cong F/G$ is an $\mathfrak{m}$-primary Bourbaki ideal of F. Thus we may take I to be $\mathfrak{a}_1$.

Now let J be the image of M in I. Clearly $J \cong M/G$ and hence $I/J \cong F/M$. Notice that $\mathfrak{b}_0$ is the image of U in I. As U is a reduction of M, it follows that $\mathfrak{b}_0$ is a reduction of J. Since by definition $\mathfrak{b}_0$ is also a reduction of $\mathfrak{a}_0$, we deduce $\mathrm{e}(J) = \mathrm{e}(\mathfrak{b}_0) = \mathrm{e}(\mathfrak{a}_0) = \mathrm{e}(\mathrm{Fitt}_0(F/M))$. Now Theorem 2.4(b) gives

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) + \mathrm{e}(\mathfrak{a}_2) + \cdots + (-1)^{r-1}\mathrm{e}(\mathfrak{a}_{r-1}). \qquad \square$$

We would like to point out that the result of Theorem 3.1 does not hold for an arbitrary pair of Bourbaki ideals $J \subset I$ of M and F satisfying $F/M \cong I/J$. What simplified the proof of Theorem 3.1 is the fact that we were able to assume that the free module G is contained in the minimal reduction U. This is no longer true in the general case that we are going to treat next. Theorem 3.3 provides an expression for $\mathrm{br}(M)$ in terms of $\mathrm{e}(I)$ and $\mathrm{e}(J)$ if I and J are already specified. This is motivated by the work in Jones [11], where it is necessary to choose I and J to be monomial ideals in order to extend the graphical computation of the Hilbert–Samuel multiplicity of monomial ideals to the Buchsbaum–Rim multiplicity of modules. Jones also provides a class of examples where the formula of Theorem 3.1 does not hold for arbitrary Bourbaki ideals $J \subset I$ [11, Theorem 7].

Assumption 3.2. Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a submodule of finite colength in a free R-module F of rank $r \geqslant 1$, and assume M has no nontrivial free direct summand. Suppose $F/M \cong I/J$, where $J \subset I$ are ideals of height $\geqslant 2$, I has finite projective dimension, and $\mu(I) \leqslant r$. Since $M \subset \mathfrak{m}F$, we have $\mu(I/J) = \mu(F/M) = \mu(F) = r \geqslant \mu(I)$ and therefore $J \subset \mathfrak{m}I$. Thus the lift $F \to I$ of the above isomorphism is surjective by Nakayama's lemma. It induces an isomorphism $I \cong F/G$, where G is a free submodule of F of rank $r - 1$. By restriction we obtain $J \cong M/G$. In particular, I and J are Bourbaki ideals of F and M, respectively.

Let $s_1, \ldots, s_{r-1}$ be generators of G and let $z_r, \ldots, z_{2r}$ be generators of a minimal reduction U of M. Thinking of $s_i \in F$ and $z_j \in F$ as column vectors we form the matrices

$$\widetilde{L} = \big(s_1|\cdots|s_{r-1}|z_r|\cdots|z_{2r}\big), \quad \widetilde{U} = \big(z_r|\cdots|z_{2r}\big), \quad \widetilde{N} = \big(s_1|\cdots|s_{r-1}|z_{2r-1}|z_{2r}\big).$$

By performing row operations on $\widetilde{L}$ and by adding suitable linear combinations of columns of $\widetilde{L}$ to later columns we may achieve these properties:

- $s_1, \ldots, s_{r-1}$ still generate G.
- $z_r, \ldots, z_{2r}$ still generate a minimal reduction U of M.
- The images of z_{2r-1}, z_{2r} in $M/G = J$ generate a minimal reduction J' of J.
- If for each i with $0 \leqslant i \leqslant r-1$, J_i denotes the ideal of maximal minors of the matrix consisting of the last $r - 2\lceil\frac{i-1}{2}\rceil$ rows and the last $r + 1 - 2\lceil\frac{i}{2}\rceil$ columns of $\widetilde{U}$, then J_i and J_{i+1} are linked with respect to a minimal reduction of J_i for $0 \leqslant i \leqslant r-2$.
- If for each i with $0 \leqslant i \leqslant r-1$, J'_i denotes the ideal of maximal minors of the matrix consisting of the last $r - 2\lceil\frac{i-1}{2}\rceil$ rows and the last $r + 1 - 2\lceil\frac{i}{2}\rceil$ columns of $\widetilde{N}$, then J'_i and J'_{i+1} are linked with respect to a minimal reduction of J'_i for $0 \leqslant i \leqslant r-2$. Notice that $J_{r-1} = J'_{r-1}$ and if r is odd then also $J_{r-2} = J'_{r-2}$.

Finally, let I' be any minimal reduction of I and $\mathrm{Fitt}_0(I/I') = I_0 \sim I_1 \sim \cdots \sim I_{r-3}$ a sequence of links as in Theorem 2.1.

Note that for the last two conditions in Assumption 3.2, one only has to check that the two minors corresponding to the first two rows or columns in the matrix of J_i (or J'_i) generate a reduction of J_i (resp. J'_i).

Theorem 3.3. *With assumptions as in* 3.2 *one has*

$$\begin{aligned}\mathrm{br}(M) = {}& \mathrm{e}(J) - \mathrm{e}(I) + \big(\mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big) + E_U\big) - \big(\mathrm{e}\big(\mathrm{Fitt}_0(I/J')\big) + E_N\big)\\ &+ \big(\mathrm{e}\big(\mathrm{Fitt}_0(I/I')\big) + E_I\big),\end{aligned}$$

where $E_U = \sum_{i=1}^{2\lceil\frac{r-3}{2}\rceil}(-1)^i\,\mathrm{e}(J_i)$, $E_N = \sum_{i=1}^{2\lceil\frac{r-3}{2}\rceil}(-1)^i\,\mathrm{e}(J'_i)$, *and* $E_I = \sum_{i=1}^{r-3}(-1)^i\mathrm{e}(I_i)$.

Proof. As U is a reduction of M, Theorem 1.1 shows that $I_r(\widetilde{U})$ is a reduction of $\mathrm{Fitt}_0(F/M) = \mathrm{Fitt}_0(I/J)$. Therefore

$$\mathrm{e}\big(I_r(\widetilde{U})\big) = \mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big).$$

The module I/J' is presented by the matrix $\widetilde{N}$, hence in particular

$$I_r(\widetilde{N}) = \mathrm{Fitt}_0(I/J').$$

Applying Theorem 2.2 to the ideals $I_r(\widetilde{U})$, $I_r(\widetilde{N})$ and $\mathrm{Fitt}_0(I/I')$ we obtain

$$\ell(R/I_r(\widetilde{U})) = \begin{cases} \mathrm{e}(\mathrm{Fitt}_0(I/J)) + E_U + (-1)^{r-1}\mathrm{e}(J_{r-1}) & \text{if } r \text{ is even},\\ \mathrm{e}(\mathrm{Fitt}_0(I/J)) + E_U + (-1)^{r-2}\mathrm{e}(J_{r-2}) + (-1)^{r-1}\mathrm{e}(J_{r-1}) & \text{if } r \text{ is odd},\end{cases} \tag{4}$$

$$-\ell\big(R/I_r(\widetilde{N})\big) = \begin{cases} -\mathrm{e}(\mathrm{Fitt}_0(I/J')) - E_N - (-1)^{r-1}\mathrm{e}(J'_{r-1}) & \text{if } r \text{ is even,} \\ -\mathrm{e}(\mathrm{Fitt}_0(I/J')) - E_N - (-1)^{r-2}\mathrm{e}(J'_{r-2}) - (-1)^{r-1}\mathrm{e}(J'_{r-1}) & \text{if } r \text{ is odd,} \end{cases} \tag{5}$$

$$\ell\big(R/\mathrm{Fitt}_0(I/I')\big) = \mathrm{e}\big(\mathrm{Fitt}_0(I/I')\big) + E_I. \tag{6}$$

Moreover by Theorem 1.2,

$$\begin{aligned} \ell\big(R/I_r(\widetilde{N})\big) - \ell\big(R/\mathrm{Fitt}_0(I/I')\big) &= \ell(I/J') - \ell(I/I') \\ &= \big(\ell(R/J') - \ell(R/I)\big) - \big(\ell(R/I') - \ell(R/I)\big) \\ &= \mathrm{e}(J') - \mathrm{e}(I') \\ &= \mathrm{e}(J) - \mathrm{e}(I). \end{aligned}$$

Thus we have

$$\ell\big(R/I_r(\widetilde{N})\big) - \ell\big(R/\mathrm{Fitt}_0(I/I')\big) = \mathrm{e}(J) - \mathrm{e}(I). \tag{7}$$

Theorem 1.2 also shows

$$\mathrm{br}(M) = \mathrm{br}(U) = \ell\big(R/I_r(\widetilde{U})\big). \tag{8}$$

Now by adding Eqs. (4)–(7) and applying (8), we obtain the multiplicity formula in Theorem 3.3. □

We state the rank two and rank three cases as a corollary. The multiplicity formulas have a simpler form in these cases.

Corollary 3.4. *We use the assumptions of* 3.2.

(a) *If* $r = 2$ *then*

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) + \mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big) - \mathrm{e}\big(\mathrm{Fitt}_0(I/J')\big).$$

(b) *If* $r = 3$ *then*

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) + \mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big) - \mathrm{e}\big(\mathrm{Fitt}_0(I/J')\big) + \mathrm{e}\big(\mathrm{Fitt}_0(I/I')\big).$$

Proof. These results follow immediately from Theorem 3.3. If $r = 2$, then the ideal I is its own minimal reduction and $\mathrm{e}(\mathrm{Fitt}_0(I/I')) = 0$. □

Remark 3.5. It should be pointed out that if a minimal reduction J' of J is sufficiently general, then there exists a minimal reduction U of M such that Assumption 3.2 is satisfied. The following example shows that the formula of Corollary 3.4(a) fails for a specific J', and therefore Assumption 3.2 does not hold for this J'.

Let $R = k[x, y]_{(x,y)}$ be a localized polynomial ring over a field and M a submodule of finite colength in a free R-module F of rank 2 such that the presenting matrix of F/M is

$$\begin{pmatrix} -y^{14} & x^{16} & 0 & 0 & x^5y^4 \\ x^{20} & 0 & y^{10} & x^8y^4 & 0 \end{pmatrix}.$$

Then $F/M \cong I/J$, where $I = (x^{20}, y^{14})$ and $J = (x^{36}, x^{25}y^4, x^8y^{18}, y^{24})$. Note that $J' = (x^{36} + y^{24}, x^{25}y^4)$ is a minimal reduction of J. The value on the right-hand side of the formula of Corollary 3.4(a) is

$$\mathrm{e}(J) - \mathrm{e}(I) + \mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big) - \mathrm{e}\big(\mathrm{Fitt}_0(I/J')\big) = 744 - 280 + 546 - 594 = 416,$$

while $\mathrm{br}(M) = 420$ (see [15, p. 50] for details).

This example also shows that $\mathrm{e}(\mathrm{Fitt}_0(I/J'))$ is not independent of the choice of a minimal reduction J' of J.

4. A graphical interpretation of the Buchsbaum–Rim multiplicity

In this section, we consider modules of rank two arising from monomial ideals. We compare our formulas to the result of E. Jones [11, p. 51], who gives a graphical computation of the Buchsbaum–Rim multiplicity in this case.

We assume $R = k[x, y]_{(x,y)}$ where k is a field, and let $\mathfrak{m}$ denote the maximal ideal of R. Let I and J be $\mathfrak{m}$-primary monomial ideals with $J \subset \mathfrak{m}I$, $\mu(I) = 2$ and $\mu(J) \leqslant 3$. Let F be a free R-module of rank 2 and M a submodule of F such that $F/M \cong I/J$. Jones computes the Buchsbaum–Rim multiplicity of M and shows that $\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I)$ with a few exceptions. For this one may assume that k is infinite.

We write $I = (x^s, y^t)$ and may assume that $J = (x^{s+i}, x^d y^{t+e}, y^{t+j})$. The module M can be taken to be the image in $F = R^2$ of the matrix

$$\widetilde{M} = \begin{pmatrix} -y^t & x^i & 0 & 0 \\ x^s & 0 & x^d y^e & y^j \end{pmatrix}.$$

In [11] the modules M are classified into seven cases: In Fig. 1, the point $T(s,t)$ corresponds to the monomial $x^s y^t$ and similarly for other points including those in Fig. 2(a)–(d) and Fig. 3(a)–(c).

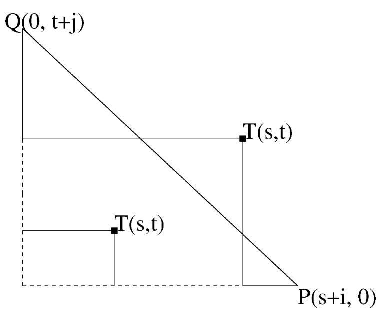

Fig. 1.

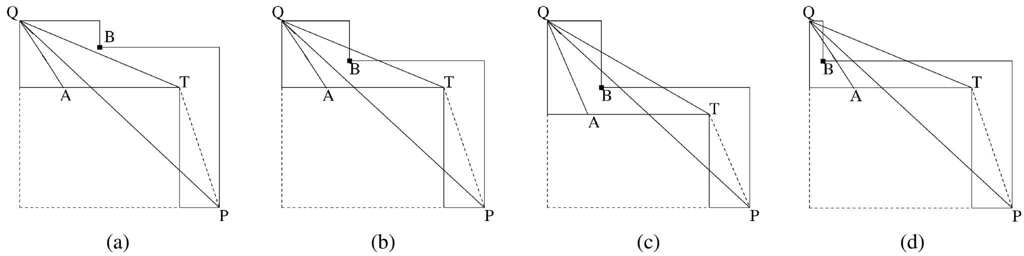

Fig. 2.

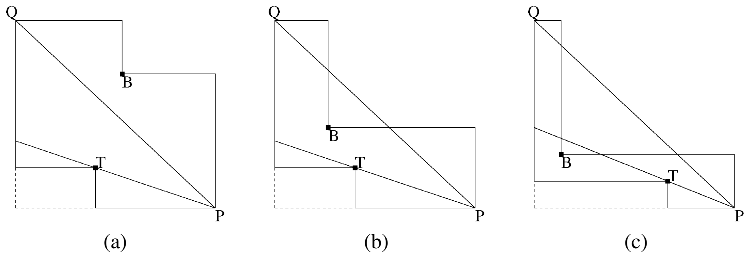

Fig. 3.

If T is above the line segment $\overline{PQ}$, then there are four cases determined by the relative positions of the point $B(d, t+e)$ and $\overline{TQ}$, $\overline{PQ}$, $\overline{AQ}$ as shown in Fig. 2(a)–(d), where $\overline{AQ}$ is parallel to $\overline{PT}$.

If T in Fig. 1 is below $\overline{PQ}$, there are three cases determined by the relative positions of B and $\overline{PQ}$, $\overline{PT}$ as shown in Fig. 3(a)–(c).

For the cases in Figs. 2(a) and 3(a), let U be the submodule of $F = R^2$ generated by the columns of the matrix

$$\widetilde{U} = \begin{pmatrix} -y^t & x^i & 0 \\ x^s & 0 & y^j \end{pmatrix}.$$

Then U is a minimal reduction of the module M. Notice that the first column in $\widetilde{U}$ is the syzygy of the ideal I and the image of U in J is a minimal reduction J' of J. Therefore in 3.2, we may take $\widetilde{N}$ to be $\widetilde{U}$ and $\widetilde{L}$ to be $\widetilde{U}$ with the first column repeated. By performing row operations on $\widetilde{U}$ and by adding suitable linear combinations of columns of $\widetilde{U}$ to later columns we have all the conditions required for Corollary 3.4. Since J' is the image of U in J and U is a reduction of M, Theorem 1.1 shows that $\mathrm{Fitt}_0(I/J')$ is a reduction of $\mathrm{Fitt}_0(I/J)$. Hence by Corollary 3.4(a),

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I).$$

This was also shown in [11].

In the cases of Figs. 2(d), 3(b) and 3(c), let U be the submodule of F generated by the columns of the matrix

$$\widetilde{U} = \begin{pmatrix} -y^t & x^i & 0 \\ x^s & y^j & x^d y^e \end{pmatrix}.$$

By the same argument, $\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I)$.

For the remaining cases, the modules of Figs. 2(b) and 2(c), we use the computation of the Buchsbaum–Rim multiplicity given in [11]. There it is shown that M is a reduction of the module generated by M itself and the vector $(0, x^s)$ in F, which is a direct sum of two monomial ideals. This allows for a computation of $\mathrm{br}(M)$. Thus in the case of Fig. 2(b),

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) - 2 \cdot \text{dark area}, \tag{9}$$

where the dark area is the area of the triangle TBQ indicated in Fig. 2(b)′. On the other hand, the modules of Fig. 2(c) have Buchsbaum–Rim multiplicity

$$\mathrm{br}(M) = \mathrm{e}(J) - \mathrm{e}(I) - 2 \cdot \text{dark area} + 2 \cdot \text{light area}, \tag{10}$$

where the dark area is the area of the triangle TBQ and the light area is the area of the triangle PBQ as indicated in Fig. 2(c)′.

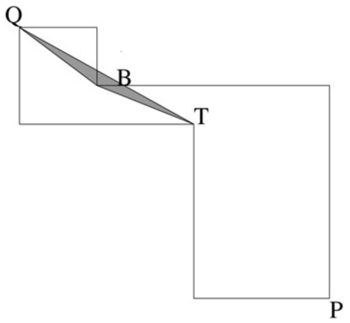

Fig. 2(b)′.

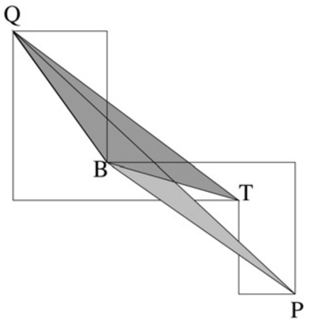

Fig. 2(c)′.

By Corollary 3.4(a), the extra terms subtracted in (9) and (10) are exactly

$$\mathrm{e}\big(\mathrm{Fitt}_0(I/J')\big) - \mathrm{e}\big(\mathrm{Fitt}_0(I/J)\big)$$

for a sufficiently general minimal reduction J' of J. We remark that in the first five cases, since $\mathrm{Fitt}_0(I/J)$ has a simple form, one can find a minimal reduction U of M that is close to being monomial. For the cases 2(b) and 2(c), this is much more complicated.

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