EXTRINSIC CURVATURE OF HYPERSURFACES IN HERMITIAN SPACE

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To my family.

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ABSTRACT

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by Tanuj Gupta

In this thesis, we study some invariants of a real hypersurface in the Hermitian space $\mathbb{C}^n$, which occur in the L^2 theory of the Cauchy-Riemann equations, a topic central to modern complex analysis. These invariants are preserved by the holomorphic isometries of $\mathbb{C}^n$ and are analogous to the second fundamental form and principal curvatures of hypersurfaces in Euclidean space $\mathbb{R}^d$, well-known in classical differential geometry. In complex analysis, these invariants arise as invariants of boundaries of domains.

The tangent space of a real hypersurface in $\mathbb{C}^n$ decomposes as the orthogonal direct sum of the *complex tangent space* (a complex linear subspace of $\mathbb{C}^n$) and a one dimensional real subspace known as the *characteristic direction*. This decomposition and the complex structure of the complex tangent space lead to a decomposition of the second fundamental form into several invariant pieces, called here the *Levi form*, the *complex-symmetric fundamental form*, the *skew functional*, and the *characteristic curvature*. After defining these quantities purely geometrically, we compute explicit formulas for them in terms of a defining function of the hypersurface, a smooth function which vanishes exactly on the hypersurface but whose gradient is never zero along the hypersurface. Such description of hypersurfaces, in terms of defining functions, is standard in complex analysis. We recapture in this way some expressions that occur in the work of Lars Hörmander and Charles Fefferman and explain their geometric significance.

Finally, we compute these curvatures for Reinhardt hypersurfaces in $\mathbb{C}^2$, which are hypersurfaces symmetric under rotations in each complex coordinate and can be thought of as analogous to surface of revolution. We try to understand the geometric significance of these curvatures by characterizing those Reinhardt hypersurfaces for which these curvatures vanish identically.

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CHAPTER I

INTRODUCTION

In complex analysis, the geometry of the boundary of a domain in $\mathbb{C}^n$ influences the function theory on that domain. The most important invariants of the boundary, from the perspective of function theory, are the so-called *CR invariants* which are invariant under biholomorphic maps (see [1] for discussion on such invariants). For example, *pseudoconvexity* (defined in Section II.2) is a CR invariant. On a domain with pseudoconvex boundary, there is a holomorphic function which is singular at each boundary point (see [21] for more discussion).

In this thesis, we focus our attention on real hypersurfaces in $\mathbb{C}^n$, since they arise naturally as boundaries of domains. We study invariants of real hypersurface in $\mathbb{C}^n$ under a much more restrictive class of mappings, namely, holomorphic isometries of $\mathbb{C}^n$, which are maps of the form $\mathbf{z} \mapsto U\mathbf{z} + \mathbf{w}$, where U is a unitary transformation and $\mathbf{w} \in \mathbb{C}^n$. The invariants studies here occur in the L^2 theory of the $\overline{\partial}$ problem (see [12]). A typical example of such invariants is an "eigenvalue of the Levi form", which plays a important role in determining the closed range property of the $\overline{\partial}$ operator (see [11, 16]).

The first chapter is preliminary. We define our objects and collect a few basic facts about them. In Chapter II, we review the classical theory of hypersurfaces in the Euclidean space $\mathbb{R}^d$. In particular, we discuss the *second fundamental form*, or equivalently the *shape operator*, which encodes the tensor curvature of such hypersurfaces. Then we discuss how the second fundamental form decomposes into pieces invariant under holomorphic isometries, which leads to definitions of tensor curvatures and numerical curvatures of real hypersurfaces in the Hermitian space $\mathbb{C}^n$. In Chapter III, we compute explicit formulas for these curvature in terms of a defining function and its derivatives. In Chapter IV, we concentrate on a special class of hypersurfaces, called Reinhardt hypersurfaces, for which computing these curvatures is easier. We also characterize such hypersurfaces for which these curvatures vanish.

Much of the content of this thesis is part of the folklore of several complex variables and here we describe it in an invariant and a systematic way. In particular, we give names to several quantities, like *complex-symmetric fundamental form*, *complex-symmetric principal curvatures*, and *skew curvature*, which occur in computations related to L^2 theory but do not have definite names in the literature. We use the name Levi principal curvature instead of the misleading but conventional name "eigenvalues of the Levi form".

I.1. Submanifolds of $\mathbb{R}^n$

Let U and V be two open sets in $\mathbb{R}^n$. Then a mapping $F : U \to V$ is called *smooth* if each component of F has continuous partial derivatives of all orders on U. A smooth map $F : U \to V$ is called a *diffeomorphism* is F is bijective and F^{-1} is also smooth.

Definition I.1. Let Ω be an open set in $\mathbb{R}^n$. Then a subset $M \subset \Omega$ is called a *k-dimensional (embedded) submanifold of* $\mathbb{R}^n$ if for each p in M, there are open neighborhoods U of p in $\mathbb{R}^n$ and V of 0 in $\mathbb{R}^n$ and a diffeomorphism $\psi : U \to V$ such that $\psi(p) = 0$ and $\psi(U \cap M) = V \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$.

A one-dimensional submanifold of $\mathbb{R}^n$ is called a *curve* in $\mathbb{R}^n$ and an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ is called a *hypersurface* in $\mathbb{R}^n$. The next proposition shows the existence of local parametrization of a submanifold and its equivalence with the definition of a submanifold.

Proposition I.2. *A subset $M \subset \mathbb{R}^n$ is a k-dimensional submanifold of $\mathbb{R}^n$ if and only if for each point p of M, there is an open set $W \subset \mathbb{R}^k$ and a smooth map $\varphi : W \to \mathbb{R}^n$ such that $p \in \varphi(W) \subset M$ and for each $q \in W$, the differential map $D\varphi_q : \mathbb{R}^k \to \mathbb{R}^n$ of φ at q is injective.*

Proof. Let us suppose that M is a k-dimensional submanifold of $\mathbb{R}^n$ and let $p \in M$. Then by Definition I.1, there are open neighborhoods $U \subset \mathbb{R}^n$ of p and $V \subset \mathbb{R}^n$ of 0 and a diffeomorphism $\psi : U \to V$ such that $\psi(p) = 0$ and $\psi(U \cap M) = V \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$. Then the set defined by $W = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid (x_1, \dots, x_k, 0, \dots, 0) \in \psi(U \cap M)\}$ is an open set in $\mathbb{R}^k$. Let $\varphi : W \to \mathbb{R}^n$ be the map defined as $\varphi(x_1, \dots, x_k) = \psi^{-1}(x_1, \dots, x_k, 0, \dots, 0)$. Then clearly φ is a smooth map, $p = \varphi(0) \in \varphi(W)$, and $\varphi(W) = \psi^{-1}(\psi(U \cap M)) \subset M$. If $J : \mathbb{R}^k \to \mathbb{R}^n$ is the inclusion map defined by

$$J(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0), \tag{I.1}$$

then we have $\varphi = \psi^{-1} \circ J$. Then by chain rule for differentiation, for each $q \in W$, we have $D\varphi_q = D\psi^{-1}_{J(q)} \circ i$. Since φ is a diffeomorphism, $D\psi^{-1}_{J(q)}$ is an isomorphism and in particular, an injection. So $D\varphi_q$ is a composition of two injective maps and hence is also injective.

For the converse, without any loss of generality we can assume the $0 \in W$ and $p = \varphi(0) \in M$. If $\{\mathbf{e_1}, \ldots, \mathbf{e_k}\}$ is the standard basis of $\mathbb{R}^k$, then by the injectivity of $D\varphi_0$, the set $\{D\varphi_0(\mathbf{e_1}), \ldots, D\varphi_0(\mathbf{e_k})\}$ is a linearly independent and hence there are vectors $\mathbf{v_{k+1}}, \ldots, \mathbf{v_n}$ in $\mathbb{R}^n$ such that $\{D\varphi_0(\mathbf{e_1}), \ldots, D\varphi_0(\mathbf{e_k}), \mathbf{v_{k+1}}, \ldots, \mathbf{v_n}\}$ is a basis of $\mathbb{R}^n$. Define $\widetilde{\varphi} : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by

$$\widetilde{\varphi}(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_k) + x_{k+1}\mathbf{v_{k+1}} + \cdots + x_n\mathbf{v_n}. \tag{I.2}$$

Then $D\widetilde{\varphi}_\mathbf{0}$ is invertible and hence by the inverse function theorem, there are neighborhoods $V \subset \mathbb{R}^n$ of $\mathbf{0}$ and $U \subset \mathbb{R}^n$ of $\widetilde{\varphi}(\mathbf{0}) = p$ such that $\widetilde{\varphi} : V \to U$ is a diffeomorphism. Now we claim that $\psi = \widetilde{\varphi}^{-1}$ is the required diffeomorphism from Definition I.1. Clearly $\psi(p) = \mathbf{0}$ and if $x \in U \cap M$ then there is a unique $w = (w_1, \ldots, w_k) \in W$ such that $\varphi(w) = x$ and since $\widetilde{\varphi}$ is one-to-one, we have $\widetilde{\varphi}(w_1, \ldots, w_k, 0, \ldots, 0) = \varphi(w) = x$. Hence $\psi(x) = \psi(\widetilde{\varphi}(w_1, \ldots, w_k, 0, \ldots, 0)) = (w_1, \ldots, w_k, 0, \ldots, 0)$. □

The map φ in the above proposition is called a *local parametrization* of M near p. Using the above proposition, every curve in $\mathbb{R}^n$ admits a parametrization over an open set in $\mathbb{R}$, which we can assume to be an open interval in $\mathbb{R}$. From now on, I will denote an open interval in $\mathbb{R}$ containing 0.

Let M be a k-dimensional submanifold in $\mathbb{R}^n$ and let $p \in M$. A vector $\mathbf{u} \in \mathbb{R}^n$ is called a *tangent vector* to M at p if there is a curve in M passing through p, with a local parametrization

$\alpha : I \to M$ near p such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{u}$. The next proposition shows that the set of all tangent vectors to M at p forms a linear subspace of $\mathbb{R}^n$ of dimension k.

Proposition I.3. *Let W be an open subset of $\mathbb{R}^k$ with $q \in W$ and let $\varphi : W \to M$ be a local parametrization of a k-dimensional submanifold M in $\mathbb{R}^n$ near $p = \varphi(q)$. Then the set of all tangent vectors to M at p is given by $D\varphi_q(\mathbb{R}^k)$.*

Proof. If $\mathbf{u} = D\varphi_q(\mathbf{v})$ for some $\mathbf{v} \in \mathbb{R}^k$, then consider the curve $\beta : I \to W$ given by $\beta(t) = t\mathbf{v} + q$ for small enough interval I. Then the curve $\alpha = \varphi \circ \beta$ is a curve in M with $\alpha(0) = \varphi(q) = p$ and $\alpha'(0) = D\varphi_q(\beta'(0)) = D\varphi_q(\mathbf{v}) = \mathbf{u}$. Hence $\mathbf{u}$ is a tangent vector to M at p.

On the other hand, if $\mathbf{u}$ is a tangent vector to M at $\varphi(q) = p$, then there is a curve $\alpha : I \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{u}$. The extension $\widetilde{\varphi}$ of φ, defined by (I.2), is a local diffeomorphism near $J(q)$, where J is defined as (I.1), we get that the curve $\beta = \varphi^{-1} \circ \alpha : I \to W$ is a smooth at 0. Then differentiating $\varphi \circ \beta = \alpha$ at 0 gives us $D\varphi_q(\beta'(0)) = \alpha'(0) = \mathbf{u}$. Hence $\mathbf{u} \in D\varphi_q(\mathbb{R}^k)$. □

Note that the above result does not depend on the choice of the local parametrization φ of M. Since $D\varphi_q$ is injective, the image $D\varphi_q(\mathbb{R}^k)$ is a k-dimensional linear subspace of $\mathbb{R}^n$. We define the *tangent space* of M at p, denoted by T_pM, as $T_pM = D\varphi_q(\mathbb{R}^k)$, where φ is a local parametrization of M near p such that $\varphi(q) = p$.

Corollary I.4. *The set of vectors $\left\{ \frac{\partial \varphi}{\partial x_1}(q), \ldots, \frac{\partial \varphi}{\partial x_k}(q) \right\}$ forms a basis of T_pM.*

Proof. The set $\{D\varphi_q(\mathbf{e_1}), \ldots, D\varphi_q(\mathbf{e_k})\}$ forms a basis of T_pM, where $\{\mathbf{e_1}, \ldots, \mathbf{e_k}\}$ is the standard basis of $\mathbb{R}^k$. Since $D\varphi_q(\mathbf{e_j}) = \partial \varphi / \partial x_j(q)$ for each $1 \le j \le k$, the result follows. □

Let $M \subset \mathbb{R}^m$ and $M' \subset \mathbb{R}^n$ be two submanifolds. A map $F : M \to M'$ is called *smooth* if there is an open neighborhood U of M in $\mathbb{R}^m$ and a smooth map $\widetilde{F} : U \to M'$ such that $\widetilde{F} = F$ on M. The differential of F at $p \in M$ is the linear map $DF_p : T_pM \to T_{F(p)}M'$ defined as follows: Let $\mathbf{u} \in T_pM$, then there is a curve $\alpha : I \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{u}$. Then $\beta = F \circ \alpha : I \to M'$ is a curve such that $\beta(0) = F(p)$. We define $DF_p(\mathbf{u}) = \beta'(0)$.

I.1.1. Orientation of a submanifold

In order to define orientation of a submanifold, we need to first define orientation of a vector space. Let V be a n dimensional vector space over $\mathbb{R}$. A linear operator $\Phi : V \to V$ is called *orientation preserving* if, with respect to a basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ of V, the matrix of Φ has positive determinant. The above definition does not depend on the choice of basis. Let $\{\mathbf{f_1}, \ldots, \mathbf{f_n}\}$ be another basis of V and let $\Psi : V \to V$ be the unique linear map such that $\Psi(\mathbf{e_j}) = \mathbf{f_j}$ for all $1 \leq j \leq n$. If $A_e = [\Phi(\mathbf{e_1}) \ \cdots \ \Phi(\mathbf{e_n})]$ is the matrix of Φ with respect to the basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$, then the matrix of Φ with respect to the basis $\{\mathbf{f_1}, \ldots, \mathbf{f_n}\}$ is given by $A_f = \left[(\Psi \circ \Phi \circ \Psi^{-1})(\mathbf{f_1}) \ \cdots \ (\Psi \circ \Phi \circ \Psi^{-1})(\mathbf{f_n})\right]$ and $\det(A_f) = \det \Psi \det(A_e) \det \Psi^{-1} = \det(A_e)$.

Let $\mathscr{B}$ be the set of all ordered bases of V. We define an relation $\sim$ on $\mathscr{B}$ as following: Two ordered bases $(\mathbf{a_1}, \ldots, \mathbf{a_n})$ and $(\mathbf{b_1}, \ldots, \mathbf{b_n})$ are related if the unique linear transformation $\Phi : V \to V$ defined by $\Phi(\mathbf{a_i}) = \mathbf{b_i}$, for all $1 \leq i \leq n$, is orientation preserving. This relation is reflexive since the matrix of the identity map is the $n \times n$ identity matrix with determinant 1. If A is the matrix of the linear transformation Φ, which is orientation preserving, then $\det(A) > 0$ and the inverse of Φ exists and the matrix of Φ^{-1} is given by A^{-1}. Since $\det(A^{-1}) = \dfrac{1}{\det(A)} > 0$, Φ^{-1} is also orientation preserving. Hence $\sim$ is symmetric. If Ψ is another orientation preserving operator on V with matrix B, then the matrix of $\Phi \circ \Psi$ is given by AB. Since $\det(AB) = \det(A)\det(B) > 0$, we get that $\sim$ is also transitive. Hence $\sim$ is an equivalence relation which divides $\mathscr{B}$ into two equivalence classes, $\mathscr{B}_+$ and $\mathscr{B}_-$. An *orientation of a vector space* is a choice of the equivalence class $\mathscr{B}_+$ or $\mathscr{B}_-$. The standard orientation on $\mathbb{R}^n$ is the equivalence class of the standard basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ of $\mathbb{R}^n$. We say that an ordered basis of $\mathbb{R}^n$ is *positively oriented* if it has the standard orientation. The standard identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by the map

$$(z_1, \ldots, z_n) \mapsto (x_1, y_1, \ldots, x_n, y_n), \tag{I.3}$$

where $z_k = x_k + iy_k$ for $1 \leq k \leq n$. Under this standard identification, every ordered basis of $\mathbb{C}^n$ is positively oriented in $\mathbb{R}^{2n}$.

Proposition I.5. *Let* $(\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_n)$ *be an ordered basis of* $\mathbb{C}^n$*. Then under the standard identification* (I.3) *of* $\mathbb{C}^n$ *with* $\mathbb{R}^{2n}$*, the ordered basis* $(\boldsymbol{\zeta}_1, i\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_n, i\boldsymbol{\zeta}_n)$ *of* $\mathbb{R}^{2n}$ *is positively oriented.*

Proof. Let $\{\mathbf{e_1},\dots,\mathbf{e_n}\}$ be the standard basis of $\mathbb{C}^n$ and let $F:\mathbb{C}^n\to\mathbb{C}^n$ be a $\mathbb{C}$-linear map such that $F(\mathbf{e_j})=\boldsymbol{\zeta}_j$ for each $1\le j\le n$. If $\boldsymbol{\zeta}_j=(\zeta_j^1,\dots,\zeta_j^n)$ for $1\le j\le n$, then the matrix of F (with respect to the standard basis of $\mathbb{C}^n$) is given by

$$F=\begin{bmatrix}\zeta_1^1 & \zeta_2^1 & \cdots & \zeta_n^1\\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_n^2\\ \vdots & \vdots & \ddots & \vdots\\ \zeta_1^n & \zeta_n^2 & \cdots & \zeta_n^n\end{bmatrix}.$$

Note that $\det(F)\neq 0$, since the columns are linearly inedpendent, and hence F is invertible. Let $\{\mathbf{f_1},\dots,\mathbf{f_{2n}}\}$ be the standard basis of $\mathbb{R}^{2n}$ and let $G:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be a $\mathbb{R}$-linear map such that under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we have $G(\mathbf{f_{2j+1}})=\boldsymbol{\zeta}_j$ and $G(\mathbf{f_{2j}})=i\boldsymbol{\zeta}_j$ for $1\le j\le n$. If $\zeta_J^k=\xi_j^k+i\eta_j^k$ for $1\le j,k\le n$, then matrix of G (with respect to the standard basis of $\mathbb{R}^{2n}$) is given by

$$G=\begin{bmatrix}\xi_1^1 & -\eta_1^1 & \xi_2^1 & -\eta_2^1 & \cdots & \xi_n^1 & -\eta_n^1\\ \eta_1^1 & \xi_1^1 & \eta_2^1 & \xi_2^1 & \cdots & \eta_n^1 & \xi_n^1\\ \xi_1^2 & -\eta_1^2 & \xi_2^2 & -\eta_2^2 & \cdots & \xi_n^2 & -\eta_n^2\\ \eta_1^2 & \xi_1^2 & \eta_2^2 & \xi_2^2 & \cdots & \eta_n^2 & \xi_n^2\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_1^n & -\eta_1^n & \xi_2^n & -\eta_2^n & \cdots & \xi_n^n & -\eta_n^n\\ \eta_1^n & \xi_1^n & \eta_2^n & \xi_2^n & \cdots & \eta_n^n & \xi_n^n\end{bmatrix}.$$

Now it suffices to show that $\det(G)>0$. So after rearranging columns, we get

$$\det(G)=(-1)^{n(n-1)/2}\det\begin{bmatrix}\xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 & -\eta_1^1 & -\eta_2^1 & \cdots & -\eta_n^1\\ \eta_1^1 & \eta_2^1 & \cdots & \eta_n^1 & \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1\\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 & -\eta_1^2 & -\eta_2^2 & \cdots & -\eta_n^2\\ \eta_1^2 & \eta_2^2 & \cdots & \eta_n^2 & \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \xi_1^n & \xi_2^n & \cdots & \xi_n^n & -\eta_1^n & -\eta_2^n & \cdots & -\eta_n^n\\ \eta_1^n & \eta_2^n & \cdots & \eta_n^n & \xi_1^n & \xi_2^n & \cdots & \xi_n^n\end{bmatrix}$$

and after rearranging the rows, we get

$$\det(G) = \det\begin{bmatrix} \xi_1^1 & \cdots & \xi_n^1 & -\eta_1^1 & \cdots & -\eta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi_1^n & \cdots & \xi_n^n & -\eta_1^n & \cdots & -\eta_n^n \\ \eta_1^1 & \cdots & \eta_n^1 & \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_1^n & \cdots & \eta_n^n & \xi_1^n & \cdots & \xi_n^n \end{bmatrix}.$$

Let $X = \begin{bmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^n & \cdots & \xi_n^n \end{bmatrix}$ and $E = \begin{bmatrix} \eta_1^1 & \cdots & \eta_n^1 \\ \vdots & \ddots & \vdots \\ \eta_1^n & \cdots & \eta_n^n \end{bmatrix}$, then we have the following block representation

$$\det(G) = \det\begin{bmatrix} X & -E \\ E & X \end{bmatrix}.$$

Multiplying second block column by i and adding to the first block column gives us

$$\det(G) = \det\begin{bmatrix} X - iE & -E \\ E + iX & X \end{bmatrix}$$

and then multiplying the first block row with $-i$ and adding to the second block row gives us

$$\det(G) = \det\begin{bmatrix} X - iE & -E \\ 0_n & X + iE \end{bmatrix},$$

where $0_n \in M_n(\mathbb{C})$ is the zero matrix. Note that $X + iE = F$ and $X - iE = \overline{F}$ and hence we get

$$\det(G) = \det\begin{bmatrix} \overline{F} & -E \\ 0_n & F \end{bmatrix} = \det\begin{bmatrix} \overline{F} & 0_n \\ 0_n & I_n \end{bmatrix} \det\begin{bmatrix} I_n & (\overline{F})^{-1}E \\ 0_n & F \end{bmatrix} = \overline{\det(F)}\det(F) = |\det(F)|^2 > 0.$$

□

Now we are ready to define an orientation for a submanifold.

Definition I.6 (Orientation of a submanifold)**.** If M be a k-dimensional submanifold in $\mathbb{R}^n$, then an orientation of M is an assignment of an orientation $\mathscr{O}_p$ to each tangent space T_pM is such a way that $\mathscr{O}_p$ varies continuously with p, that is, for each $q \in M$, there is a open set $U \subset \mathbb{R}^n$ containing

q and continuous linearly independent vector fields $\mathbf{e_1}, \ldots, \mathbf{e_k}$ such that for each $p \in U \cap M$, we have $(\mathbf{e_1}(p), \ldots, \mathbf{e_k}(p)) \in \mathscr{O}_p$.

A submanifold is said to be *orientable* is there exists an orientation of that submanifold. A submanifold along with an orientation is called an *oriented* submanifold.

I.2. Isometries

If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are two vectors in $\mathbb{R}^n$, then the standard inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k. \tag{I.4}$$

This standard inner product defines a metric on $\mathbb{R}^n$, given by $d(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle$. Similarly, if $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ are two vectors in $\mathbb{C}^n$, then the standard Hermitian inner product of ξ and η in $\mathbb{C}^n$, denoted by $(\boldsymbol{\xi}, \boldsymbol{\eta})$, is defined as

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{k=1}^{n} \xi_k \overline{\eta}_k. \tag{I.5}$$

Under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the Hermitian space $\mathbb{C}^n$ has a real inner product structure, given by: If $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$ such that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ with $\xi_k = p_k + iq_k$ and $\eta_k = u_k + iv_k$ for $1 \leq k \leq n$, then $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \sum_{k=1}^{n} p_k u_k + q_k v_k$. Note that $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{k=1}^{n} \xi_k \overline{\eta}_k = \sum_{k=1}^{n} (x_k u_k + y_k v_k) + i(-x_k v_k + y_k u_k)$ and hence we have

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \operatorname{Re}(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{I.6}$$

where $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ is the real part of z. Hence we get a metric on $\mathbb{C}^n$ induced by the real inner product. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called an *rigid motion* if $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, for some $A \in SO_n$ and $\mathbf{b} \in \mathbb{R}^n$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $F : X \to Y$ is called an *isometry* if F is a bijection and it preserve distances, i.e., for any $x_1, x_2 \in X$, we have $d_Y(F(x_1), F(x_2)) =$

$d_X(x_1,x_2)$. It is clear from the definition that composition of two isometries is again an isometry and hence the set of all isometries from a metric space (X,d_X) to itself forms a group under composition. We say that an isometry $F:\mathbb{R}^n\to\mathbb{R}^n$ is *orientation preserving* if the ordered basis $(F(\mathbf{e_1}),\dots,F(\mathbf{e_n}))$ of $\mathbb{R}^n$ is positively oriented, where $(\mathbf{e_1},\dots,\mathbf{e_n})$ is the standard basis of $\mathbb{R}^n$. It is clear from the definition that every rigid motion of $\mathbb{R}^n$ is an orientation preserving isometry. The converse is also true since every isometry of $\mathbb{R}^n$ is of the form $F(\mathbf{x})=A\mathbf{x}+\mathbf{b}$, where $A\in O_n$ is an orthogonal matrix and $\mathbf{b}\in\mathbb{R}^n$. A proof for $n=3$ can be found in [20, Chapter 3, Theroem 1.7] and the same arguments generalizes to higher dimensions. Hence the orientation preserving isometries of $\mathbb{R}^n$ are precisely the rigid motions of $\mathbb{R}^n$.

If $\Omega\subset\mathbb{C}^n$ is an open set, then a function $f:\Omega\to\mathbb{C}$ is called a *holomorphic function* if $f\in\mathscr{C}^1(\Omega)$ and it is holomorphic in each variable separately, that is, for each $1\le j\le n$ and every $\boldsymbol{\zeta}\in\Omega$, we have $\partial f/\partial\bar{z}_j(\boldsymbol{\zeta})=0$. A map $F:\Omega\to\mathbb{C}^m$, with $F=(f_1,\dots,f_m)$ is called a *holomorphic map* if each f_j is a holomorphic function on Ω for $1\le j\le n$. Under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the Hermitian space $\mathbb{C}^n$ is also equipped with the Euclidean metric of $\mathbb{R}^{2n}$, so we can define isometries on $\mathbb{C}^n$. We are particularly interested in those isometries of $\mathbb{C}^n$ that are also holomorphic maps.

Lemma I.7. *Every $\mathbb{R}$-linear holomorphic map $G:\mathbb{C}^n\to\mathbb{C}^n$ is also $\mathbb{C}$-linear.*

Proof. It suffices to show that for each $\boldsymbol{\zeta}\in\mathbb{C}^n$, we have $G(i\boldsymbol{\zeta})=iG(\boldsymbol{\zeta})$ for each $\boldsymbol{\zeta}\in\mathbb{C}^n$. Let $G=(G_1,\dots,G_n)$ and for each $1\le j\le n$, suppose $G_j(\boldsymbol{\zeta})=u_j(\boldsymbol{\zeta})+iv_j(\boldsymbol{\zeta})$, where u_j and v_j are real-valued functions. Let $\boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$ and let $\zeta_j=\xi_j+i\eta_j$ for each $1\le j\le n$. Under the standard identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, since G is a $\mathbb{R}$-linear map, there are real numbers $\{a_{jk},b_{jk},c_{jk},d_{jk}\}_{j,k=1}^n\subset\mathbb{R}$ such that

$$u_j(\boldsymbol{\zeta})=\sum_{k=1}^n a_{jk}\xi_k+b_{jk}\eta_k\quad\text{and}\quad v_j(\boldsymbol{\zeta})=\sum_{k=1}^n c_{jk}\xi_k+d_{jk}\eta_k.$$

Also, since G is holomorphic map, each G_j is a holomorphic function on $\mathbb{C}^n$ and using the Cauchy-Riemann equations, we get $a_{jk} = d_{jk}$ and $b_{jk} = -c_{jk}$ for each $1 \le j,k \le n$. Hence

$$u_j(\boldsymbol{\zeta}) = \sum_{k=1}^{n} a_{jk}\xi_k + b_{jk}\eta_k \quad \text{and} \quad v_j(\boldsymbol{\zeta}) = \sum_{k=1}^{n} -b_{jk}\xi_k + a_{jk}\eta_k.$$

Note that $i\boldsymbol{\zeta} = (i\zeta_j, \ldots, i\zeta_n)$ is identified with $(-\eta_1, \xi_1, \ldots, -\eta_n, \xi_n) \in \mathbb{R}^{2n}$. Hence for each $1 \le j \le n$, we get

$$\begin{aligned} G_j(i\boldsymbol{\zeta}) &= \sum_{k=1}^{n} \left(-a_{jk}\eta_k + b_{jk}\xi_k\right) + i\left(b_{jk}\eta_k + a_{jk}\xi_k\right) = i\sum_{k=1}^{n} -i\left(-a_{jk}\eta_k + b_{jk}\xi_k\right) + \left(b_{jk}\eta_k + a_{jk}\xi_k\right) \\ &= i\sum_{k=1}^{n} \left(a_{jk}\xi_k + b_{jk}\eta_k\right) + i\left(-b_{jk}\xi_k + a_{jk}\eta_k\right) = iG_j(\boldsymbol{\zeta}). \end{aligned}$$

□

Definition I.8 (Holomorphic isometry). A map $G : \mathbb{C}^n \to \mathbb{C}^n$ is called a holomorphic isometry if G is an isometry of $\mathbb{C}^n$ and also G is a holomorphic map.

Proposition I.9. *If $G : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic isometry that fixes $\mathbf{0}$, then G preserves Hermitian inner product of $\mathbb{C}^n$, that is, $(G(\boldsymbol{\xi}), G(\boldsymbol{\eta})) = (\boldsymbol{\xi}, \boldsymbol{\eta})$ for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$.*

Proof. Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic isometry such that $G(\mathbf{0}) = \mathbf{0}$. Since G is an isometry of $\mathbb{R}^{2n}$ under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, there exists an orthogonal matrix $A \in O_{2n}$ such that in the real coordinates, $G(\boldsymbol{\xi}) = A\boldsymbol{\xi}$. Hence $|G(\xi)| = |\boldsymbol{\xi}|$ for each $\boldsymbol{\xi} \in \mathbb{C}^n$. From Lemma I.7, G is also $\mathbb{C}$-linear. Let U be the matrix of G with respect to the standard basis of $\mathbb{C}^n$. Then for every $\boldsymbol{\xi} \in \mathbb{C}^n$, we have $(\boldsymbol{\xi}, \boldsymbol{\xi}) = (U\boldsymbol{\xi}, U\boldsymbol{\xi}) = (U^*U\boldsymbol{\xi}, \boldsymbol{\xi})$, where $(\cdot,\cdot)$ is the Hermitian inner product on $\mathbb{C}^n$. Hence for each $\boldsymbol{\xi} \in \mathbb{C}^n$, we have $((U^*U - I_n)\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$, where $I_n \in M_n(\mathbb{C})$ is the identity matrix. Let $B = U^*U - I_n$. Then for each $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$, the polarization identity gives us

$$\begin{aligned} (B\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{4}&[(B(\boldsymbol{\xi}+\boldsymbol{\eta}), (\boldsymbol{\xi}+\boldsymbol{\eta})) + i(B(\boldsymbol{\xi}+i\boldsymbol{\eta}), (\boldsymbol{\xi}+i\boldsymbol{\eta})) \\ &- (B(\boldsymbol{\xi}-\boldsymbol{\eta}), (\boldsymbol{\xi}-\boldsymbol{\eta})) - i(B(\boldsymbol{\xi}-i\boldsymbol{\eta}), (\boldsymbol{\xi}-i\boldsymbol{\eta}))] \\ = 0.& \end{aligned}$$

Hence $B = 0_n$, where $0_n \in M_n(\mathbb{C})$ is the zero matrix and hence $U^*U = I_n$. Using similar arguments, one gets $UU^* = I_n$. So U is a unitary matrix and G is a unitary transformation. Hence for any $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$, we have $(G(\boldsymbol{\xi}), G(\boldsymbol{\eta})) = (U\boldsymbol{\xi}, U\boldsymbol{\eta}) = (U^*U\boldsymbol{\xi}, \boldsymbol{\eta}) = (\boldsymbol{\xi}, \boldsymbol{\eta})$. □

Corollary I.10. *If $G : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic isometry, then there exist an unique $\mathbf{w} \in \mathbb{C}^n$ and a unique unitary matrix $U \in U_n$ such that for each $\mathbf{z} \in \mathbb{C}^n$, we have $G(\mathbf{z}) = U\mathbf{z} + \mathbf{w}$.*

Proof. Let $\mathbf{w} = G(\mathbf{0})$. Then the map $\mathbf{z} \mapsto G(\mathbf{z}) - \mathbf{w}$ is a holomorphic isometry of $\mathbb{C}^n$ that fixes $\mathbf{0}$ and hence by previous proposition, there is a unitary transformation U such that $G(\mathbf{z}) - \mathbf{w} = U\mathbf{z}$ and hence we get $G(\mathbf{z}) = U\mathbf{z} + \mathbf{w}$ for each $\mathbf{z} \in \mathbb{C}^n$. □

I.3. Some Linear Algebra

Let V be a d-dimensional vector space over a field $\mathbb{K}$. A quadratic form is a homogeneous polynomial of degree 2. More precisely, a *quadratic form* on V is a map $\mathsf{Q} : V \to \mathbb{K}$ such that, for any basis $\{\mathbf{e_1}, \dots, \mathbf{e_d}\}$ of V, there exist scalars $\{a_{jk}\}_{j,k=1}^d$ in $\mathbb{K}$ such that for every $\mathbf{v} = v^1\mathbf{e^1} + \cdots + v^d\mathbf{e_d} \in V$, we have

$$\mathsf{Q}(\mathbf{v}) = \sum_{j,k=1}^{d} a_{jk} v^j v^k.$$

If $\{\mathbf{f_1}, \dots, \mathbf{f_d}\}$ is another basis of V, then for every $1 \le i \le d$, $\mathbf{f_i} = \sum_{j=1}^{d} \alpha_i^j \mathbf{e_j}$, for some scalars $\{\alpha_i^j\}_{i,j=1}^d$ in $\mathbb{K}$. If $\mathbf{v} = \sum_{i=1}^{d} \widetilde{v}^i \mathbf{f_i}$, then we get $\mathbf{v} = \sum_{i=1}^{d} \widetilde{v}^i \sum_{j=1}^{d} \alpha_i^j \mathbf{e_j} = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} \widetilde{v}^i \alpha_i^j \right) \mathbf{e_j}$ and

$$\mathsf{Q}(\mathbf{v}) = \sum_{j,k=1}^{d} a_{jk} \left(\sum_{p=1}^{d} \widetilde{v}^p \alpha_p^j \right) \left(\sum_{q=1}^{d} \widetilde{v}^q \alpha_q^k \right) = \sum_{p,q=1}^{d} \left(\sum_{j,k=1}^{d} a_{jk} \alpha_p^j \alpha_q^k \right) \widetilde{v}^p \widetilde{v}^q = \sum_{p,q=1}^{d} b_{pq} \widetilde{v}^p \widetilde{v}^q,$$

where $\{b_{pq}\}_{p,q=1}^d$ are scalars in $\mathbb{K}$. Hence the above definition of quadratic form in independent of the choice of the basis. Also if $\operatorname{char}(\mathbb{K}) \neq 2$, by replacing a_{jk} with $\frac{1}{2}(a_{jk} + a_{kj})$, we can assume that $a_{jk} = a_{kj}$.

A *bilinear form* on V is a map $\mathscr{B} : V \times V \to \mathbb{K}$ such that $\mathscr{B}$ is $\mathbb{K}$-linear in each variable separately, that is, for each $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{K}$, we have $\mathscr{B}(\mathbf{u} + \lambda\mathbf{v}, \mathbf{w}) = \mathscr{B}(\mathbf{u}, \mathbf{w}) + \lambda\mathscr{B}(\mathbf{v}, \mathbf{w})$

and $\mathscr{B}(\mathbf{u}, \mathbf{v}+\lambda\mathbf{w}) = \mathscr{B}(\mathbf{u},\mathbf{v}) + \lambda\mathscr{B}(\mathbf{u},\mathbf{w})$. A bilinear form $\mathscr{B}$ is called *symmetric* if $\mathscr{B}(\mathbf{u},\mathbf{v}) = \mathscr{B}(\mathbf{v},\mathbf{u})$ for each $\mathbf{u},\mathbf{v} \in V$. Given a symmetric bilinear form $\mathscr{B} : V \times V \to \mathbb{K}$, the map $\mathsf{Q} : V \to \mathbb{K}$ defined by $\mathsf{Q}(\mathbf{v}) = \mathscr{B}(\mathbf{v},\mathbf{v})$ is a quadratic form on V. Conversely, given a quadratic form Q, we define $\mathscr{B}_{\mathsf{Q}} : V \times V \to \mathbb{K}$ as

$$\mathscr{B}_{\mathsf{Q}}(\mathbf{u},\mathbf{v}) = \frac{1}{4}\left(\mathsf{Q}(\mathbf{u}+\mathbf{v}) - \mathsf{Q}(\mathbf{u}-\mathbf{v})\right) = \sum_{j,k=1}^{d} a_{jk}u^j v^k. \tag{I.7}$$

Then for vectors $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ and $\lambda \in \mathbb{K}$ we have $\mathscr{B}_{\mathsf{Q}}(\mathbf{v},\mathbf{v}) = \mathsf{Q}(\mathbf{v})$, $\mathscr{B}_{\mathsf{Q}}(\mathbf{u},\mathbf{v}) = \mathscr{B}_{\mathsf{Q}}(\mathbf{v},\mathbf{u})$, and $\mathscr{B}_{\mathsf{Q}}(\mathbf{u}+\lambda\mathbf{w},\mathbf{v}) = \sum_{j,k=1}^{d} a_{jk}(u^j+\lambda w^j)v^k = \sum_{j,k=1}^{d} a_{jk}u^j v^k + \lambda \sum_{j,k=1}^{d} a_{jk}w^j v^k = \mathscr{B}_{\mathsf{Q}}(\mathbf{u},\mathbf{v}) + \lambda\mathscr{B}_{\mathsf{Q}}(\mathbf{w},\mathbf{v})$. Hence $\mathscr{B}_{\mathsf{Q}}$ is a symmetric bilinear form on V. We say that $\mathscr{B}_{\mathsf{Q}}$ is the bilinear form of the quadratic form Q on V. The matrix B of a symmetric bilinear form $\mathscr{B}_{\mathsf{Q}}$ with respect to the basis $\{\mathbf{e_1},\dots,\mathbf{e_d}\}$ is the symmetric matrix whose $(j,k)^{th}$ entry is $a_{jk} = \mathscr{B}_{\mathsf{Q}}(\mathbf{e_j},\mathbf{e_k})$. There is a one-to-one correspondence between $\mathbb{K}$-quadratic forms and symmetric $\mathbb{K}$-bilinear forms. A bilinear form $\mathscr{B} : V \times V \to \mathbb{K}$ is called non-degenerate if $\mathscr{B}(\mathbf{u},\mathbf{v}) = 0$ for all $\mathbf{u} \in V$ implies $\mathbf{v} = 0$.

Proposition I.11. *A symmetric bilinear form $\mathscr{B} : V \times V \to \mathbb{K}$ is non-degenerate if and only if the matrix B of $\mathscr{B}$ is invertible.*

Proof. If $\mathbf{x} = (x^1,\dots,x^d) \in \mathbb{K}^d$, then

$$\begin{aligned} B\mathbf{x} = \mathbf{0} &\iff \sum_{j,k=1}^{d} \mathscr{B}(\mathbf{e_j},\mathbf{e_k})x^k = \mathbf{0} \text{ for each } 1 \le j \le d \\ &\iff \mathscr{B}(\mathbf{e_j}, \sum_{k=1}^{d} x^k\mathbf{e_k}) = \mathbf{0} \text{ for each } 1 \le j \le d \\ &\iff \mathscr{B}(\mathbf{u}, \sum_{k=1}^{d} x^k\mathbf{e_k}) = \mathbf{0} \text{ for each } \mathbf{u} \in V \end{aligned}$$

Hence B is not invertible if and only if there is a non-zero $\mathbf{x} \in \mathbb{K}^d$ satisfying $B\mathbf{x} = \mathbf{0}$, that is, $\mathscr{B}$ is not non-degenerate. $\square$

Lemma I.12 (Riesz representation theorem)**.** *Let* $\Phi : V \to \mathbb{K}$ *be a* $\mathbb{K}$*-linear functional. Given a non-degenerate symmetric bilinear form* $\mathscr{B} : V \times V \to \mathbb{K}$*, there is a unique vector* $\mathbf{v_0} \in V$ *such that* $\Phi(\mathbf{v}) = \mathscr{B}(\mathbf{v}, \mathbf{v_0})$ *for every* $\mathbf{v} \in V$*.*

Proof. Let $\{\mathbf{e_1}, \dots, \mathbf{e_d}\}$ be a basis of V and let B be the matrix of the $\mathscr{B}$ with respect to this basis. Let $\mathbf{v} = v^1\mathbf{e_1} + \cdots + v^d\mathbf{e_d}$ and then let $\mathbf{y} = (\Phi(\mathbf{e_1}), \dots, \Phi(\mathbf{e_d})) \in \mathbb{K}^d$. By Proposition I.11, B is invertible and the linear system $B\mathbf{x} = \mathbf{y}$ has a solution, that is, there exists $\mathbf{x} = (x^1, \dots, x^d) \in \mathbb{K}^d$ such that $\sum_{j,k=1}^{d} B(\mathbf{e_j}, \mathbf{e_k})x^k = \Phi(\mathbf{e_j})$ for $1 \leq j \leq d$. Setting $\mathbf{v_0} = x^1\mathbf{e_1} + \cdots + x^d\mathbf{e_d}$ we get

$$\Phi(\mathbf{v}) = \sum_{j=1}^{d} v^j \Phi(\mathbf{e_j}) = \sum_{j,k=1}^{d} v^j x^k B(\mathbf{e_j}, \mathbf{e_k}) = B(\sum_{j=1}^{d} v^j \mathbf{e_j}, \sum_{k=1}^{d} x^k \mathbf{e_k}) = B(\mathbf{v}, \mathbf{v_0}).$$

The uniqueness of $\mathbf{v_0}$ follows from the non-degeneracy condition of $\mathscr{B}$. □

Proposition I.13. *Given a non-degenerate symmetric bilinear form* $\mathscr{B} : V \times V \to \mathbb{K}$*, for each quadratic form* Q *on* V*, there is a unique* $\mathbb{K}$*-linear map* $S_{\mathsf{Q}} : V \to V$ *such that for every* $\mathbf{u}, \mathbf{v} \in V$*, we have* $\mathscr{B}_{\mathsf{Q}}(\mathbf{u}, \mathbf{v}) = \mathscr{B}(S_{\mathsf{Q}}(\mathbf{u}), \mathbf{v})$*.*

Proof. Consider the linear functional $\Phi_{\mathbf{u}} : V \to \mathbb{K}$ defined as $\Phi_{\mathbf{u}}(\mathbf{v}) = \mathscr{B}_{\mathsf{Q}}(\mathbf{u}, \mathbf{v})$. Then by Lemma I.12, there is unique vector $\mathbf{u_0}$ such that $\mathscr{B}_{\mathsf{Q}}(\mathbf{u}, \mathbf{v}) = \mathscr{B}(\mathbf{u_0}, \mathbf{v})$. Define $S_{\mathsf{Q}} : V \to V$ as $S_{\mathsf{Q}}(\mathbf{u}) = \mathbf{u_0}$. The $\mathbb{K}$-linearity of S_{Q} follows from the linearity of $\mathscr{B}_{\mathsf{Q}}$ and $\mathscr{B}$ is the first coordinate. □

Remark I.14. Since $B_{\mathsf{Q}}(\mathbf{u}, \mathbf{v}) = B_{\mathsf{Q}}(\mathbf{v}, \mathbf{u})$, we get $\langle S_{\mathsf{Q}}(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, S_{\mathsf{Q}}(\mathbf{v}) \rangle$.

CHAPTER II

TENSOR AND NUMERICAL CURVATURES OF HYPERSURFACES

In the first part of this chapter, we recall the theory of hypersurfaces in the Euclidean space $\mathbb{R}^d$. We recall the definitions of the "tensor curvature" of such hypersurfaces, namely the shape operator and the second fundamental form as well as the "numerical curvatures" known as the principal curvatures. These are invariants of Euclidean geometry given by the standard inner product and are invariant under rigid motions of $\mathbb{R}^d$. This material is classical and standard (see e.g. [24]).

In the next section, we study real hypersurfaces in the Hermitian space $\mathbb{C}^n$, where the geometry is given by the standard Hermitian inner product. We describe the decomposition of the tensor curvature into smaller pieces invariant under the smaller group of symmetries of $\mathbb{C}^n$ given by holomorphic isometries. We define the corresponding numerical curvatures. This includes the quantities classically known as "eigenvalues of the Levi form" (which we call *Levi principal curvatures*) as well as other quantities.

II.1. Hypersurfaces in $\mathbb{R}^d$

Let M be an oriented hypersurface in $\mathbb{R}^d$ and let $p \in M$. Recall from Proposition I.3, that the tangent space T_pM of M at p is a $(d-1)$ dimensional linear subspace of $\mathbb{R}^d$. The standard inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^d$ restricts to T_pM, making T_pM an inner product space. The quadratic form $\mathsf{I}_p : T_pM \to \mathbb{R}$, defined as

$$\mathsf{I}_p(\mathbf{u}) = \langle \mathbf{u}, \mathbf{u} \rangle = |\mathbf{u}|^2 \geq 0, \tag{II.1}$$

for $\mathbf{u} \in T_pM$ is called the *first fundamental form* of M at p. Since T_pM is a $(d-1)$ dimensional vector space, its orthogonal complement in $\mathbb{R}^d$, given by $T_pM^{\perp} = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{t} \rangle = 0 \text{ for each } \mathbf{t} \in T_pM\}$, is a one dimensional subspace of $\mathbb{R}^d$. So there are exactly two unit vectors in $T_pM^{\perp}$. By the definition of orientation (cf. Definition I.6), each tangent space T_pM has an orientation $\mathscr{O}_p$ which varies continuously as p varies over M, that is, for each $q \in M$, there is a open set

$U \subset \mathbb{R}^d$ containing q and continuous linearly independent vector fields $\mathbf{e_1}, \ldots, \mathbf{e_{n-1}}$ such that for each $p \in U \cap M$, we have $(\mathbf{e_1}(p), \ldots, \mathbf{e_{n-1}}(p)) \in \mathscr{O}_p$. We choose a unit normal vector $N(p)$ in $T_pM^\perp$ such that for a positively oriented ordered basis $(\mathbf{e_1}(p), \ldots, \mathbf{e_{d-1}}(\mathbf{p})) \in \mathscr{O}_p$ of T_pM, the ordered basis $(N(p), \mathbf{e_1}(p), \ldots, \mathbf{e_{d-1}}(p))$ of $\mathbb{R}^d$ is also positively oriented. We will now show that this choice of the unit vector does not depend on the choice of the positively oriented basis of T_pM. If $(\mathbf{f_1}, \ldots, \mathbf{f_{d-1}})$ is another positively oriented basis of T_pM, then there is a unique linear operator $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi(N(p)) = N(p)$ and $\Phi(\mathbf{e_j}) = \mathbf{f_j}$ for each $1 \leq j \leq d-1$. Since $N(p) \in T_pM^\perp$, the matrix of Φ with respect to the basis $(N(p), \mathbf{e_1}, \ldots, \mathbf{e_{d-1}})$ has the form

$$A = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & A' \end{bmatrix},$$

where $\mathbf{0} \in \mathbb{R}^{d-1}$ is the zero column vector and A' is the matrix of the linear operator Φ restricted to T_pM. Note that the restriction of Φ to T_pM is orientation preserving, so we get $\det(A) = \det(A') > 0$. Hence Φ is orientation preserving. This defines a map $N : M \to \mathbb{R}^d$ given by $p \mapsto N(p)$ called the unit normal vector field of M. Note that N takes its values in the unit sphere $S^{d-1} \subset \mathbb{R}^d$ and hence N can also be thought as a map $N : M \to S^{d-1}$, which is called the *Gauss map* of M. The tangent space T_pM can now be written as

$$T_pM = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, N(p) \rangle = 0\}. \tag{II.2}$$

We will give an explicit formula for the Gauss map in terms of the local parametrization using the notion of generalized cross product in $\mathbb{R}^d$. Let $\mathbf{v_1}, \ldots, \mathbf{v_{d-1}}$ be vectors in $\mathbb{R}^d$ such that for each $1 \leq j \leq d-1$, we have $\mathbf{v_j} = (v_j^1, \ldots, v_j^d)$. The *cross product* of vectors $\mathbf{v_1}, \ldots, \mathbf{v_{d-1}}$ is defined as

$$\mathbf{v_1} \times \cdots \times \mathbf{v_{d-1}} = \det \begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_{d-1}^1 & \mathbf{e_1} \\ v_1^2 & v_2^2 & \cdots & v_{d-1}^2 & \mathbf{e_2} \\ \vdots & \vdots & & \vdots & \vdots \\ v_1^n & v_2^n & \cdots & v_{d-1}^n & \mathbf{e_d} \end{bmatrix}, \tag{II.3}$$

where $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$ is the standard basis of $\mathbb{R}^d$ and det is the formal determinant calculated by Laplace expansion about the last column. For $d = 2$, the cross product of $\mathbf{v_1}$ is given by $J\mathbf{v_1}$, where in the standard basis of $\mathbb{R}^2$, the matrix of J is given by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{II.4}$$

After expanding (II.3) about the last column, we get

$$\mathbf{v_1} \times \cdots \times \mathbf{v_{d-1}} = \sum_{i=1}^{d} (-1)^{d-i} \det(\hat{\mathbf{v}}_{\mathbf{1}}^i, \ldots, \hat{\mathbf{v}}_{\mathbf{d-1}}^i) \mathbf{e_i}, \tag{II.5}$$

where $\hat{\mathbf{v}}_{\mathbf{j}}^i = (v_j^1, \ldots, v_j^{i-1}, v_j^{i+1}, \ldots, v_j^d) \in \mathbb{R}^{d-1}$ and $\det(\hat{\mathbf{v}}_{\mathbf{1}}^i, \ldots, \hat{\mathbf{v}}_{\mathbf{d-1}}^i)$ is the determinant of the matrix whose j^{th} column vector is $\hat{\mathbf{v}}_{\mathbf{j}}^i \in \mathbb{R}^{d-1}$ for $1 \leq j \leq d-1$ and $1 \leq i \leq d$. Note that if $\mathbf{v} = (v^1, \ldots, v^d) \in \mathbb{R}^d$, then using (II.5) and Laplace expansion for determinants, we get

$$\langle \mathbf{v_1} \times \cdots \times \mathbf{v_{d-1}}, \mathbf{v} \rangle = \sum_{i=1}^{d} (-1)^{d-i} \det(\hat{\mathbf{v}}_{\mathbf{1}}^i, \ldots, \hat{\mathbf{v}}_{\mathbf{d-1}}^i) v^i = \det(\mathbf{v_1}, \ldots, \mathbf{v_{d-1}}, \mathbf{v}). \tag{II.6}$$

In the next proposition, we give an explicit formula for the Gauss map, which will also show that the Gauss map is smooth.

Proposition II.1. *Let $W \subset \mathbb{R}^{d-1}$ be an open set and let $\varphi : W \to M$ be a parametrization of M. Then there is a choice of sign $\pm$ on W such that for each $q \in W$,*

$$N(p) = \pm \begin{cases} \dfrac{J\varphi_1}{|\varphi_1|}, & d = 2 \\ \dfrac{\varphi_1 \times \cdots \times \varphi_{d-1}}{|\varphi_1 \times \cdots \times \varphi_{d-1}|}, & d \geq 3 \end{cases} \tag{II.7}$$

where $p = \varphi(q)$, the linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (II.4)*, and $\varphi_i = \dfrac{\partial \varphi}{\partial x_i}(q)$ for $1 \leq i \leq d-1$.*

Proof. Recall from Corollary I.4 that $\{\varphi_1, \ldots, \varphi_{d-1}\}$ forms a basis of T_pM. Clearly $N(p)$ defined by (II.7) is a unit vector. Using (II.6), for each $1 \leq j \leq d-1$, we get $\langle \varphi_1 \times \cdots \times \varphi_{d-1}, \varphi_j \rangle = \det(\varphi_1, \cdots, \varphi_{d-1}, \varphi_j) = 0$. So $N(p)$ is orthogonal to each φ_i for each $1 \leq i \leq d-1$ and hence $N(p)$ is a unit vector in $T_pM^{\perp}$. Since φ is a smooth map, each component of $N(p)$ in the expansion of the form (II.5) is also smooth. Hence $N : M \to S^{d-1}$ is a smooth map. The orientation of M gives an orientation of the tangent space T_pM which varies continuously with p. If the ordered basis $(\varphi_1, \ldots, \varphi_{d-1})$ of T_pM has the same (resp. opposite) orientation as the orientation of M, then choose the sign of N so that the ordered basis $(N(p), \varphi_1, \ldots, \varphi_{d-1})$ of $\mathbb{R}^d$ is positively (resp. negatively) oriented. By the continuity of N, it is easy to see that the sign of N stays the same for each q in W. □

From the above proof, we get that the Gauss map $N : M \to S^{d-1}$ is smooth, and hence the differential of N at p is a linear map given by $DN_p : T_pM \to T_{N(p)}S^{d-1}$. Note that both T_pM and $T_{N(p)}S^{d-1}$ are $(d-1)$ dimensional vector subspaces of $\mathbb{R}^d$ and are also the orthogonal complement of the vector $N(p)$. Hence, T_pM and $T_{N(p)}S^{d-1}$ are the same as vector subspaces of $\mathbb{R}^d$ and this gives us that DN_p is a linear operator on T_pM.

Definition II.2. The linear map $\mathsf{S}_p : T_pM \to T_pM$, defined by

$$\mathsf{S}_p(\mathbf{u}) = -DN_p(\mathbf{u}) \tag{II.8}$$

for $\mathbf{u} \in T_pM$, is called the *shape operator* of M at p. The real quadratic form on T_pM defined by

$$\mathsf{II}_p(\mathbf{u}) = \langle \mathsf{S}_p(\mathbf{u}), \mathbf{u} \rangle \tag{II.9}$$

is called the *second fundamental form* of M at p. The function $K_G : M \to \mathbb{R}$ defined by

$$K_G(p) = \det \mathsf{S}_p \tag{II.10}$$

is called the *Gaussian curvature* of M.

Let M and M' be two hypersurfaces in $\mathbb{R}^d$, then a smooth bijection $F : M \to M'$ is called a *(Riemannian) isometry* if for every $p \in M$, the map $DF_p : T_pM \to T_{F(p)}M'$ is a linear isometry of inner product spaces, that is, for each $\mathbf{u} \in T_pM$, we have $|DF_p(\mathbf{u})| = |\mathbf{u}|$. Note that such an F need not extend to an isometry of $\mathbb{R}^d$. The Gaussian curvature is invariant under orientation preserving isometries of hypersurface (under all isometries if d is odd). For $d = 3$, this is Gauss' Theorema Egregium [9] and the general case can be found in [24].

The next theorem is well-known and describes the structure of the second fundamental form of a smooth oriented hypersurface in $\mathbb{R}^d$.

Theorem 1. *Let M be a smooth oriented hypersurface in $\mathbb{R}^d$ and let $p \in M$. Then there is an orthonormal basis $\{\mathbf{x_1}, \ldots, \mathbf{x_{d-1}}\}$ of T_pM and scalars $\kappa_1, \ldots, \kappa_{d-1}$ such that for each $\mathbf{u} = \sum_{j=1}^{d-1} u_j \mathbf{x_j}$ in T_pM, we have*

$$\mathsf{II}_p(\mathbf{u}) = \sum_{j=1}^{d-1} \kappa_j u_j^2.$$

Moreover, the Gaussian curvature of M at p is given by

$$K_G = \kappa_1 \cdots \kappa_{d-1}. \tag{II.11}$$

The vectors in the orthonormal basis $\{\mathbf{x_1}, \ldots, \mathbf{x_{d-1}}\}$ of T_pM are called the *principal directions* of M at p and the scalars κ_j are called the *principal curvatures* of M at p. In the proof, we will see that S_p is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$ on T_pM and principal directions are the eigenvectors of S_p and the principal curvatures are the eigenvalues of S_p.

Proof. Let $W \subset \mathbb{R}^{d-1}$ be an open set and let $\varphi : W \to M$ be a parametrization of M near $p \in M$. Let $q \in W$ be such that $\varphi(q) = p$. Then by Corollary I.4, we know that $\{\varphi_1, \ldots, \varphi_{d-1}\}$ forms a basis of T_pM, where $\varphi_i = \partial \varphi / \partial x_i(q)$ for $1 \leq i \leq d-1$.We claim that S_p is self-adjoint. Since $\mathsf{S}_p = -DN_p$ is linear, it suffices to show $\langle \mathsf{S}_p(\varphi_j), \varphi_k \rangle = \langle \mathsf{S}_p(\varphi_k), \varphi_j \rangle$, for each $1 \leq j,k \leq d-1$. Let $\varepsilon > 0$ be small enough so that for each $1 \leq i \leq d-1$, the curves $\beta_j : (-\varepsilon, \varepsilon) \to \mathbb{R}^{d-1}$ defined by $\beta_j(t) = t\mathbf{e_j} + q$, where $\{\mathbf{e_1}, \ldots, \mathbf{e_{d-1}}\}$ is the standard basis of $\mathbb{R}^{d-1}$, lies in W. Then define the curves $\alpha_j : (-\varepsilon, \varepsilon) \to M$ by $\alpha_j = \varphi \circ \beta_j$ for each $1 \leq j \leq d-1$. Then we have, $\alpha_j(0) = p$ and $\alpha_j'(0) = \varphi_j$ for each $1 \leq j \leq d-1$. Since N is orthogonal to the tangent space of M at each point, for each $t \in (-\varepsilon, \varepsilon)$ and for each $1 \leq j,k \leq d-1$, we have $0 = \big\langle (N \circ \varphi)(\beta_j(t)), \varphi_k(\beta_j(t)) \big\rangle = \big\langle (N \circ \alpha_j)(t), \varphi_k(\beta_j(t)) \big\rangle$. Differentiating at $t = 0$ gives us

$$\big\langle DN_p(\varphi_j), \varphi_k \big\rangle + \big\langle N(p), \varphi_{kj} \big\rangle = 0, \tag{II.12}$$

where $\varphi_{kj} = \dfrac{\partial^2 \varphi}{\partial x_j \partial x_k}(q)$. Similarly, differentiating $\left\langle (N \circ \alpha_k)(t), \varphi_j(\beta_k(t)) \right\rangle = 0$ at $t = 0$ gives us

$$\left\langle DN_p(\varphi_k), \varphi_j \right\rangle + \left\langle N(p), \varphi_{jk} \right\rangle = 0. \tag{II.13}$$

Since φ is smooth, $\varphi_{jk} = \varphi_{kj}$ and comparing (II.12) and (II.13), we get

$$\left\langle -DN_p(\varphi_j), \varphi_k \right\rangle = \left\langle N(p), \varphi_{kj} \right\rangle = \left\langle -DN_p(\varphi_k), \varphi_j \right\rangle.$$

Hence S_p is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$ and by the spectral theorem for self-adjoint operators in a real inner product space, there is an orthonormal basis $\{\mathbf{x_1}, \dots, \mathbf{x_{d-1}}\}$ of T_pM, consisting of eigenvectors of S_p, and the eigenvalues κ_j's of S_p are real. If $\mathbf{u} = \sum_{j=1}^{d-1} u_j \mathbf{x_j} \in T_pM$ and $\mathsf{S}_p(\mathbf{x_j}) = \kappa_j \mathbf{x_j}$ for $1 \le j \le d-1$, then we have

$$\mathsf{II}_p(\mathbf{u}) = \left\langle \mathsf{S}_p \left(\sum_{j=1}^{d-1} u_j \mathbf{x_j} \right), \sum_{k=1}^{d-1} u_k \mathbf{x_k} \right\rangle = \sum_{j,k=1}^{d-1} u_j u_k \left\langle \mathsf{S}_p(\mathbf{x_j}), \mathbf{x_k} \right\rangle = \sum_{j,k=1}^{d-1} u_j u_k \left\langle \kappa_j \mathbf{x_j}, \mathbf{x_k} \right\rangle = \sum_{j=1}^{d-1} \kappa_j u_j^2.$$

Since the shape operator is self-adjoint, $\det \mathsf{S}_p$ is equal to the product of its eigenvalues, and hence we get $K_G = \det \mathsf{S}_p = \kappa_1 \cdots \kappa_{d-1}$. $\square$

Corollary II.3. *The matrix of the second fundamental form with respect to the basis* $\{\varphi_1, \dots, \varphi_{n-1}\}$ *of* T_pM *has the* $(j,k)^{th}$ *entry given by* $h_{jk} = \langle N(p), \varphi_{jk} \rangle$.

Proof. With respect to the basis $\{\varphi_1, \dots, \varphi_{n-1}\}$ of T_pM, the $(j,k)^{th}$ entry of the matrix of the second fundamental form is $h_{jk} = \left\langle \mathsf{S}_p(\varphi_j), \varphi_k \right\rangle = \left\langle -DN_p(\varphi_j), \varphi_k \right\rangle$. From (II.12), we know that $\left\langle -DN_p(\varphi_j), \varphi_k \right\rangle = \langle N(p), \varphi_{jk} \rangle$. Hence $h_{jk} = \langle N(p), \varphi_{jk} \rangle$. $\square$

II.2. Hypersurfaces in $\mathbb{C}^n$

II.2.1. The complex tangent space

Let M be a real oriented hypersurface in $\mathbb{C}^n$, that is, under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, there is a real inner product structure on $\mathbb{C}^n$ given by (I.6) and the hypersurface M can be thought of as an oriented hypersurface in $\mathbb{R}^{2n}$. Therefore all the considerations of the previous section apply to M for $d = 2n$. Consequently, we have at each point $p \in M$,

1. a unit normal $N(p) \in T_pM^{\perp}$ which defines the smooth Gauss map $N : M \to S^{2n-1}$,
2. its differential, the shape operator $\mathsf{S}_p : T_pM \to T_pM$, and
3. the second fundamental form $\mathsf{II}_p(\boldsymbol{\zeta}) = \langle \mathsf{S}_p(\boldsymbol{\zeta}), \boldsymbol{\zeta} \rangle$, $\boldsymbol{\zeta} \in T_pM$.

We now concentrate on the extra structure that arises due to the fact that the ambient space $\mathbb{C}^n$ is also a complex vector space and the fact that the real inner product $\langle \cdot, \cdot \rangle$ is the real part of a Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$. We define the *complex tangent space* H_pM at a point $p \in M$ by the formula

$$H_pM = \{\boldsymbol{\xi} \in \mathbb{C}^n \mid (\boldsymbol{\xi}, N(p)) = 0\}, \tag{II.14}$$

where $(\cdot, \cdot)$ is the standard Hermitian inner product of $\mathbb{C}^n$. We then have the following properties of the complex tangent space H_pM:

1. H_pM is a complex vector space of complex dimension $(n-1)$ since it is the orthogonal complement of the vector $N(p) \in \mathbb{C}^n$ with respect to the Hermitian structure.

2. The Hermitian inner product of $\mathbb{C}^n$ restricts to H_pM. This makes H_pM an Hermitian inner product space (this maybe thought of as a complex analog of the first fundamental form).

3. Since $\langle \boldsymbol{\xi}, N(p) \rangle = \mathrm{Re}\,(\boldsymbol{\xi}, N(p))$, we get that H_pM is contained in T_pM. As a real vector subspace of T_pM, the complex tangent space H_pM has co-dimension 1. Hence for each $p \in M$, there is a one dimensional subspace X_pM of T_pM such that

$$T_pM = H_pM \oplus X_pM \tag{II.15}$$

is a real orthogonal decomposition of T_pM. The subspace X_pM is called the *characteristic direction* of M at p.

4. Note that from (II.2) and (II.14), we have $iN(p) \in T_pM$ but $iN(p) \notin H_pM$ and hence from (II.15) we get that $iN(p) \in X_pM$. The vector $T(p) = iN(p)$ is a unit vector in X_pM is called the *characteristic vector* of M at p. This allows us to define a vector field $T : M \to \mathbb{C}^n$ on M, called

the *characteristic vector field*, defined as

$$T(p) = iN(p). \tag{II.16}$$

Since the unit normal vector field N is smooth, so is the characteristic vector field T.

II.2.2. Decomposition of the second fundamental form

In this subsection, we study how the second fundamental form decomposes into several pieces invariant under the Hermitian structure of $\mathbb{C}^n$ with respect to the decomposition (II.15) of T_pM. As a preliminary, we will need the following notions from Hermitian linear algebra.

Definition II.4. Let $(H, (\cdot,\cdot))$ be a Hermitian inner product space. A map $\chi : H \to H$ is a called a *conjugation* on H if for each $\boldsymbol{\xi} \in H$, we have

1. χ is conjugate-linear, that is, χ is real-linear and $\chi(\lambda\boldsymbol{\xi}) = \overline{\lambda}\chi(\boldsymbol{\xi})$ for each $\lambda \in \mathbb{C}$,
2. χ is an isometry, that is, $|\chi\boldsymbol{\xi}| = |\boldsymbol{\xi}|$, and
3. χ is an involution, that is, $\chi^2(\boldsymbol{\xi}) = \boldsymbol{\xi}$.

We say that an operator $\Sigma : H \to H$ is *χ-symmetric* if $\Sigma = \chi\Sigma^*\chi$, where Σ^* is the adjoint of Σ with respect to the Hermitian inner product $(\cdot,\cdot)$ of H.

Example II.5. In $\mathbb{C}^n$, with the standard Hermitian structure, the map $(z_1,\dots,z_n) \mapsto (\bar{z}_1,\dots,\bar{z}_n)$ is a conjugation, where $\bar{z}_j$ is the complex conjugate of z_j for $1 \le j \le n$. A $\mathbb{C}$-linear operator $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is symmetric with respect to this conjugation if and only if the matrix of Φ with respect to the standard basis of $\mathbb{C}^n$ is a symmetric matrix.

Note that the above example shows that every Hermitian space admits at least one conjugation since it is linearly isometric to $\mathbb{C}^n$.

Proposition II.6. *If χ is a conjugation on a Hermitian space $(H, (\cdot,\cdot))$, then for any $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ in H, we have $(\chi\boldsymbol{\xi}_1, \chi\boldsymbol{\xi}_2) = \overline{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} = (\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)$.*

Proof. Since χ is conjugate linear and an isometry, the polarization identity on H gives us

$$\begin{aligned}(\chi\xi_1,\chi\xi_2) &= |\chi\xi_1+\chi\xi_2|^2 + i\,|\chi\xi_1+i\chi\xi_2|^2 - |\chi\xi_1-\chi\xi_2|^2 - i\,|\chi\xi_1-i\chi\xi_2|^2 \\ &= |\chi(\xi_1+\xi_2)|^2 + i\,|\chi(\xi_1-i\xi_2)|^2 - |\chi(\xi_1-\xi_2)|^2 - i\,|\chi(\xi_1+i\xi_2)|^2 \\ &= |\xi_1+\xi_2|^2 + i\,|\xi_1-i\xi_2|^2 - |\xi_1-\xi_2|^2 - i\,|\xi_1+i\xi_2|^2 \\ &= \overline{(\xi_1,\xi_2)} = (\xi_2,\xi_1)\,.\end{aligned}$$

□

We are now ready to state our main theorem which describes the structure of the second fundamental form of a smooth oriented real hypersurface in $\mathbb{C}^n$.

Theorem 2. *Let M be a smooth oriented real hypersurface in $\mathbb{C}^n$ and let $p \in M$. Then the second fundamental form II_p of M at p decomposes uniquely as*

$$\mathrm{II}_p(\zeta) = -2\,(L(\xi),\xi) + 2\mathrm{Re}\left(\Sigma_\chi(\xi),\chi\xi\right) + 2\eta W(\xi) + \eta^2 K_T, \tag{II.17}$$

where

1. $\zeta \in T_pM$ and $\zeta = \xi + \eta T(p)$, where $\xi \in H_pM$, $\eta \in \mathbb{R}$, and $T(p)$ is the characteristic vector of M at p,

2. $L : H_pM \to H_pM$ is a linear operator which is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM,

3. χ is an arbitrarily chosen conjugation on H_pM and Σ_χ is a χ-symmetric operator on H_pM,

4. $W : H_pM \to \mathbb{R}$ is a $\mathbb{R}$-linear functional, and

5. $K_T \in \mathbb{R}$.

Further, the Hermitian form $\xi \mapsto (L(\xi),\xi)$ on H_pM, the complex quadratic form $\xi \mapsto \left(\Sigma_\chi(\xi),\chi\xi\right)$, the linear functional $\xi \mapsto W(\xi)$, and the number K_T are invariant under holomorphic isometries of $\mathbb{C}^n$.

While the presence of a decomposition as above is implicitly understood in the literature on several complex variables as well as on CR geometry, we give here a unified treatment of this result. We also give names to the various parts of the decomposition.

1. The negative sign in front of the first term is chosen so that the real quadratic form $\mathsf{L}(\boldsymbol{\xi}) = (L(\boldsymbol{\xi}), \boldsymbol{\xi})$ on H_pM, which is called the *Levi form* of M at p, has the sign which is conventional in complex analysis (see [17, 21]). We will see in the proof that L is self-adjoint with respect to the Hermitian inner product $(\cdot, \cdot)$ of H_pM. Hence by the spectral theorem for self-adjoint operators on Hermitian spaces, there is an orthonormal basis $\{\mathbf{w_1}, \ldots, \mathbf{w_{n-1}}\}$ of H_pM consisting of eigenvectors of L, such that for $\boldsymbol{\xi} = \sum_{j=1}^{n-1} \xi_j \mathbf{w_j} \in H_pM$, we have $\mathsf{L}(\boldsymbol{\xi}) = \sum_{j=1}^{n-1} \lambda_j^2 \left|\xi_j\right|^2$, where λ_j's are the eigenvalues of L. We will call the eigenvectors of L the *Levi principal directions* of M at p and the eigenvalues of L the *Levi principal curvatures* of M at p.

The *signature* of the Levi form, that is, the numbers of positive, negative, and zero eigenvalues of the Levi form, is a *biholomorphic invariant*, that is, if M and M' are hypersurfaces in $\mathbb{C}^n$, with $p \in M$ and $q \in M'$, and $\Phi : U \to V$ is a biholomorphic map of a neighborhood U of p to a neighborhood V of q, such that $\Phi(p) = q$ and $\Phi(U \cap M) = V \cap M'$, then the Levi form of M at p and the Levi form of M' at q have the same signature. A hypersurface is said to be *pseudoconvex* (resp. *strongly pseudoconvex*) if all the Levi principal curvatures are non-negative (resp. positive) at each point of the hypersurface. A domain is (strongly) pseudoconvex if its boundary is (strongly) pseudoconvex. These notions are central to complex analysis (see [21]).

In the several complex variables literature, the quantities that we call Levi principal curvatures are usually referred to as the "eigenvalues of the Levi form" or "Levi eigenvalues". These play a crucial role in the quantitative aspects of the existence and regularity of solutions of the Cauchy-Riemann equations (see e.g. [11, 16, 12]).

The determinant of the linear operator $L : H_pM \to H_pM$ will be called the Hörmander curvature of M at p, written as

$$K_H = \det L = \lambda_1 \cdots \lambda_{n-1}. \tag{II.18}$$

The name Hörmander curvature pays tribute to [12, Theorem 3.5.1]. Authors have also called it *(Total) Levi curvature* (see [14]). Recall that a Bergman space $A^2(\Omega)$ of a domain $\Omega \subset \mathbb{C}^n$ is the closed subspace of $L^2(\Omega)$ consisting of holomorphic functions. The orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$ is called the Bergman projection and maybe represented by an integral formula

$$(Pf)(z) = \int_\Omega K(z,w)f(w)dV(w),$$

where dV denotes the Lebesgure measure and $K : \Omega \times \Omega \to \mathbb{C}$ is the *Bergman kernel* of Ω, one of the most important invariants of a domain in complex analysis. The theorem of Hörmander states that if $\Omega \subset \mathbb{C}^n$ is a smoothly bounded strongly pseudoconvex domain, p is a point on the boundary of Ω, K is its Bergman kernel, and $d_\Omega(z)$ is the distance from a point $z \in \Omega$ to the boundary of Ω, then

$$d_\Omega(z)^{n+1}K(z,z) \to \frac{n!}{4\pi^n} K_H(p) \text{ as } z \to p.$$

2. The complex quadratic form $\mathsf{S}(\boldsymbol{\xi}) = \left(\Sigma_\chi(\boldsymbol{\xi}), \chi\boldsymbol{\xi}\right)$ on H_pM will be called the *complex-symmetric fundamental form* of M at p. Recall that the *singular values* of an operator A on a Hermitian space H are the positive square roots of the eigenvalues of the self-adjoint non-negative operator AA^*. We will also see later (Proposition II.8) that there is an orthonormal basis $\{\mathbf{z_1}, \dots, \mathbf{z_{n-1}}\}$ of H_pM such that for $\boldsymbol{\xi} = \sum_{j=1}^{n-1} \xi_j \mathbf{z_j} \in H_pM$, we have

$$\mathsf{S}(\boldsymbol{\xi}) = \sigma_j \xi_j^2, \tag{II.19}$$

where σ_j are the singular values of the operator Σ_χ. The numbers σ_j's are independent of the choice of the conjugation (see Corollary II.9). The numbers $\sigma_1, \dots, \sigma_{n-1}$ will be called the *complex-symmetric principal curvatures* of M at p and the vectors in the orthonormal basis $\{\mathbf{z_1}, \dots, \mathbf{z_{n-1}}\}$ of H_pM will be called the *complex-symmetric principal directions* of M at p.

Even though the complex-symmetric principal curvatures occur in many computations (for example in the recent paper [3]), their geometric and function theoretic significance is not well understood.

3. The $\mathbb{R}$-linear functional $W : H_pM \to \mathbb{R}$ is given by

$$W(\boldsymbol{\xi}) = \left\langle \boldsymbol{\xi}, \mathsf{S}_p(T(p)) \right\rangle = \left\langle \mathsf{S}_p(\boldsymbol{\xi}), T(p) \right\rangle, \tag{II.20}$$

where $\langle \cdot, \cdot \rangle$ is the real inner product in T_pM will be called the *skew functional* of M at p. The norm of the skew functional, denoted by

$$K_{\text{skew}} = \|W\|, \tag{II.21}$$

will be called the *skew curvature* of M at p. This is clearly an invariant of the Hermitian geometry of the hypersurface M and in fact the only numerical invariant associated to W.

4. The scalar K_T is given by

$$K_T = \mathsf{II}_p(T(p)) = \left\langle \mathsf{S}_p(T(p)), T(p) \right\rangle \tag{II.22}$$

and is called the *characteristic curvature* of M at p. The geometric properties of the characteristic curvature has been studied by several authors (see e.g. [19]).

5. This theorem provides us with three tensor invariants L, Σ_χ, W and a scalar invariant K_T associated to each point of the hypersurface. From the tensor invariants, we can also extract numerical and vector invariants namely, the Levi principal curvatures and the corresponding Levi principal directions, the complex-symmetric principal curvatures and the corresponding complex-symmetric principal directions, and the skew curvature.

II.3. Proof of Theorem 2

The proof involves the decomposition of the second fundamental form in two stages. In the first stage, we look at the decomposition under the real orthogonal splitting of the tangent space into the complex tangent space and the characteristic direction

$$T_pM = H_pM \oplus X_pM = H_pM \oplus \mathbb{R}T(p),$$

where $T(p)$ is the characteristic vector of M at p. In the second stage, the restriction of the second fundamental form to the complex tangent space H_pM splits into a Hermitian part and the real part of a complex quadratic form. We note that all the constructions here are uniquely determined by the Hermitian structure. This justifies the last statement of the theorem, that is, the various pieces of the decomposition are invariant under holomorphic isometries.

II.3.1. The skew functional and the characteristic curvature

Since $\boldsymbol{\zeta} = \boldsymbol{\xi} + \eta T(p) \in T_pM$, where $\boldsymbol{\xi} \in H_pM$ and $T(p)$ is the characteristic vector of M at p, we have

$$\begin{aligned}\mathsf{II}_p(\boldsymbol{\zeta}) &= \left\langle \mathsf{S}_p(\boldsymbol{\xi} + \eta T(p)), \boldsymbol{\xi} + \eta T(p) \right\rangle \\ &= \left\langle \mathsf{S}_p(\boldsymbol{\xi}), \boldsymbol{\xi} \right\rangle + 2\eta \left\langle \mathsf{S}_p(\boldsymbol{\xi}), T(p) \right\rangle + \eta^2 \left\langle \mathsf{S}_p(T(p)), T(p) \right\rangle.\end{aligned}$$

Defining W by (II.20) and K_T by (II.22), this becomes

$$\mathsf{II}_p(\boldsymbol{\zeta}) = \mathsf{II}_p(\boldsymbol{\xi}) + 2\eta W(\boldsymbol{\xi}) + \eta^2 K_T. \tag{II.23}$$

The first term of this sum is the restriction of the second fundamental form to the complex tangent space H_pM. It is therfore a real quadratic form on the complex vector space H_pM with a Hermitian inner product. The proof Theorem 2 will be completed by showing that for each $\boldsymbol{\xi} \in H_pM$, we have

$$\mathsf{II}_p(\boldsymbol{\xi}) = -2\left(L(\boldsymbol{\xi}), \boldsymbol{\xi}\right) + 2\mathrm{Re}\left(\Sigma_\chi(\boldsymbol{\xi}), \chi\boldsymbol{\xi}\right). \tag{II.24}$$

II.3.2. The Levi form

For each $\boldsymbol{\xi} \in H_pM$, we have the following unique decomposition

$$\mathsf{II}_p(\boldsymbol{\xi}) = -2\mathsf{L}(\boldsymbol{\xi}) + 2\mathsf{R}(\boldsymbol{\xi}), \tag{II.25}$$

where $\mathsf{L}(\boldsymbol{\xi}) = \dfrac{-1}{4}\left(\mathsf{II}_p(\boldsymbol{\xi}) + \mathsf{II}_p(i\boldsymbol{\xi})\right)$ and $\mathsf{R}(\boldsymbol{\xi}) = \dfrac{1}{4}\left(\mathsf{II}_p(\boldsymbol{\xi}) - \mathsf{II}_p(i\boldsymbol{\xi})\right)$. Note that $\mathsf{L}(i\boldsymbol{\xi}) = \mathsf{L}(\boldsymbol{\xi})$. We now define a bilinear form $\mathscr{L}(\boldsymbol{\xi_1}, \boldsymbol{\xi_2}) = \dfrac{1}{4}\left(\mathsf{L}(\boldsymbol{\xi_1} + \boldsymbol{\xi_2}) + i\mathsf{L}(\boldsymbol{\xi_1} + i\boldsymbol{\xi_2}) - \mathsf{L}(\boldsymbol{\xi_1} - \boldsymbol{\xi_2}) - i\mathsf{L}(\boldsymbol{\xi_1} - i\boldsymbol{\xi_2})\right)$.

Then $\mathscr{L}(\boldsymbol{\xi},\boldsymbol{\xi}) = \mathsf{L}(\boldsymbol{\xi})$ and $\mathscr{L}$ is Hermitian symmetric, that is, for every $\boldsymbol{\xi_1},\boldsymbol{\xi_2} \in H_pM$, we have $\mathscr{L}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = \overline{\mathscr{L}(\boldsymbol{\xi_2},\boldsymbol{\xi_1})}$ because

$$\begin{aligned}
4\mathscr{L}(\boldsymbol{\xi_2},\boldsymbol{\xi_1}) &= \mathsf{L}(\boldsymbol{\xi_2}+\boldsymbol{\xi_1}) + i\mathsf{L}(\boldsymbol{\xi_2}+i\boldsymbol{\xi_1}) - \mathsf{L}(\boldsymbol{\xi_2}-\boldsymbol{\xi_1}) - i\mathsf{L}(\boldsymbol{\xi_2}-i\boldsymbol{\xi_1}) \\
&= \mathsf{L}(\boldsymbol{\xi_1}+\boldsymbol{\xi_2}) + i\mathsf{L}(i(\boldsymbol{\xi_1}-i\boldsymbol{\xi_2})) - \mathsf{L}(\boldsymbol{\xi_1}-\boldsymbol{\xi_2}) - i\mathsf{L}(-i(\boldsymbol{\xi_1}+i\boldsymbol{\xi_2})) \\
&= \mathsf{L}(\boldsymbol{\xi_1}+\boldsymbol{\xi_2}) + i\mathsf{L}(\boldsymbol{\xi_1}-i\boldsymbol{\xi_2}) - \mathsf{L}(\boldsymbol{\xi_1}-\boldsymbol{\xi_2}) - i\mathsf{L}(\boldsymbol{\xi_1}+i\boldsymbol{\xi_2}) \\
&= 4\overline{\mathscr{L}(\boldsymbol{\xi_1},\boldsymbol{\xi_2})}.
\end{aligned}$$

If $\boldsymbol{\omega} \in H_pM$ is fixed, then consider the linear functional $\Phi_\omega : H_pM \to \mathbb{C}$ given by $\Phi_\omega(\boldsymbol{\xi}) = \mathscr{L}(\boldsymbol{\xi},\boldsymbol{\omega})$. Then by Riesz representation theorem, there is a unique element $L(\boldsymbol{\omega}) \in H_pM$ such that for each $\boldsymbol{\xi} \in H_pM$, we have $\Phi_\omega(\boldsymbol{\xi}) = \mathscr{L}(\boldsymbol{\xi},\boldsymbol{\omega}) = (\boldsymbol{\xi},L(\boldsymbol{\omega}))$. It now easily follows that the map $L : H_pM \to H_pM$ is $\mathbb{C}$-linear. For each $\boldsymbol{\xi_1},\boldsymbol{\xi_2} \in H_pM$, we have $(L(\boldsymbol{\xi_1}),\boldsymbol{\xi_2}) = \overline{(\boldsymbol{\xi_2},L(\boldsymbol{\xi_1}))} = \overline{\mathscr{L}(\boldsymbol{\xi_2},\boldsymbol{\xi_1})} = \mathscr{L}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = (\boldsymbol{\xi_1},L(\boldsymbol{\xi_2}))$ and hence L is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM and for each $\boldsymbol{\xi} \in H_pM$, we have

$$\mathsf{L}(\boldsymbol{\xi}) = \mathscr{L}(\boldsymbol{\xi},\boldsymbol{\xi}) = (L(\boldsymbol{\xi}),\boldsymbol{\xi}). \tag{II.26}$$

II.3.3. The complex-symmetric fundamental form

Now we look at the second term of (II.25). Note that $\mathsf{R}(i\boldsymbol{\xi}) = -\mathsf{R}(\boldsymbol{\xi})$ and we define $\mathsf{S}(\boldsymbol{\xi}) = \mathsf{R}(\boldsymbol{\xi}) - i\mathsf{R}(e^{i\pi/4}\boldsymbol{\xi})$. Then $\mathsf{R} = \mathrm{Re}\ \mathsf{S}$ and $\mathsf{S}(e^{i\pi/4}\boldsymbol{\xi}) = i\mathsf{S}(\boldsymbol{\xi})$. We will show that S is a complex quadratic form on H_pM. Let $\{\mathbf{z_1},\ldots,\mathbf{z_{n-1}}\}$ be a basis of H_pM as a vector space over $\mathbb{C}$. Then $\{\mathbf{z_1},i\mathbf{z_1},\ldots,\mathbf{z_n},i\mathbf{z_n}\}$ is a basis of H_pM as a vector space over $\mathbb{R}$. Since R is a real quadratic form on H_pM, if $\boldsymbol{\xi} = \sum_{j=1}^{n-1} \xi_j\mathbf{z_j} \in H_pM$ such that $\xi_j = \lambda_j + i\mu_j$ for $1 \le j \le n-1$, then

$$\mathsf{S}(\boldsymbol{\xi}) = \sum_{j,k=1}^{n-1} \left(a_{jk}\lambda_j\lambda_k + 2b_{jk}\lambda_j\mu_k + c_{jk}\mu_j\mu_k\right), \tag{II.27}$$

for some complex numbers a_{jk}, b_{jk}, c_{jk}. Writing $e^{i\pi/4} = \dfrac{(1+i)}{\sqrt{2}}$, we get

$$\begin{aligned}\mathsf{S}(e^{i\pi/4}\boldsymbol{\xi}) = \sum_{j,k=1}^{n} \frac{a_{jk}}{2}(\lambda_j - \mu_j)(\lambda_k - \mu_k) + 2\sum_{j,k=1}^{n} \frac{b_{jk}}{2}(\lambda_j - \mu_j)(\lambda_k + \mu_k) \\ + \sum_{j,k=1}^{n} \frac{c_{jk}}{2}(\lambda_j + \mu_j)(\lambda_k + \mu_k).\end{aligned} \tag{II.28}$$

Comparing the coefficients of $\lambda_j\lambda_k, \lambda_j\mu_k$, and $\mu_j\mu_k$ from (II.27) and (II.28), we get $a_{jk} + c_{jk} = 0$ and $b_{jk} = ia_{jk}$ for $1 \leq j,k \leq n-1$. Hence we can write $\mathsf{S}(\boldsymbol{\xi}) = \sum_{j,k=1}^{n} a_{jk}\xi^j\xi^k$ which shows that S is a complex quadratic form on H_pM. Also, this S is the unique complex quadratic form such that $\mathsf{R}(\boldsymbol{\xi}) = \operatorname{Re}\mathsf{S}(\boldsymbol{\xi})$ for each $\boldsymbol{\xi} \in H_pM$.

Now as in (I.7), we can associate with the complex quadratic form S, the symmetric $\mathbb{C}$-bilinear form S given by $\mathscr{S}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = \dfrac{1}{4}(\mathsf{S}(\boldsymbol{\xi_1}+\boldsymbol{\xi_2}) - \mathsf{S}(\boldsymbol{\xi_1}-\boldsymbol{\xi_2}))$. It follows that $\mathscr{S}(\boldsymbol{\xi},\boldsymbol{\xi}) = \mathsf{S}(\boldsymbol{\xi})$ and $\mathscr{S}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = \mathscr{S}(\boldsymbol{\xi_2},\boldsymbol{\xi_1})$. If $\boldsymbol{\omega} \in H_pM$ is fixed then consider the linear functional $\Psi_{\omega} : H_pM \to \mathbb{C}$ defined as $\Psi_{\omega}(\boldsymbol{\xi}) = \mathscr{S}(\boldsymbol{\xi},\boldsymbol{\omega})$. Then by Riesz representation theorem, there is a unique $\Theta(\boldsymbol{\omega}) \in H_pM$ such that

$$\Psi_{\omega}(\boldsymbol{\xi}) = \mathscr{S}(\boldsymbol{\xi},\boldsymbol{\omega}) = (\boldsymbol{\xi},\Theta(\boldsymbol{\omega})). \tag{II.29}$$

It can be checked easily that $\Theta : H_pM \to H_pM$ is conjugate-linear. Now let $\chi : H_pM \to H_pM$ be any conjugation of H_pM. Then $\Theta\chi : H_pM \to H_pM$ is a $\mathbb{C}$-linear map since it is a composition of two conjugate-linear maps. Now we define

$$\Sigma_{\chi} = (\Theta\chi)^*, \tag{II.30}$$

where $(\Theta\chi)^*$ is the adjoint of $\Theta\chi$ with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM. We claim that Σ_{χ} is χ-symmetric. Using Proposition II.6, we have for each $\boldsymbol{\xi_1},\boldsymbol{\xi_2} \in H_pM$

$$\begin{aligned}(\boldsymbol{\xi_1},\chi\Sigma_{\chi}(\boldsymbol{\xi_2})) = (\Sigma_{\chi}(\boldsymbol{\xi_2}),\chi\boldsymbol{\xi_1}) = \left(\boldsymbol{\xi_2},\Sigma_{\chi}^*\chi(\boldsymbol{\xi_1})\right) = (\boldsymbol{\xi_2},\Theta(\boldsymbol{\xi_1})) \\ = \mathscr{S}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = \mathscr{S}(\boldsymbol{\xi_2},\boldsymbol{\xi_1}) = (\boldsymbol{\xi_1},\Theta(\boldsymbol{\xi_2})) = \left(\boldsymbol{\xi_1},\Sigma_{\chi}^*\chi(\boldsymbol{\xi_2})\right)\end{aligned}$$

which shows that $\chi\Sigma_\chi = \Sigma_\chi^*\chi$ so that Σ_χ is χ-symmetric. Finally, $\mathscr{S}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = (\boldsymbol{\xi_1}, T(\boldsymbol{\xi_2})) = \Big(\boldsymbol{\xi_1}, \Sigma_\chi^*\chi(\boldsymbol{\xi_2})\Big) = \big(\Sigma_\chi(\boldsymbol{\xi_1}), \chi\boldsymbol{\xi_2}\big)$ and hence for each $\boldsymbol{\xi} \in H_pM$, we get

$$\mathsf{R}(\boldsymbol{\xi}) = \mathrm{Re}\mathsf{S}(\boldsymbol{\xi}) = \mathrm{Re}\mathscr{S}(\boldsymbol{\xi},\boldsymbol{\xi}) = \mathrm{Re}\left(\Sigma_\chi(\boldsymbol{\xi}), \chi\boldsymbol{\xi}\right). \tag{II.31}$$

Hence using (II.23), (II.25), (II.26), and (II.31), we get (II.17), which finishes the proof of Theorem 2.

II.3.4. Symmetric operators

We will now justify (II.19) by showing the existence of an orthonormal basis of H_pM which diagonalizes the complex-symmetric fundamental form $\mathsf{S}(\boldsymbol{\xi}) = \left(\Sigma_\chi(\boldsymbol{\xi}), \chi\boldsymbol{\xi}\right)$. This result is known in the matrix analysis literature as *Autonne's theorem* (see [13, Corollary 4.4.4]) but it is difficult to find a coordinate-free presentation, which we will now give.

Lemma II.7. *Let $(H,(\cdot,\cdot))$ be a Hermitian space and let $\Theta : H \to H$ be a conjugate linear map such that the complex linear map $\Theta^2 : H \to H$ is self-adjoint and positive definite with respect to the inner product $(\cdot,\cdot)$. Then there exists $\sigma \geq 0$ and $\mathbf{z} \in H$ such that $\mathbf{z} \neq 0$ and $\Theta\mathbf{z} = \sigma\mathbf{z}$.*

Proof. Let $\lambda \geq 0$ be an eigenvalue of Θ^2 and let the corresponding eigenvector be $\mathbf{v}$. Consider the subspace $P = \mathrm{span}_{\mathbb{C}}\{\Theta\mathbf{v}, \mathbf{v}\}$ of H. If the dimension of P (over $\mathbb{C}$) is 1, then there exist $\mu \in \mathbb{C}$ such that $\Theta\mathbf{v} = \mu\mathbf{v}$. If $\mu = 0$, take $\sigma = 0$ and $\mathbf{z} = \mathbf{v}$. Otherwise, let $\mu = |\mu|\, e^{i\theta}$ and set $\mathbf{z} = e^{i\theta/2}\mathbf{v}$. Since Θ is conjugate linear, we get $\Theta\mathbf{z} = \Theta(e^{i\theta/2}\mathbf{v}) = e^{-i\theta/2}\Theta\mathbf{v} = e^{-i\theta/2}\mu\mathbf{v} = |\mu|\, e^{i\theta/2}\mathbf{v} = |\mu|\,\mathbf{z}$. Hence $\sigma = |\mu| > 0$. On the other hand, if the dimension of P is 2, then set σ to be the positive square root of λ and $\mathbf{z} = \Theta\mathbf{v} + \sigma\mathbf{v}$. Then we have $\Theta\mathbf{z} = \Theta^2\mathbf{v} + \Theta(\sigma\mathbf{v}) = \sigma^2\mathbf{v} + \sigma\Theta\mathbf{v} = \sigma(\sigma\mathbf{v} + \Theta\mathbf{v}) = \sigma\mathbf{z}$. □

Proposition II.8. *Let $(H,(\cdot,\cdot))$ be a Hermitian space and let $\chi : H \to H$ be a conjugation on H. If $\Sigma : H \to H$ is linear and χ-symmetric, then there is an orthonormal basis $\{\mathbf{z_1}, \ldots, \mathbf{z_n}\}$ of H and non-negative numbers $\sigma_1, \ldots, \sigma_n$ such that $\Sigma\chi\mathbf{z_j} = \sigma_j\mathbf{z_j}$ for $1 \leq j \leq n$.*

Proof. The operator $\Theta = \Sigma\chi$ is conjugate linear and $\Theta^2 = \Sigma\chi\Sigma\chi = \Sigma\Sigma^*$ is self-adjoint and positive definite. Then Lemma II.7 ensures existence of $\sigma_1 \geq 0$ and $\mathbf{z} \in H$ such that $\mathbf{z} \neq 0$ and $\Sigma\chi\mathbf{z} = \sigma_1\mathbf{z}$.

We set $\mathbf{z_1} = \mathbf{z}/|\mathbf{z}|$ and $H_1 = \{\boldsymbol{\zeta} \in H \mid (\boldsymbol{\zeta}, \mathbf{z_1}) = 0\}$. Now we claim that H_1 is invariant under Θ. If $\boldsymbol{\zeta} \in H_1$ then $(\Theta\boldsymbol{\zeta}, \mathbf{z_1}) = (\Sigma\chi\boldsymbol{\zeta}, \mathbf{z_1}) = (\chi\boldsymbol{\zeta}, \Sigma^*\mathbf{z_1}) = (\chi\boldsymbol{\zeta}, \chi\Sigma\chi\mathbf{z_1}) = (\chi\boldsymbol{\zeta}, \chi\sigma_1\mathbf{z_1}) = \overline{(\boldsymbol{\zeta}, \sigma_1\mathbf{z_1})} = 0$. Hence we can apply the Lemma again to H_1 to obtain σ_2 and a non-zero vector $\widetilde{\mathbf{z}} \in H_1$ such that $\Sigma\chi\widetilde{\mathbf{z}} = \sigma_2\widetilde{\mathbf{z}}$ and set $\mathbf{z_2} = \widetilde{\mathbf{z}}/|\widetilde{\mathbf{z}}|$. Hence $\{\mathbf{z_1}, \mathbf{z_2}\}$ forms the first two vectors of the orthonormal basis of H. Continuing this way, we obtain an orthonormal basis $\{\mathbf{z_1}, \dots, \mathbf{z_n}\}$ of H with the desired property. □

Corollary II.9. *$\sigma_1, \dots, \sigma_n$ are independent of the choice of the conjugation χ on H.*

Proof. If $\widetilde{\chi}$ is another conjugation on H, then $U = \chi\widetilde{\chi}$ is an isometry and also for every $\boldsymbol{\zeta}, \boldsymbol{\xi} \in H$, we have $(U\boldsymbol{\zeta}, \boldsymbol{\xi}) = (\chi\widetilde{\chi}\boldsymbol{\zeta}, \boldsymbol{\xi}) = (\chi\boldsymbol{\xi}, \widetilde{\chi}\boldsymbol{\zeta}) = (\boldsymbol{\zeta}, \widetilde{\chi}\chi\boldsymbol{\xi})$. Hence $U^* = \widetilde{\chi}\chi = U^{-1}$ and U is unitary. Also, U is both χ-symmetric and $\widetilde{\chi}$-symmetric. If $\Sigma = \Theta\chi$ and $\widetilde{\Sigma} = \Theta\widetilde{\chi}$, then we have $\widetilde{\Sigma} = \Theta\widetilde{\chi} = \Theta\chi\chi\widetilde{\chi} = \Sigma U$. Hence $\widetilde{\Sigma}^*\widetilde{\Sigma} = U^*\Sigma^*\Sigma U$ hence $\widetilde{\Sigma}^*\widetilde{\Sigma}$ and $\Sigma^*\Sigma$ are similar matrices and have same eigenvalues. Hence the result follows. □

Applying Proposition II.8 with $\Sigma = \Sigma^*_\chi$ from (II.30), we get that there exisits an orthonormal basis $\{\mathbf{z_1}, \dots, \mathbf{z_{n-1}}\}$ of H_pM and non-negative numbers $\sigma_1, \dots, \sigma_{n-1}$ such that $\Sigma^*_\chi\chi(\mathbf{z_j}) = \sigma_j\mathbf{z_j}$ for $1 \le j \le n-1$. Therefore for each $\boldsymbol{\xi} = \sum_{j=1}^{n-1} \xi_j\mathbf{z_j}$ in H_pM, the complex-symmetric fundamental form $\mathsf{S}(\boldsymbol{\xi})$ is represented as

$$\mathsf{S}(\boldsymbol{\xi}) = \left(\Sigma_\chi(\boldsymbol{\xi}), \chi\boldsymbol{\xi}\right) = \left(\boldsymbol{\xi}, \Sigma^*_\chi\chi\boldsymbol{\xi}\right) = \left(\sum_{j=1}^{n-1} \xi_j\mathbf{z_j}, \Sigma^*_\chi\chi(\sum_{k=1}^{n-1} \xi_k\mathbf{z_k})\right)$$
$$= \sum_{j,k=1}^{n-1} \xi_j\xi_k\left(\mathbf{z_j}, \sigma_k\mathbf{z_k}\right) = \sum_{j=1}^{n-1} \sigma_j\xi_j^2.$$

CHAPTER III

CALCULATIONS FOR IMPLICIT HYPERSURFACES

In complex analysis, hypersurfaces arise primarily as boundaries of domains. It is traditional and convenient to represent these hypersurfaces in terms of defining functions. In this chapter, we compute the quantities introduced in Chapter II in terms of a defining function and its derivatives. As expected, we recover expressions that are familiar in complex analysis.

Definition III.1 (Defining function)**.** Let M be a smooth hypersurface in $\mathbb{R}^d$ and let $U \subset \mathbb{R}^d$ be an open set containing M. A defining function of M is a smooth function $\rho : U \to \mathbb{R}$ such that $M = \{x \in U \mid \rho(x) = 0\}$ and $|\nabla\rho(x)| \neq 0$ whenever $\rho(x) = 0$.

For each point p in a smooth hypersurface M of $\mathbb{R}^d$, there is an open set U of $\mathbb{R}^d$ containing p, an open set $V \subset \mathbb{R}^d$, and a diffeomorphism $\psi : U \to V$ such that $\psi(U \cap M) = V \cap \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d = 0\}$. If $\pi_d : \mathbb{R}^d \to \mathbb{R}$ is the orthogonal projection map $\pi_d(x) = x_d$, where $x = (x_1, \dots, x_d)$, then $\pi_d \circ \psi : U \to \mathbb{R}$ is a defining function of $U \cap M$. Hence a local defining function always exists. If M is also orientable, using a partition of unity, one can also obtain a (global) defining function of M. More about defining functions can be found in [18, Section 1.2].

Further, let $U \subset \mathbb{R}^d$ be open and let $\rho : U \to \mathbb{R}$ be a smooth function such that $|\nabla\rho(p)| \neq 0$ whenever $\rho(p) = 0$, then the set $M = \{x \in U \mid \rho(x) = 0\}$ is a hypersurface in $\mathbb{R}^d$. To see this, without loss of generality, we can assume that $\partial\rho/\partial x_d(p) \neq 0$. If $p = (p_1, \dots, p_d)$, then by the implicit function theorem, there are open neighborhoods A of $(p_1, \dots, p_{d-1})$ in $\mathbb{R}^{d-1}$ and B of p_d in $\mathbb{R}$ and a smooth map $H : A \to B$ such that $\rho(x) = 0$ for $x \in A \times B$ if and only if $x_d = H(x_1, \dots, x_{d-1})$. Hence $\varphi : A \to M$ defined by $\varphi(x_1, \dots, x_{d-1}) = (x_1, \dots, x_{d-1}, H(x_1, \dots, x_{d-1}))$ is a local parametrization of M and hence M is a hypersurface in $\mathbb{R}^d$ using Proposition I.2. Therefore the notions of orientable hypersurface and defining function are, in some sense, the same.

In this chapter there are two sections. In the first section we study hypersurfaces in the Euclidean space $\mathbb{R}^d$ and compute the quantities defined in Section II.1 in terms of the defining function. We will obtain explicit expression for the shape operator, the second fundamental form, the principal curvatures, and the Gaussian curvature. Much of the contents of the first section are classical and some of it can be found in Gauss' 1827 paper [9] for $d = 3$. The only exception seems to be the method for calculating the principal curvatures in Theorem 3, which is motivated by the calculation in [24, Page 137] but avoids the use of constrained optimization with Lagrange multipliers.

In the second section, we compute the quantities defined in section II.2. We will use the method used in computing the principal curvatures and the Gaussian curvature to get explicit formulas for the Levi principal curvatures, the product of the Levi principal curvatures (which we called the Hörmander curvature in (II.18)), the complex-symmetric principal curvatures, and the product of the complex-symmetric principal curvatures.

III.1. Hypersurfaces in $\mathbb{R}^d$

Definition III.2. If U is an open subset of $\mathbb{R}^d$ and $\rho : U \to \mathbb{R}$ is a smooth function, then for each p in U, we define

1. The *gradient* vector of ρ defined as

$$\nabla\rho(p) = \left(\frac{\partial\rho}{\partial x_1}(p), \ldots, \frac{\partial\rho}{\partial x_n}(p)\right). \tag{III.1}$$

2. The *Hessian* of ρ is the linear map $\mathrm{Hess}_p(\rho) : \mathbb{R}^d \to \mathbb{R}^d$ such that the $(j,k)^{th}$ entry of the matrix of $\mathrm{Hess}_p(\rho)$ with respect to the standard basis of $\mathbb{R}^d$ is $\dfrac{\partial^2\rho}{\partial x_j \partial x_k}(p)$. We will denote the matrix of the Hessian of ρ by $\mathrm{Hess}_p(\rho)$ again.

Note that the matrix $\mathrm{Hess}_p(\rho)$ is symmetric and hence the operator $\mathrm{Hess}_p(\rho)$ is a self-adjoint with respect to the standard inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^d$.

Let M be a smooth oriented hypersurface in $\mathbb{R}^d$ and $p \in M$. Also let $\rho : U \to \mathbb{R}$ be a smooth defining function for M, where U is a neighborhood of M. We begin by computing the tangent space T_pM of M in terms of ρ.

Proposition III.3. *The tangent space of M at p is the subspace of $\mathbb{R}^d$ given by $T_pM = \nabla\rho(p)^\perp = \{\mathbf{t} \in \mathbb{R}^d \mid \langle \mathbf{t}, \nabla\rho(p)\rangle = 0\}$.*

Proof. Let $\mathbf{t} \in T_pM$. Then there is a smooth curve $\alpha : I \to M$ in M such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{t}$. Then for each $s \in I$, we have $\rho(\alpha(s)) = 0$ and differentiating at $s = 0$ gives us

$$\frac{\partial\rho}{\partial x_1}(p)\cdot\alpha_1'(0) + \cdots + \frac{\partial\rho}{\partial x_d}(p)\cdot\alpha_d'(0) = 0,$$

where $\alpha(s) = (\alpha_1(s), \ldots, \alpha_d(s))$ for each $s \in I$ and $\mathbf{t} = \alpha'(0) = (\alpha_1'(0), \ldots, \alpha_d'(0))$. This tells us that T_pM is a subspace of $\nabla\rho(p)^\perp$. But the subspaces T_pM and $\nabla\rho(p)^\perp$ of $\mathbb{R}^d$ are both $d-1$ dimensional and hence $T_pM = \nabla\rho(p)^\perp$. □

From above proposition, the vectors $\pm\nabla\rho$ are normal to M at p. After replacing ρ with $-\rho$ if necessary, we can assume that the vector

$$N(p) = \frac{1}{|\nabla\rho(p)|}\nabla\rho(p). \tag{III.2}$$

is the positively oriented unit normal to M at p, that is, if $(\mathbf{e_1}, \ldots, \mathbf{e_{d-1}})$ is a positively oriented basis of T_pM, with respect to the orientation of M, then the ordered basis $(N(p), \mathbf{e_1}, \ldots, \mathbf{e_{d-1}})$ of $\mathbb{R}^d$ is also positively oriented. The following result from linear algebra will be useful.

Lemma III.4. *Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space. If A and B are two self-adjoint operators on V such that $\langle A\mathbf{v}, \mathbf{v}\rangle = \langle B\mathbf{v}, \mathbf{v}\rangle$ for each $\mathbf{v} \in V$, then $A = B$.*

Proof. It suffices to show that if C is a self-adjoint operator on V such that $\langle C\mathbf{v}, \mathbf{v}\rangle = 0$ for each $\mathbf{v} \in V$, then $C = 0$. For each $\mathbf{u}, \mathbf{v} \in V$, we have

$$0 = \langle C(\mathbf{u}+\mathbf{v}), \mathbf{u}+\mathbf{v}\rangle - \langle C(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle = 2\langle C\mathbf{u}, \mathbf{v}\rangle + 2\langle C\mathbf{v}, \mathbf{u}\rangle = 4\langle C\mathbf{u}, \mathbf{v}\rangle.$$

Hence for each $\mathbf{u} \in V$, we have $C\mathbf{u} = \mathbf{0}$. □

Now we are ready to compute the shape operator and the second fundamental form of M.

Proposition III.5. *The shape operator of M at a point p in M is given by*

$$\mathsf{S}_p = \frac{-1}{|\nabla\rho(p)|}\pi_p \circ \mathrm{Hess}_p(\rho) \circ i_p, \tag{III.3}$$

where $\pi_p : \mathbb{R}^d \to T_pM$ is the real orthogonal projection and $i_p : T_pM \to \mathbb{R}^d$ is the inclusion map. Moreover, for each $\mathbf{t} \in T_pM$, the second fundamental form of M at p is given by

$$\mathsf{II}_p(\mathbf{t}) = \frac{-1}{|\nabla\rho(p)|}\left\langle \mathrm{Hess}_p(\rho)(\mathbf{t}), \mathbf{t}\right\rangle. \tag{III.4}$$

Proof. Since $\nabla\rho(p) \neq \mathbf{0}$ along the hypersurface $M = \{\rho = 0\}$, by continuity of $\nabla\rho$, there is a neighborhood V of M on which $\nabla\rho \neq \mathbf{0}$ as well. Define a map $\widetilde{N} : V \to S^{d-1}$ by setting

$$\widetilde{N}(x) = \frac{1}{|\nabla\rho(x)|}\nabla\rho(x).$$

Note that if $x \in M$, then $\widetilde{N}(x) = N(x)$, that is, $\widetilde{N}$ is a smooth extension of the Gauss map N of M to a neighborhood V of M. The differential of $\widetilde{N}$ is the linear map $D\widetilde{N}_p : \mathbb{R}^d \to T_{\widetilde{N}(p)}S^{d-1}$ whose matrix with respect to the standard basis of $\mathbb{R}^d$ has the $(j,k)^{th}$ entry given by

$$s_{jk} = \frac{\partial}{\partial x_k}\left(\frac{1}{|\nabla\rho|}\frac{\partial\rho}{\partial x_j}\right) = \frac{-1}{|\nabla\rho(p)|^3}\sum_{i=1}^{d}(\rho_i\rho_{ik})\rho_j + \frac{1}{|\nabla\rho(p)|}\rho_{jk},$$

where for each $1 \leq j,k \leq n$,

$$\rho_j = \frac{\partial\rho}{\partial x_j}(p) \quad \text{and} \quad \rho_{jk} = \frac{\partial^2\rho}{\partial x_k \partial x_j}(p).$$

Note that the vector $N(p)$ is normal to M at p and also $N(p)$ is normal to S^{d-1} at $N(p)$. Hence both the vector subspaces T_pM and $T_{N(p)}S^{d-1}$ of $\mathbb{R}^d$, being the orthogonal complement of $\nabla\rho(p)$ in $\mathbb{R}^d$, are the same. Then the shape operator of M at p is the restriction of $-D\widetilde{N}_p$ to T_pM and for each $\mathbf{t} = (t_1, \ldots, t_d) \in T_pM$, we have

$$\mathsf{S}_p(\mathbf{t}) = -D\widetilde{N}_p(\mathbf{t}) = \sum_{k=1}^{d}\left(\frac{1}{|\nabla\rho(p)|^3}\sum_{i=1}^{d}\rho_i\rho_{ik}\rho_j t_k - \frac{1}{|\nabla\rho(p)|}\rho_{jk}t_k\right). \tag{III.5}$$

Also note that,

$$\left\langle \mathsf{S}_p(\mathbf{t}),\mathbf{t}\right\rangle = \sum_{j,k=1}^{d}\left(\frac{1}{|\nabla\rho(p)|^3}\sum_{i=1}^{d}\rho_i\rho_{ik}\rho_j t_k t_j - \frac{1}{|\nabla\rho(p)|}\rho_{jk}t_k t_j\right).$$

For $\mathbf{t}=(t_1,\ldots,t_d)\in T_pM$, Proposition III.3 gives us $\sum_{j=1}^{d}\rho_j t_j = 0$, and hence the first term in the expression $\left\langle \mathsf{S}_p(\mathbf{t}),\mathbf{t}\right\rangle$ vanish as

$$\sum_{j,k=1}^{d}\left(\frac{1}{|\nabla\rho(p)|^3}\sum_{i=1}^{d}\rho_i\rho_{ik}\rho_j t_k t_j\right) = \frac{1}{|\nabla\rho(p)|^3}\sum_{i,k=1}^{d}\rho_i\rho_{ik}t_k\sum_{j=1}^{d}\rho_j t_j = 0.$$

Hence for each $\mathbf{t}\in T_pM$,

$$\left\langle \mathsf{S}_p(\mathbf{t}),\mathbf{t}\right\rangle = \frac{-1}{|\nabla\rho(p)|}\sum_{j,k=1}^{d}\rho_{jk}t_j t_k. \tag{III.6}$$

Note that for each $\mathbf{t_1},\mathbf{t_2}\in T_pM$,

$$\begin{aligned}\left\langle(\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p)(\mathbf{t_1}),\mathbf{t_2}\right\rangle &= \left\langle\mathrm{Hess}_p(\rho)(\mathbf{t_1}),\mathbf{t_2}\right\rangle = \left\langle\mathbf{t_1},\mathrm{Hess}_p(\rho)(\mathbf{t_2})\right\rangle\\ &= \left\langle\mathbf{t_1},(\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p)(\mathbf{t_2})\right\rangle\end{aligned}$$

and hence the operator $\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p : T_pM\to T_pM$ is self-adjoint with respect to the standard inner product $\langle\cdot,\cdot\rangle$ of T_pM and for each $\mathbf{t}\in T_pM$,

$$\left\langle(\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p)(\mathbf{t}),\mathbf{t}\right\rangle = \sum_{j,k=1}^{d}\rho_{jk}t_j t_k. \tag{III.7}$$

From (III.6) and (III.7), for each $\mathbf{t}\in T_pM$, we have

$$\left\langle \mathsf{S}_p(\mathbf{t}),\mathbf{t}\right\rangle = \frac{-1}{|\nabla\rho(p)|}\left\langle(\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p)(\mathbf{t}),\mathbf{t}\right\rangle$$

and hence by Lemma III.4, we get

$$\mathsf{S}_p = \frac{-1}{|\nabla\rho(p)|}\pi_p\circ\mathrm{Hess}_p(\rho)\circ i_p.$$

If $\mathbf{t} = (t_1, \ldots, t_d) \in T_pM$, then from (III.6) we have

$$\mathsf{II}_p(\mathbf{t}) = \left\langle \mathsf{S}_p(\mathbf{t}), \mathbf{t} \right\rangle = \frac{-1}{|\nabla\rho(p)|} \sum_{i,k=1}^{n} \rho_{jk} t_j t_k = \frac{-1}{|\nabla\rho(p)|} \left\langle \mathrm{Hess}_p(\rho)(\mathbf{t}), \mathbf{t} \right\rangle.$$

□

III.1.1. Computation of principal curvatures and Gaussian curvature

The following result from linear algebra will be useful.

Lemma III.6. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, $A \in M_n(\mathbb{K})$, $V \in M_{n,k}(\mathbb{K})$, $U \in M_{k,n}(\mathbb{K})$, and let x be an indeterminate. Suppose that $\lambda \neq 0$ and $\mathbf{u_j v_j} \neq 0$ for each $1 \leq j \leq n$, where $\mathbf{u_j} \in M_{1,n}(\mathbb{K})$ is the j^{th} row of U and $\mathbf{v_j} \in M_{n,1}(\mathbb{K})$ is the j^{th} column of V, then the function*

$$P(x) = \det \begin{bmatrix} A + x\lambda I_n & V \\ U & 0_k \end{bmatrix},$$

where $I_n \in M_n(\mathbb{K})$ is the identity matrix and $0_k \in M_k(\mathbb{K})$ is the zero matrix, is a polynomial in x of degree $n-k$ and the leading coefficient of P is $(-1)^k \lambda^{n-k} \prod_{j=1}^{k} \mathbf{u_j v_j}$.

Proof. For a non-zero polynomial f, the degree of f is the unique non-negative integer d such that the limit $\lim_{x\to\infty} f(x)/x^d$ exists and is non-zero, and this limit is the leading coefficient of f. Since the determinant of a matrix is a polynomial in its entries, we get that P is a polynomial in x. Taking a common factor of x from the first n columns, we get

$$P(x) = \det \begin{bmatrix} A + x\lambda I_n & V \\ U & 0_k \end{bmatrix} = x^n \det \begin{bmatrix} (1/x)A + \lambda I_n & V \\ (1/x)U & 0_k \end{bmatrix}$$

now taking a common factor of $1/x$ from last k rows, we get $P(x) = x^{n-k} \det \begin{bmatrix} (1/x)A + \lambda I_n & V \\ U & 0_k \end{bmatrix}$. Hence $\lim_{x\to\infty} \frac{P(x)}{x^{n-k}}$ exists and is equal to $\det \begin{bmatrix} \lambda I_n & V \\ U & 0_k \end{bmatrix}$. Since we have the following decomposition into an upper triangular and an lower triangular matrices

$$\begin{bmatrix} \lambda I_n & V \\ U & 0_k \end{bmatrix} = \begin{bmatrix} \lambda I_n & 0_{n\times k} \\ U & I_k \end{bmatrix} \begin{bmatrix} I_n & (1/\lambda)V \\ 0_{k\times n} & G \end{bmatrix},$$

where $0_{n\times k} \in M_{n,k}(\mathbb{K})$ is the zero matrix, $I_k \in M_k(\mathbb{K})$ is the identity matrix, and G is the diagonal matrix

$$G = \frac{-1}{\lambda}\mathrm{diag}(\mathbf{u_1 v_1}, \mathbf{u_2 v_2}, \ldots, \mathbf{u_k v_k}).$$

So we get $\det\begin{bmatrix}\lambda I_n & V \\ U & 0_k\end{bmatrix} = \lambda^n \cdot \det G = \lambda^n \cdot \left(\frac{-1}{\lambda}\right)^k \prod_{j=1}^{k} \mathbf{u_j v_j} = (-1)^k \lambda^{n-k} \prod_{j=1}^{k} \mathbf{u_j v_j} \neq 0$. Hence the degree of P is $n-k$ and the leading coefficient is $(-1)^k \lambda^{n-k} \prod_{j=1}^{k} \mathbf{u_j v_j}$. □

We now compute principal curvatures and the Gaussian curvature of M.

Theorem 3. *The principal curvatures of M at p are the roots of the polynomial*

$$P(t) = \det\begin{bmatrix}\mathrm{Hess}_p(\rho) + t\,|\nabla\rho(p)|\, I_d & \nabla\rho(p) \\ \nabla\rho(p)^T & 0\end{bmatrix}, \tag{III.8}$$

where $I_d \in M_d(\mathbb{R})$ is the identity matrix and $\nabla\rho(p)$ is thought of as a column vector. Moreover, the product of the principal curvatures, known as the Gaussian curvature *of M at p, is given by*

$$K_G = \frac{(-1)^d}{|\nabla\rho(p)|^{d+1}} \det\begin{bmatrix}\mathrm{Hess}_p(\rho) & \nabla\rho(p) \\ \nabla\rho(p)^T & 0\end{bmatrix}. \tag{III.9}$$

The formula III.9 for $d = 3$ is due to Gauss and can be found in [9, Section 9] with a different proof and of course without the determinant notation.

Proof. Let $\pi_p : \mathbb{R}^d \to T_pM$ be the orthogonal projection. Then for each $\mathbf{t} \in T_pM$, we have

$$\mathrm{Hess}_p(\rho)(\mathbf{t}) - (\pi_p \circ \mathrm{Hess}_p(\rho))(\mathbf{t}) \in T_pM^{\perp}.$$

By (III.3), we know that $(\pi_p \circ \mathrm{Hess}_p(\rho))(\mathbf{t}) = -|\nabla\rho(p)|\,\mathsf{S}_p(\mathbf{t})$ and hence there exists $c(\mathbf{t}) \in \mathbb{R}$, such that $\mathrm{Hess}_p(\rho)(\mathbf{t}) + |\nabla\rho(p)|\,\mathsf{S}_p(\mathbf{t}) = c(\mathbf{t})\nabla\rho(p)$. In particular, if $\mathbf{v}$ is a principal direction of M at p and $\mathsf{S}_p(\mathbf{v}) = \kappa\mathbf{v}$, where κ is the corresponding principal curvature, then we get $\mathrm{Hess}_p(\rho)(\mathbf{v}) + \kappa|\nabla\rho(p)|\,\mathbf{v} = c(\mathbf{v})\nabla\rho(p)$. The vector representation of this equation is

$$\begin{bmatrix}\mathrm{Hess}_p(\rho) + \kappa|\nabla\rho|\, I_d & \nabla\rho \\ \nabla\rho^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{v} \\ -c(\mathbf{v})\end{bmatrix} = \mathbf{0}. \tag{III.10}$$

Since there is at least one non-zero principal direction, the above system of linear equations has a non-trivial solution and hence

$$\det \begin{bmatrix} \mathrm{Hess}_p(\rho) + \kappa |\nabla\rho| I_d & \nabla\rho \\ \nabla\rho^T & 0 \end{bmatrix} = 0.$$

Hence each principal curvature of M at p is a root of the polynomial $P(t)$, defined as (III.8). Using Lemma III.6, we get that $P(t)$ is a polynomial in t of degree $d-1$ and hence the roots of P are precisely the principal curvatures of M. Lemma III.6 also gives us that the leading coefficient of $P(t)$ is $l_P = (-1)|\nabla\rho(p)|^{d-1}|\nabla\rho(p)|^2 = -|\nabla\rho(p)|^{d+1}$. Note that the product of the principal curvatures is same as the product of the roots of the polynomial $P(t)$. Hence the Gaussian curvature of M at p is

$$K_G = (-1)^{d-1}\frac{P(0)}{l_p} = \frac{(-1)^d}{|\nabla\rho(p)|^{d+1}} \det \begin{bmatrix} \mathrm{Hess}_p(\rho) & \nabla\rho(p) \\ \nabla\rho(p)^T & 0 \end{bmatrix}.$$

□

Example III.7 (Sphere)**.** Consider $S^{d-1} = \{(x_1,\ldots,x_n) \in \mathbb{R}^d \mid x_1^2 + \cdots + x_n^2 = R^2\}$. Then the function $\rho(x_1,\ldots,x_d) = x_1^2+\cdots+x_d^2 - R^2$ is a defining function for S^{d-1} with $\nabla\rho(p) = 2p$ for each $p \in S^{d-1}$. Note that the normal vector is outward pointing, and hence this $\nabla\rho(p)$ is a positively oriented normal. For each $p \in S^{d-1}$, we have $|\nabla\rho(p)| = 2R$ and $\mathrm{Hess}_p(\rho) = 2I_d$, where $I_d \in M_d(\mathbb{R})$ is the identity matrix. From (III.8),

$$P(t) = \det \begin{bmatrix} \mathrm{Hess}_p(\rho) + t|\nabla\rho| I_d & \nabla\rho \\ \nabla\rho^T & 0 \end{bmatrix} = \det \begin{bmatrix} 2(1+Rt)I_d & 2p \\ 2p^T & 0 \end{bmatrix},$$

where $p = (p_1,\ldots,p_d)^T$ is a column vector. Taking a common factor of $2(1+Rt)$ from first d columns, we get $P(t) = 2^d(1+Rt)^d \det \begin{bmatrix} I_d & 2p \\ (1+Rt)^{-1}p^T & 0 \end{bmatrix}$ and now taking a common factor of $(1+Rt)^{-1}$ from the last row and a common factor of 2 from the last column, we get

$$P(t) = 2^{d+1}(1+Rt)^{d-1} \det \begin{bmatrix} I_d & p \\ p^T & 0 \end{bmatrix}. \tag{III.11}$$

Since the determinant in (III.11) is independent of t, all the roots of P are $-1/R$. The Gaussian curvature of the sphere, using (III.9), is

$$K_G = \frac{(-1)^d}{(2R)^{d+1}} \det \begin{bmatrix} 2I_d & 2p \\ 2p^T & 0 \end{bmatrix} = \frac{(-1)^d}{R^{d+1}} \det \begin{bmatrix} I_d & p \\ p^T & 0 \end{bmatrix} \tag{III.12}$$

Using the fact $p_1^2 + \cdots + p_d^2 = R^2$,t he matrix in (III.12) can be written as the following product of an upper triangular and a lower triangular matrix

$$\begin{bmatrix} I_d & p \\ p^T & 0 \end{bmatrix} = \begin{bmatrix} I_d & \mathbf{0} \\ p^T & 1 \end{bmatrix} \begin{bmatrix} I_d & p \\ \mathbf{0}^T & -R^2 \end{bmatrix},$$

where $\mathbf{0} \in \mathbb{R}^d$ is the zero vector. This shows that the determinant in (III.12) is equal to $-R^2$ and hence we get $K_G = (-1/R)^{d-1}$.

Example III.8 (Gaussian curvature of an ellipsoid)**.** For some non-zero numbers $a_1, \ldots, a_d$, consider the hypersurface given by $M = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1^2/a_1^2 + \cdots + x_d^2/a_d^2 = 1\right\}$. A defining function for M is $\rho(x_1, \ldots, x_d) = x_1^2/a_1^2 + \cdots + x_d^2/a_d^2 - 1$. If $p = (p_1, \ldots, p_d) \in M$, then $\nabla\rho(p) = 2\left(p_1/a_1^2, \ldots, p_d/a_d^2\right)$ is an outward pointing normal and the Hessian of ρ at p is the diagonal matrix $\mathrm{Hess}_p(\rho) = \mathrm{diag}(2/a_1^2, \ldots, 2/a_d^2)$. Then the Gaussian curvature of M at p, using the formula (III.9), is

$$K_G = \frac{(-1)^d}{2^{d+1}\left(p_1^2/a_1^4 + \cdots + p_d^2/a_d^4\right)^{(d+1)/2}} \det \begin{bmatrix} 2/a_1^2 & 0 & \cdots & 0 & 2p_1/a_1^2 \\ 0 & 2/a_2^2 & \cdots & 0 & 2p_2/a_2^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2/a_d^2 & 2p_d/a_d^2 \\ 2p_1/a_1^2 & 2p_2/a_2^2 & \cdots & 2p_d/a_d^2 & 0 \end{bmatrix}.$$

For each $1 \le j \le d$, taking a common factor of $2/a_j^2$ from the j^{th} row and a common factor of 2 from the last row gives us

$$K_G = \frac{(-1)^d}{\left(p_1^2/a_1^4 + \cdots + p_d^2/a_d^4\right)^{(d+1)/2}} \cdot \frac{1}{a_1^2 \cdots a_d^2} \det \begin{bmatrix} 1 & 0 & \cdots & 0 & p_1 \\ 0 & 1 & \cdots & 0 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & p_d \\ p_1/a_1^2 & p_2/a_2^2 & \cdots & p_d/a_d^2 & 0 \end{bmatrix}. \tag{III.13}$$

Using the fact that $p \in M$, the matrix in (III.13) can be written as a product of an upper triangular and a lower triangular matrix as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & p_1 \\ 0 & 1 & \cdots & 0 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & p_d \\ p_1/a_1^2 & p_2/a_2^2 & \cdots & p_d/a_d^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ p_1/a_1^2 & p_2/a_2^2 & \cdots & p_d/a_d^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & p_1 \\ 0 & 1 & \cdots & 0 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & p_d \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

and hence the determinant in (III.13) becomes -1. Hence we get

$$K_G = \frac{(-1)^{d+1}}{a_1^2 \cdots a_d^2 \left(p_1^2/a_1^4 + \cdots + p_d^2/a_d^4\right)^{(d+1)/2}}.$$

III.2. Hypersurfaces in $\mathbb{C}^n$

Let M be a smooth oriented real hypersurface in $\mathbb{C}^n$, that is, under the standard identification (I.3) of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the hypersurface M of $\mathbb{R}^{2n}$ is smooth and oriented. Then for each p in M, we apply the results from previous section with $d = 2n$ and we get the following:

1. A positively oriented unit normal to M (cf. (III.2)) given by

$$N(p) = \frac{1}{|\nabla\rho(p)|}\nabla\rho(p), \tag{III.14}$$

where $\nabla\rho = \left(\frac{\partial\rho}{\partial x_1}, \frac{\partial\rho}{\partial y_1}, \ldots, \frac{\partial\rho}{\partial x_n}, \frac{\partial\rho}{\partial y_n}\right)$. Using the identification (I.3), we can identify $\nabla\rho \in \mathbb{R}^{2n}$ with the vector

$$\nabla\rho = 2\left(\frac{\partial\rho}{\partial\bar{z}_1}, \ldots, \frac{\partial\rho}{\partial\bar{z}_n}\right) \in \mathbb{C}^n, \tag{III.15}$$

where we use the standard notation

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial f}{\partial\bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right). \tag{III.16}$$

2. The tangent space of M (cf. Proposition III.3) given by

$$T_pM = \{\boldsymbol{\zeta} \in \mathbb{C}^n \mid \langle\boldsymbol{\zeta}, \nabla\rho(p)\rangle = 0\}, \tag{III.17}$$

where $\langle \cdot, \cdot \rangle$ is the standard real inner product in $\mathbb{C}^n$.

3. The shape operator $\mathsf{S}_p : T_pM \to T_pM$ of M (cf. Proposition III.5) given by

$$\mathsf{S}_p = \frac{-1}{|\nabla \rho(p)|} \pi_p \circ \mathrm{Hess}_p(\rho) \circ i_p, \tag{III.18}$$

where $\pi_p : \mathbb{C}^n \to T_pM$ is the real orthogonal projection, $\mathrm{Hess}_p(\rho)$ is the Hessian of ρ when viewed as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}$ (cf. Definition III.2), and $i_p : T_pM \to \mathbb{C}^n$ is the inclusion map.

Proposition III.9. *If $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in T_pM$, then the second fundamental form of M at p is*

$$\mathsf{II}_p(\boldsymbol{\zeta}) = \frac{-2}{|\nabla \rho(p)|} \sum_{j,k=1}^{n} \left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta}_k + \mathrm{Re}\, \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) \zeta_j \zeta_k \right). \tag{III.19}$$

Proof. Let $\zeta_j = t_j + is_j$ for $1 \leq j \leq n$. From (III.4), the second fundamental form of M at p is

$$\mathsf{II}_p(\boldsymbol{\zeta}) = \frac{-1}{|\nabla \rho(p)|} \left\langle \mathrm{Hess}_p(\rho)(\boldsymbol{\zeta}), \boldsymbol{\zeta} \right\rangle$$

and after expressing $\mathrm{Hess}_p(\rho)$ and $\boldsymbol{\zeta}$ in real coordinates, we get

$$\mathsf{II}_p(\boldsymbol{\zeta}) = \frac{-1}{|\nabla \rho(p)|} \sum_{j,k=1}^{n} \left(\rho_{x_j x_k} t_j t_k + 2\rho_{x_j y_k} t_j s_k + \rho_{y_j y_k} s_j s_k \right), \tag{III.20}$$

where for $1 \leq j,k \leq n$,

$$\rho_{x_j x_k} = \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p), \quad \rho_{x_j y_k} = \frac{\partial^2 \rho}{\partial x_j \partial y_k}(p), \text{ and } \quad \rho_{y_j y_k} = \frac{\partial^2 \rho}{\partial y_j \partial y_k}(p).$$

To complete the proof, we need to show that the expressions (III.20) and (III.19) are the same. Using (III.16), we have

$$\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta}_k = \frac{1}{4} \left(\rho_{x_j x_k} + i\rho_{x_j y_k} - i\rho_{y_j x_k} + \rho_{y_j y_k} \right) (t_j + is_j)(t_k - is_k).$$

Note that $\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta}_k$ is real since

$$\overline{\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta}_k} = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta}_k.$$

Therefore

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\overline{\zeta}_k &= \operatorname{Re}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\overline{\zeta}_k \\ &= \frac{1}{4}\sum_{j,k=1}^{n}\left[\left(\rho_{x_jx_k}+\rho_{y_jy_k}\right)\left(t_jt_k+s_js_k\right)+\left(\rho_{x_jy_k}-\rho_{y_jx_k}\right)\left(t_js_k-s_jt_k\right)\right] \\ &= \frac{1}{4}\sum_{j,k=1}^{n}\left[\left(\rho_{x_jx_k}+\rho_{y_jy_k}\right)\left(t_jt_k+s_js_k\right)+2\rho_{x_jy_k}\left(t_js_k-s_jt_k\right)\right]. \qquad \text{(III.21)}\end{aligned}$$

Similarly using (III.16),

$$\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\zeta_j\zeta_k = \frac{1}{4}\left(\rho_{x_jx_k}-i\rho_{x_jy_k}-i\rho_{y_jx_k}-\rho_{y_jy_k}\right)(t_j+is_j)(t_k+is_k)$$

and its real part is given by

$$\begin{aligned}\operatorname{Re}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\zeta_j\zeta_k &= \frac{1}{4}\sum_{j,k=1}^{n}\left[\left(\rho_{x_jx_k}-\rho_{y_jy_k}\right)\left(t_jt_k-s_js_k\right)+\left(\rho_{x_jy_k}+\rho_{y_jx_k}\right)\left(t_js_k+s_jt_k\right)\right] \\ &= \frac{1}{4}\sum_{j,k=1}^{n}\left[\left(\rho_{x_jx_k}-\rho_{y_jy_k}\right)\left(t_jt_k-s_js_k\right)+2\rho_{x_jy_k}(t_js_k+s_jt_k)\right]. \qquad \text{(III.22)}\end{aligned}$$

Using (III.21) and (III.22), we get

$$\begin{aligned}\sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\overline{\zeta}_k+\operatorname{Re}\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\zeta_j\zeta_k\right) &= \frac{1}{4}\sum_{j,k=1}^{n}\left[\left(\rho_{x_jx_k}+\rho_{y_jy_k}\right)\left(t_jt_k+s_js_k\right)\right. \\ &\quad +2\rho_{x_jy_k}\left(t_js_k-s_jt_k\right) \\ &\quad +\left(\rho_{x_jx_k}-\rho_{y_jy_k}\right)\left(t_jt_k-s_js_k\right) \\ &\quad \left.+2\rho_{x_jy_k}(t_js_k+s_jt_k)\right] \\ &= \frac{1}{4}\sum_{j,k=1}^{n}\left(2\rho_{x_jx_k}t_jt_k+4\rho_{x_jy_k}t_js_k+2\rho_{y_jy_k}s_js_k\right) \\ &= \frac{1}{2}\sum_{j,k=1}^{n}\left(\rho_{x_jx_k}t_jt_k+2\rho_{x_jy_k}t_js_k+\rho_{y_jy_k}s_js_k\right).\end{aligned}$$

Hence the expressions (III.20) and (III.19) are same. □

III.2.1. The complex tangent space and the splitting of the second fundamental form

By definition, the complex tangent space is given by

$$H_pM = \{\boldsymbol{\xi} \in \mathbb{C}^n \mid (\boldsymbol{\xi}, N(p)) = 0\} = \{\boldsymbol{\xi} \in \mathbb{C}^n \mid (\boldsymbol{\xi}, \nabla\rho(p)) = 0\}, \tag{III.23}$$

where $(\cdot,\cdot)$ is the standard Hermitian inner product in $\mathbb{C}^n$ and $N(p)$ is the unit normal to M at p. Using (III.15) and the fact that if ρ is real, then $\partial\rho/\partial\bar{z} = \overline{\partial\rho/\partial z}$, we can write $H_pM = \left\{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \;\middle|\; \xi_1 \frac{\partial\rho}{\partial z_1}(p) + \cdots + \xi_n \frac{\partial\rho}{\partial z_n}(p) = 0\right\}$. The characteristic vector (cf. (II.16)) given by

$$T(p) = iN(p) = \frac{i}{|\nabla\rho(p)|}\nabla\rho(p) = \frac{2i}{|\nabla\rho(p)|}\left(\frac{\partial\rho}{\partial\bar{z}_1}(p), \ldots, \frac{\partial\rho}{\partial\bar{z}_n}(p)\right). \tag{III.24}$$

We will use the following notations for complex derivatives

$$\rho_j = \frac{\partial\rho}{\partial z_j}(p), \quad \rho_{\bar{j}} = \frac{\partial\rho}{\partial\bar{z}_j}(p), \quad \rho_{jk} = \frac{\partial^2\rho}{\partial z_j\partial z_k}(p), \text{ and } \quad \rho_{j\bar{k}} = \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(p). \tag{III.25}$$

Let $\mathrm{Hess}_p^{\mathbb{C}}(\rho) : \mathbb{C}^n \to \mathbb{C}^n$ be the linear map whose matrix with respect to the standard basis of $\mathbb{C}^n$ has the $(j,k)^{th}$ entry $\rho_{j\bar{k}}$ and let $\mathrm{Symm}_p(\rho) : \mathbb{C}^n \to \mathbb{C}^n$ be the linear map whose matrix with respect to the standard basis of $\mathbb{C}^n$ has the $(j,k)^{th}$ entry ρ_{jk}. In matrix notation, we write

$$\mathrm{Hess}_p^{\mathbb{C}}(\rho) = \left[\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(p)\right]_{j,k} = \left[\rho_{j\bar{k}}\right]_{j,k} \quad \text{and} \quad \mathrm{Symm}_p(\rho) = \left[\frac{\partial^2\rho}{\partial z_j\partial z_k}(p)\right]_{j,k} = [\rho_{jk}]_{j,k} \tag{III.26}$$

Since the matrix of $\mathrm{Hess}_p^{\mathbb{C}}(\rho)$ with respect to the standard basis of $\mathbb{C}^n$ is Hermitian, the operator $\mathrm{Hess}_p^{\mathbb{C}}(\rho)$ is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of $\mathbb{C}^n$. Also, $\mathrm{Symm}_p(\rho)$ is a symmetric matrix and hence is symmetric with respect to the standard conjugation χ_s of $\mathbb{C}^n$, defined in Example II.5. We will need the following result from linear algebra (cf. Lemma III.4).

Lemma III.10. *Let $(H, (\cdot,\cdot))$ be a Hermitian inner product space. If A and B are two self-adjoint operators on H such that $(A(\boldsymbol{\xi}), \boldsymbol{\xi}) = (B(\boldsymbol{\xi}), \boldsymbol{\xi})$ for each $\boldsymbol{\xi} \in H$, then $A = B$.*

Proof. It suffices to show that if C is a self-adjoint operator on H such that $(H\boldsymbol{\zeta},\boldsymbol{\zeta})=0$ for each $\boldsymbol{\zeta}\in H$, then $C=0$. For each $\boldsymbol{\zeta},\boldsymbol{\xi}\in H$, we have

$$\begin{aligned}0&=(C(\boldsymbol{\zeta}+\boldsymbol{\xi}),\boldsymbol{\zeta}+\boldsymbol{\xi})+i\,(C(\boldsymbol{\zeta}+i\boldsymbol{\xi}),\boldsymbol{\zeta}+i\boldsymbol{\xi})-(C(\boldsymbol{\zeta}-\boldsymbol{\xi}),\boldsymbol{\zeta}-\boldsymbol{\xi})-i\,(C(\boldsymbol{\zeta}-i\boldsymbol{\xi}),\boldsymbol{\zeta}-i\boldsymbol{\xi})\\&=(C\boldsymbol{\zeta},\boldsymbol{\xi})+(C\boldsymbol{\xi},\boldsymbol{\zeta})+(C\boldsymbol{\zeta},\boldsymbol{\xi})-(C\boldsymbol{\xi},\boldsymbol{\zeta})+(C\boldsymbol{\zeta},\boldsymbol{\xi})+(C\boldsymbol{\xi},\boldsymbol{\zeta})+(C\boldsymbol{\zeta},\boldsymbol{\xi})-(C\boldsymbol{\xi},\boldsymbol{\zeta})\\&=4\,(C\boldsymbol{\zeta},\boldsymbol{\xi})\,.\end{aligned}$$

Hence for each $\boldsymbol{\zeta}\in H$, we have $C\boldsymbol{\zeta}=\mathbf{0}$. □

We will now compute the various quantities from Theorem 2 in terms of ρ.

Theorem 4. *Let M be a smooth oriented real hypersurface in $\mathbb{C}^n$ and $p\in M$. Then*

1. the characteristic curvature of M at p is

$$K_T=\frac{-2}{|\nabla\rho(p)|^3}\left[\Big(\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\nabla\rho(p)),\nabla\rho(p)\Big)-\Big\langle\mathrm{Symm}_p(\rho)(\nabla\rho(p)),\overline{\nabla\rho(p)}\Big\rangle\right],\qquad\text{(III.27)}$$

2. if $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)\in H_pM$, then the Levi form of M at p is

$$\mathsf{L}(\boldsymbol{\xi})=\frac{1}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\rho_{j\bar{k}}\xi_j\overline{\xi}_k\qquad\text{(III.28)}$$

and the operator $L:H_pM\to H_pM$, which is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM, such that $\mathsf{L}(\boldsymbol{\xi})=(L(\boldsymbol{\xi}),\boldsymbol{\xi})$ is given by

$$L=\frac{1}{|\nabla\rho(p)|}\pi_p^{\mathbb{C}}\circ\mathrm{Hess}_p^{\mathbb{C}}(\rho)\circ i_p^{\mathbb{C}},\qquad\text{(III.29)}$$

where $\pi_p^{\mathbb{C}}:\mathbb{C}^n\to H_pM$ is the Hermitian orthogonal projection, $\mathrm{Hess}_p^{\mathbb{C}}(\rho)$ *is as* (III.26)*, and $i_p^{\mathbb{C}}:H_pM\to\mathbb{C}^n$ is the inclusion map,*

3. if $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)\in H_pM$, then the complex-symmetric fundamental form of M at p is

$$\mathsf{S}(\boldsymbol{\xi})=\frac{-1}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\rho_{jk}\xi_j\xi_k,\qquad\text{(III.30)}$$

and

4. the skew functional $W : H_pM \to \mathbb{R}$ of M at p is given by

$$W(\boldsymbol{\xi}) = \frac{-1}{|\nabla\rho(p)|^2} \left\langle \boldsymbol{\xi}, \mathrm{Hess}_p(\rho)(i\nabla\rho(p)) \right\rangle, \tag{III.31}$$

where $\mathrm{Hess}_p(\rho)$ *is the real Hessian of ρ when viewed as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}$ (cf. Definition III.2).*

Proof. 1. The characteristic curvature of M a p is given by $K_T = \mathsf{II}_p(T(p))$. Substituting the expression of the characteristic vector (III.24) in the expression of the second fundamental form (III.19), we get

$$\begin{aligned}
K_T &= \frac{-2}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} \left(\rho_{j\bar{k}} \frac{i}{|\nabla\rho(p)|} 2\rho_{\bar{j}} \frac{-i}{|\nabla\rho(p)|} 2\rho_k + \mathrm{Re}\ \rho_{jk} \frac{i}{|\nabla\rho(p)|} 2\rho_{\bar{j}} \frac{i}{|\nabla\rho(p)|} 2\rho_{\bar{k}} \right) \\
&= \frac{-2}{|\nabla\rho(p)|^3} \sum_{j,k=1}^{n} \left(\rho_{j\bar{k}} 2\rho_{\bar{j}} 2\rho_k - \mathrm{Re}\ \rho_{jk} 2\rho_{\bar{j}} 2\rho_{\bar{k}} \right) \\
&= \frac{-2}{|\nabla\rho(p)|^3} \left[\left(\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\nabla\rho(p)), \nabla\rho(p) \right) - \mathrm{Re} \left(\mathrm{Symm}_p(\rho)(\nabla\rho(p)), \overline{\nabla\rho(p)} \right) \right] \\
&= \frac{-2}{|\nabla\rho(p)|^3} \left[\left(\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\nabla\rho(p)), \nabla\rho(p) \right) - \left\langle \mathrm{Symm}_p(\rho)(\nabla\rho(p)), \overline{\nabla\rho(p)} \right\rangle \right].
\end{aligned}$$

2. If $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in H_pM$, (II.25) gives us that the restriction of the second fundamental form to H_pM splits as $\mathsf{II}_p(\boldsymbol{\xi}) = -2\mathsf{L}(\boldsymbol{\xi}) + 2\mathsf{R}(\boldsymbol{\xi})$ and (III.19) gives us

$$\begin{aligned}
\mathsf{L}(\boldsymbol{\xi}) &= \frac{-1}{4} \left(\mathsf{II}_p(\boldsymbol{\xi}) + \mathsf{II}_p(i\boldsymbol{\xi}) \right) \\
&= \frac{-1}{4} \left[\frac{-2}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} \left(\rho_{j\bar{k}} \xi_j \overline{\xi}_k + \mathrm{Re}\ \rho_{jk} \xi_j \xi_k \right) \right. \\
&\qquad \left. + \frac{-2}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} \left(\rho_{j\bar{k}} i\xi_j \overline{i\xi}_k + \mathrm{Re}\ \rho_{jk} i\xi_j i\xi_k \right) \right] \\
&= \frac{-1}{4} \left[\frac{-2}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} 2\rho_{j\bar{k}} \xi_j \overline{\xi}_k \right] \\
&= \frac{1}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} \rho_{j\bar{k}} \xi_j \overline{\xi}_k.
\end{aligned}$$

Note that the operator $\pi_p^{\mathbb{C}} \circ \text{Hess}_p^{\mathbb{C}}(\rho) \circ i_p^{\mathbb{C}} : H_pM \to H_pM$ is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM and using above expression for the Levi form, for each $\boldsymbol{\xi} \in H_pM$, we have

$$\mathsf{L}(\boldsymbol{\xi}) = \frac{1}{|\nabla\rho(p)|}\Big((\pi_p^{\mathbb{C}} \circ \text{Hess}_p^{\mathbb{C}}(\rho) \circ i_p^{\mathbb{C}})(\boldsymbol{\xi}), \boldsymbol{\xi}\Big).$$

Note that for for each $\boldsymbol{\xi} \in H_pM$, we also have $\mathsf{L}(\boldsymbol{\xi}) = (L(\boldsymbol{\xi}), \boldsymbol{\xi})$. So using Lemma III.10, we get

$$L = \frac{1}{|\nabla\rho(p)|}\pi_p^{\mathbb{C}} \circ \text{Hess}_p^{\mathbb{C}} \circ i_p^{\mathbb{C}}.$$

3. For each $\boldsymbol{\xi} \in H_pM$, (II.25) gives us $\mathsf{R}(\boldsymbol{\xi}) = \frac{1}{2}\mathsf{II}_p(\boldsymbol{\xi}) + \mathsf{L}(\boldsymbol{\xi})$. Using (III.19) and (III.21), we get

$$\mathsf{R}(\boldsymbol{\xi}) = \text{Re}\left(\frac{-1}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\xi_j\xi_k\right). \tag{III.32}$$

Since

$$\mathsf{R}(e^{i\pi/4}\boldsymbol{\xi}) = \text{Re}\left(\frac{-i}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\xi_j\xi_k\right) = \text{Im}\left(\frac{1}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\xi_j\xi_k\right),$$

the complex-symmetric fundamental form of M at p is given by $\mathsf{S}(\boldsymbol{\xi}) = \mathsf{R}(\boldsymbol{\xi}) - i\mathsf{R}(e^{i\pi/4}\boldsymbol{\xi}) = \frac{-1}{|\nabla\rho(p)|}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j \partial z_k}(p)\xi_j\xi_k$.

4. This follows from the fact that $W(\boldsymbol{\xi}) = \langle \boldsymbol{\xi}, \mathsf{S}_p(T(p))\rangle$ and $\mathsf{S}_p(T(p)) = \frac{-1}{|\nabla\rho(p)|}(\pi_p \circ \text{Hess}_p(\rho))$, where $\pi_p : \mathbb{C}^n \to T_pM$ is the real orthogonal projection.

□

Example III.11 (Sphere)**.** Consider the sphere $S^{2n-1} = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = R^2\}$ with defining function $\rho(z_1,\dots,z_n) = |z_1|^2 + \cdots + |z_n|^2 - R^2$. Then $\nabla\rho(p) = 2p$ is an outward pointing normal at each $p \in S^{2n-1}$. Then, $|\nabla\rho(p)| = 2R$ for each $p \in S^{2n-1}$, $\text{Hess}_p^{\mathbb{C}}(\rho) = I_n$, the $n \times n$ identity matrix, and $\text{Symm}_p(\rho) = 0_n$, the $n \times n$ zero matrix. Then (III.27), the characteristic curvature is given by

$$K_T = \frac{-2}{|\nabla\rho(p)|^3}\left[\Big(\text{Hess}_p^{\mathbb{C}}(\rho)(\nabla\rho(p)), \nabla\rho(p)\Big) - \Big\langle \text{Symm}_p(\rho)(\nabla\rho(p)), \overline{\nabla\rho(p)}\Big\rangle\right]$$

$$= \frac{-2}{(2R)^3}(2p, 2p)$$
$$= \frac{-1}{R}.$$

For $\boldsymbol{\xi} \in H_pM$, the skew functional is given by

$$W(\boldsymbol{\xi}) = \frac{-1}{|\nabla\rho(p)|^2}\left\langle \boldsymbol{\xi}, \mathrm{Hess}_p(\rho)(i\nabla\rho(p))\right\rangle = \frac{-1}{(2R)^2}\langle \boldsymbol{\xi}, i\nabla\rho(p)\rangle = 0$$

since $\boldsymbol{\xi}$ and $i\nabla\rho(p)$ are real orthogonal. Hence $K_{\mathrm{skew}} = \|W\| = 0$.

The next theorem is very similar to Theorem 3 and describes a method for calculating the Levi principal curvature of M at p.

Theorem 5. *The Levi principal curvatures of M at p are the roots of the polynomial*

$$P(z) = \det\begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) - z|\nabla\rho(p)|I_n & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix}, \tag{III.33}$$

where I_n is the $n\times n$ identity matrix, and the column vector $\nabla\rho(p)$ is given by (III.15). *Moreover, the* Hörmander curvature *(cf.* (II.18)*) is given by*

$$K_H = \frac{-1}{|\nabla\rho(p)|^{n+1}}\det\begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix}. \tag{III.34}$$

The operator M on the space of smooth real-valued functions on $\mathbb{C}^n$, defined by

$$M(\rho) = -\det\begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & \rho \end{bmatrix},$$

is called the *Fefferman Monge-Ampère operator* (see [8, 7, 4]). It plays a crucial role in the modern theory of the Bergman kernel.

Proof. If $\pi_p^{\mathbb{C}} : \mathbb{C}^n \to H_pM$ is the Hermitian orthogonal projection, then for each $\boldsymbol{\xi} \in H_pM$, the vector $\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\boldsymbol{\xi}) - (\pi_p^{\mathbb{C}} \circ \mathrm{Hess}_p^{\mathbb{C}}(\rho))(\boldsymbol{\xi})$ is Hermitian orthogonal to H_pM and hence there exists $\mu(\boldsymbol{\xi}) \in \mathbb{C}$ such that $\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\boldsymbol{\xi}) - (\pi_p^{\mathbb{C}} \circ \mathrm{Hess}_p^{\mathbb{C}}(\rho))(\boldsymbol{\xi}) = \mu(\boldsymbol{\xi})\nabla\rho(p)$. Using (III.29), we get

$$\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\boldsymbol{\xi}) - |\nabla\rho(p)|L(\boldsymbol{\xi}) = \mu(\boldsymbol{\xi})\nabla\rho(p). \tag{III.35}$$

If $\mathbf{w} \in H_pM$ is a Levi principal direction of M at p and λ is the corresponding Levi principal curvature, that is, $L(\mathbf{w}) = \lambda \mathbf{w}$, then (III.35) becomes $\mathrm{Hess}_p^{\mathbb{C}}(\rho)(\mathbf{w}) - \lambda\, |\nabla\rho(p)|\, \mathbf{w} = \mu(\mathbf{w})\nabla\rho(p)$ and in vector notation, we have

$$\begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) - \lambda\, |\nabla\rho(p)|\, I_n & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ -\mu(\mathbf{w}) \end{bmatrix} = \mathbf{0}.$$

Since there is at least one non-zero Levi principal direction $\mathbf{w}$, we have

$$\det \begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) - \lambda\, |\nabla\rho(p)|\, I_n & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix} = 0,$$

and hence each Levi principal curvature is a root of the polynomial $P(z)$, defined by (III.33). Using Lemma III.6, we get that $P(z)$ is a polynomial of degree $n-1$ and hence the roots of $P(z)$ are precisely the Levi principal curvatures. Also from Lemma III.6, the leading coefficient of P is given by $l_P = (-1)(-|\nabla\rho(p)|)^{n-1}\, |\nabla\rho(p)|^2 = (-1)^n\, |\nabla\rho(p)|^{n+1}$. Hence the product of the Levi principal curvatures, the Hörmander curvature, is given by

$$K_H = (-1)^{n-1}\frac{P(0)}{l_P} = \frac{-1}{|\nabla\rho(p)|^{n+1}} \det \begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix}.$$

□

Example III.12 (Sphere)**.** Consider the sphere $S^{2n-1} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = R^2\}$ with defining function $\rho(z_1,\ldots,z_n) = |z_1|^2 + \cdots + |z_n|^2 - R^2$. Then $\nabla\rho(p) = 2p$ is an outward pointing normal at each $p \in S^{2n-1}$. Then, $|\nabla\rho(p)| = 2R$ for each $p \in S^{2n-1}$ and $\mathrm{Hess}_p^{\mathbb{C}}(\rho) = I_n$, where I_n is the $n \times n$ identity matrix. Then the polynomial, defined by (III.33) is

$$P(z) = \det \begin{bmatrix} \mathrm{Hess}_p^{\mathbb{C}}(\rho) - z|\nabla\rho(p)|\, I_n & \nabla\rho(p) \\ \overline{\nabla\rho(p)}^T & 0 \end{bmatrix} = \det \begin{bmatrix} (1-2Rz)I_n & 2p \\ 2\overline{p}^T & 0 \end{bmatrix},$$

where $p = (p_1,\ldots,p_n)^T$ is thought of as a column vector. Taking a common factor of $(1-2Rz)$ from the first n columns and then taking a common factor of $(1-2Rz)^{-1}$ from the last row, the above expression becomes $P(z) = (1-2Rz)^{n-1} \det \begin{bmatrix} I_n & 2p \\ 2\overline{p}^T & 0 \end{bmatrix}$ and since the determinant is independent of z, all the roots of $P(z)$ are $1/2R$. Hence all the Levi principal curvatures of the sphere are $1/2R$. The Hörmander curvature of the sphere S^{2n-1}, defined by (III.34), is

$$K_H = \frac{-1}{(2R)^{n+1}} \det \begin{bmatrix} I_n & 2p \\ 2\overline{p}^T & 0 \end{bmatrix}. \tag{III.36}$$

Since $|p_1|^2 + \cdots + |p_n|^2 = R^2$, we can write the matrix in (III.36) as a product of an upper triangular and a lower triangular matrix as

$$\begin{bmatrix} I_n & 2p \\ 2\overline{p}^T & 0 \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ 2\overline{p}^T & 1 \end{bmatrix} \begin{bmatrix} I_n & 2p \\ \mathbf{0}^T & -4R^2 \end{bmatrix}$$

which shows that the determinant in (III.36) is $-4R^2$. Hence we get $K_H = (1/2R)^{n-1}$.

We can obtain a similar result for the complex-symmetric principal curvatures. For that, we will need the following result.

Lemma III.13. *Let $\chi : H_pM \to H_pM$ be a conjugation on H_pM. Then χ extends to a conjugation $\widetilde{\chi} : \mathbb{C}^n \to \mathbb{C}^n$ on $\mathbb{C}^n$*

Proof. Using Proposition II.8 with H as the complex tangent space H_pM and Σ as the identity map Id on H_pM, we get an orthonormal basis $\{\mathbf{z_1}, \ldots, \mathbf{z_{n-1}}\}$ of H_pM such that $\chi\mathbf{z_j} = \sigma_j\mathbf{z_j}$ for $1 \le j \le n-1$, where σ_j are the eigenvalues of the operator $\Theta^2 = \Sigma\chi\Sigma\chi = Id$. Hence for each $1 \le j \le n-1$, we have $\sigma_j = 1$ and $\chi\mathbf{z_j} = \mathbf{z_j}$. Hence if $\boldsymbol{\xi} = \lambda_1\mathbf{z_1} + \cdots + \lambda_{n-1}\mathbf{z_{n-1}} \in H_pM$ for scalars $\lambda_1, \ldots, \lambda_{n-1}$ in $\mathbb{C}$, then $\chi(\boldsymbol{\xi}) = \overline{\lambda}_1\mathbf{z_1} + \cdots + \overline{\lambda}_{n-1}\mathbf{z_{n-1}}$, where $\overline{\lambda}_j$ is the complex conjugate of λ_j for each $1 \le j \le n$. Now the orthonormal basis $\{\mathbf{z_1}, \ldots, \mathbf{z_{n-1}}\}$ extends to an orthonormal basis $\{\mathbf{z_1}, \ldots, \mathbf{z_{n-1}}, \mathbf{z_n}\}$ of $\mathbb{C}^n$, for some $\mathbf{z_n} \in \mathbb{C}^n$. If $\mathbf{z} = \mu_1\mathbf{z_1} + \cdots + \mu_n\mathbf{z_n} \in \mathbb{C}^n$ for scalars $\mu_1, \ldots, \mu_n$ in $\mathbb{C}$, then we define the map $\widetilde{\chi} : \mathbb{C}^n \to \mathbb{C}^n$ as $\widetilde{\chi}(\mathbf{z}) = \overline{\mu}_1\mathbf{z_1} + \cdots + \overline{\mu}_n\mathbf{z_n}$, where $\overline{\mu}_j$ is the complex conjugate of μ_j for each $1 \le j \le n$. From the definition, it follows that the map $\widetilde{\chi}$ is conjugate-linear, an involution, and an isometry. Hence $\widetilde{\chi}$ is a conjugation on $\mathbb{C}^n$ and $\widetilde{\chi} = \chi$ on H_pM. □

Theorem 6. *For each p in M and conjugation $\chi : H_pM \to H_pM$, there exists a unitary operator $U_p : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\Sigma_\chi = \frac{-1}{|\nabla\rho(p)|}\pi_p^{\mathbb{C}} \circ U_p \circ \mathrm{Symm}_p(\rho) \circ i_p^{\mathbb{C}}, \tag{III.37}$$

where $\pi_p^{\mathbb{C}} : \mathbb{C}^n \to H_pM$ *is the Hermitian orthogonal projection,* $\mathrm{Symm}_p(\rho)$ *is defined as* (III.26), *and* $i_p^{\mathbb{C}} : H_pM \to \mathbb{C}^n$ *is the inclusion map.*

Proof. From Lemma III.13, there is a χ extends to a conjugation $\widetilde{\chi} : \mathbb{C}^n \to \mathbb{C}^n$, hence $\widetilde{\chi} \circ i_p^{\mathbb{C}} = i_p^{\mathbb{C}} \circ \chi$. Let us denote by χ_s, the standard conjugation on $\mathbb{C}^n$, which is given by $\chi_s(z_1,\dots,z_n) = (\bar{z}_1,\dots,\bar{z}_n)$, where $\bar{z}_j$ is the complex conjugate of z_j for $1 \le j \le n$. Consider the $\mathbb{C}$-linear map $U_p = \widetilde{\chi} \circ \chi_s$ which is a composition of isometries and hence is also an isometry. From Proposition II.6, we get $(U_p(\boldsymbol{\xi}), U_p(\boldsymbol{\eta})) = (\chi_s(\boldsymbol{\eta}), \chi_s(\boldsymbol{\xi})) = (\boldsymbol{\xi}, \boldsymbol{\eta})$ for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in H_pM$ and hence U_p is a unitary operator. From (III.30), we have for each $\boldsymbol{\xi} \in H_pM$, we have

$$\mathsf{S}(\boldsymbol{\xi}) = \frac{-1}{|\nabla\rho(p)|} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) \xi_j \xi_k = \frac{-1}{|\nabla\rho(p)|} \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}), \chi_s(\boldsymbol{\xi})\right).$$

For each $\boldsymbol{\xi}, \boldsymbol{\eta} \in H_pM$, the bilinear form of S is given by

$$\begin{aligned}
\mathscr{S}(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \frac{1}{4}\left[\mathsf{S}(\boldsymbol{\xi} + \boldsymbol{\eta}) - \mathsf{S}(\boldsymbol{\xi} - \boldsymbol{\eta})\right] \\
&= \frac{-1}{4|\nabla\rho(p)|} \left[\left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi} + \boldsymbol{\eta}), \chi_s(\boldsymbol{\xi} + \boldsymbol{\eta})\right) - \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi} - \boldsymbol{\eta}), \chi_s(\boldsymbol{\xi} - \boldsymbol{\eta})\right)\right] \\
&= \frac{-1}{4|\nabla\rho(p)|} \left[2\left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}), \chi_s \boldsymbol{\eta}\right) + 2\left(\mathrm{Symm}_p(\rho)(\boldsymbol{\eta}), \chi_s \boldsymbol{\xi}\right)\right].
\end{aligned}$$

Note that if $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n)$ and $\boldsymbol{\eta} = (\eta_1,\dots,\eta_n)$, then

$$\left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}), \chi_s \boldsymbol{\eta}\right) = \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\eta}), \chi_s \boldsymbol{\xi}\right) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} \xi_j \eta_k \tag{III.38}$$

and hence Proposition II.6 give us that $\mathrm{Symm}_p(\rho)$ is χ_s-symmetric, because

$$\begin{aligned}
\left(\boldsymbol{\eta}, (\chi_s \circ \mathrm{Symm}_p(\rho))(\boldsymbol{\xi})\right) &= \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}), \chi_s \boldsymbol{\eta}\right) = \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\eta}), \chi_s \boldsymbol{\xi}\right) \\
&= \left(\boldsymbol{\eta}, (\mathrm{Symm}_p(\rho)^* \circ \chi_s)\boldsymbol{\xi}\right),
\end{aligned} \tag{III.39}$$

where $\mathrm{Symm}_p(\rho)^*$ is the adjoint of $\mathrm{Symm}_p(\rho)$ with respect to the Hermitian inner product $(\cdot,\cdot)$ of $\mathbb{C}^n$. Hence (III.38) and Proposition II.6 gives us

$$\mathscr{S}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{-1}{|\nabla\rho(p)|} \left(\mathrm{Symm}_p(\rho)(\boldsymbol{\eta}), \chi_s \boldsymbol{\xi}\right) = \frac{-1}{|\nabla\rho(p)|} \left(\boldsymbol{\xi}, (\chi_s \circ \mathrm{Symm}_p(\rho))(\boldsymbol{\eta})\right). \tag{III.40}$$

From (II.29), there is a conjugate linear map $\Theta : H_pM \to H_pM$ such that for every $\boldsymbol{\xi},\boldsymbol{\eta} \in H_pM$, the bilinear form of S is given by $\mathscr{S}(\boldsymbol{\xi},\boldsymbol{\eta}) = (\boldsymbol{\xi},\Theta(\boldsymbol{\eta}))$. Hence by Lemma III.10, we have

$$\Theta = \frac{-1}{|\nabla\rho(p)|}\pi_p^{\mathbb{C}} \circ \chi_s \circ \mathrm{Symm}_p(\rho) \circ i_p \tag{III.41}$$

for each $\boldsymbol{\xi} \in H_pM$, where $\pi_p^{\mathbb{C}} : \mathbb{C}^n \to H_pM$ is the Hermitian orthogonal projection and $i_p : H_pM \to \mathbb{C}^n$ is the inclusion map. From (II.30), we have $\Sigma_\chi = (\Theta\chi)^*$, where $(\Theta\chi)^*$ is the adjoint of the operator $\Theta\chi$ with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM. Hence we have

$$\begin{aligned}
\Sigma_\chi = (\Theta\chi)^* &= \frac{-1}{|\nabla\rho(p)|}(\pi_p^{\mathbb{C}} \circ \chi_s \circ \mathrm{Symm}_p(\rho) \circ i_p \circ \chi)^* \\
&= \frac{-1}{|\nabla\rho(p)|}(\pi_p^{\mathbb{C}} \circ \chi_s \circ \mathrm{Symm}_p(\rho) \circ \widetilde{\chi} \circ i_p)^* \quad (\text{since } \widetilde{\chi} \circ i_p = i_p \circ \chi) \\
&= \frac{-1}{|\nabla\rho(p)|}(\pi_p^{\mathbb{C}} \circ \mathrm{Symm}_p(\rho)^* \circ \chi_s \circ \widetilde{\chi} \circ i_p)^* \quad \text{from (III.39)} \\
&= \frac{-1}{|\nabla\rho(p)|}(\pi_p^{\mathbb{C}} \circ \mathrm{Symm}_p(\rho)^* \circ U_p^* \circ i_p)^* \quad (\text{since } U_p = \widetilde{\chi} \circ \chi_s) \\
&= \frac{-1}{|\nabla\rho(p)|}\pi_p^{\mathbb{C}} \circ U_p \circ \mathrm{Symm}_p(\rho) \circ i_p.
\end{aligned}$$

□

In the next theorem, we will present a method for computing the complex-symmetric principal curvatures of M at p.

Theorem 7. *Let $[\chi_s \circ \mathrm{Symm}_p(\rho)]_{\mathbb{R}} \in M_{2n}(\mathbb{R})$ be the matrix of the operator $\chi_s \circ \mathrm{Symm}_p(\rho) : C^n \to \mathbb{C}^n$ as a $\mathbb{R}$-linear operator, where χ_s is the standard conjugation on $\mathbb{C}^n$ and let $[\nabla\rho(p)]_{\mathbb{R}}$ and $[i\nabla\rho(p)]_{\mathbb{R}}$ be $2n \times 1$ column vectors $\nabla\rho(p)$ and $i\nabla\rho(p)$, respectively, in real coordinates. Then the the roots of the polynomial*

$$P(t) = \det\begin{bmatrix} [\chi_s \circ \mathrm{Symm}_p(\rho)]_{\mathbb{R}} + t\,|\nabla\rho(p)|\,I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}} \\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \end{bmatrix} \tag{III.42}$$

are the complex-symmetric principal curvatures and their negatives. Moreover, the product of the complex-symmetric principal curvatures is given by

$$\frac{1}{|\nabla\rho(p)|^{n+1}}\left|\det\begin{bmatrix}[\chi_s\circ\mathrm{Symm}_p(\rho)]_{\mathbb{R}} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}}\\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\end{bmatrix}\right|^{1/2}. \tag{III.43}$$

Proof. If $\pi_p^{\mathbb{C}}:\mathbb{C}^n\to H_pM$ is the Hermitian orthogonal projection, then for each $\boldsymbol{\xi}\in H_pM$, the vector $(\chi_s\circ\mathrm{Symm}_p(\rho))(\boldsymbol{\xi})-(\pi_p^{\mathbb{C}}\circ\chi_s\circ\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}))$ is Hermitian orthogonal to H_pM and hence there exists $\mu(\boldsymbol{\xi})\in\mathbb{C}$ such that $(\chi_s\circ\mathrm{Symm}_p(\rho))(\boldsymbol{\xi})-(\pi_p^{\mathbb{C}}\circ\chi_s\circ\mathrm{Symm}_p(\rho)(\boldsymbol{\xi}))=\mu(\boldsymbol{\xi})\nabla\rho(p)$. From (III.41), for each $\boldsymbol{\xi}\in H_pM$, we have $-|\nabla\rho(p)|\Theta(\boldsymbol{\xi})=(\pi_p^{\mathbb{C}}\circ\chi_s\circ\mathrm{Symm}_p(\rho))(\boldsymbol{\xi})$ and hence we get

$$(\chi_s\circ\mathrm{Symm}_p(\rho))(\boldsymbol{\xi})+|\nabla\rho(p)|\Theta(\boldsymbol{\xi})=\mu(\boldsymbol{\xi})\nabla\rho(p). \tag{III.44}$$

Note that $\Theta: H_pM\to H_pM$ is a conjugate-linear map and $\Theta^2=\Sigma_\chi^*\Sigma_\chi$ is self-adjoint with respect to the Hermitian inner product $(\cdot,\cdot)$ of H_pM. Using Lemma II.7, there is a vector $\mathbf{z}\in H_pM$ and $\sigma\geq 0$ such that $\Theta\mathbf{z}=\sigma\mathbf{z}$. Since Θ is conjugate-linear, we get

$$\Theta(i\mathbf{z})=-i\Theta(\mathbf{z})=-\sigma(i\mathbf{z}). \tag{III.45}$$

Substituting $\boldsymbol{\xi}=\mathbf{z}$ in (III.44), we get $\chi_s\circ\mathrm{Symm}_p(\rho)(\mathbf{z})+\sigma|\nabla\rho(p)|\mathbf{z}=\mu(\mathbf{z})\nabla\rho(p)$. Expressing everything in real coordinates, with $\mu(\mathbf{z})=p(\mathbf{z})+iq(\mathbf{z})$, we can write

$$\begin{bmatrix}[\chi_s\circ\mathrm{Symm}_p(\rho)]_{\mathbb{R}}+\sigma|\nabla\rho(p)|I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}}\\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ -p(\mathbf{z})\\ -q(\mathbf{z})\end{bmatrix}=\mathbf{0}.$$

Since there is a non-zero vector $\mathbf{z}$ satisfying the above equation, we get

$$\det\begin{bmatrix}[\chi_s\circ\mathrm{Symm}_p(\rho)]_{\mathbb{R}}+\sigma|\nabla\rho(p)|I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}}\\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\end{bmatrix}=0. \tag{III.46}$$

Hence all the complex-symmetric principal curvatures σ are the roots of the polynomial $P(t)$ defined by (III.42). Using Lemma III.6, we get that $P(t)$ is a polynomial of degree $2n-2$. Note that replacing $\mathbf{z}$ with $i\mathbf{z}$, (III.45) gives us that $-\sigma$ is also a root of P. Hence the roots of P are $\{\sigma_1, \ldots, \sigma_{n-1}, -\sigma_1, \ldots, -\sigma_{n-1}\}$, where $\sigma_1, \ldots, \sigma_{n-1}$ are the complex-symmetric principal curvatures of M at p. Lemma III.6 also gives us that the leading coefficient of P is $l_P = (-1)^2 |\nabla\rho(p)|^{2n-2} |\nabla\rho(p)|^2 |i\nabla\rho(p)|^2 = |\nabla\rho(p)|^{2n+2}$. Then the product of the roots of P is

$$(-1)^{n-1}\prod_{j=1}^{n-1}\sigma_j^2 = (-1)^{2n-2}\frac{P(0)}{l_P} = \frac{1}{|\nabla\rho(p)|^{2n+2}}\det\begin{bmatrix}\chi_s\circ\mathrm{Symm}_p(\rho) & \nabla\rho(p) & i\nabla\rho(p)\\ \nabla\rho(p)^T & 0 & 0\\ i\nabla\rho(p)^T & 0 & 0\end{bmatrix}$$

and hence the product of the complex-symmetric principal curvature is

$$\frac{1}{|\nabla\rho(p)|^{n+1}}\left|\det\begin{bmatrix}[\chi_s\circ\mathrm{Symm}_p(\rho)]_{\mathbb{R}} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}}\\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0\end{bmatrix}\right|^{1/2}.$$

□

Example III.14 (Sphere)**.** Consider the sphere $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = R^2\}$ with defining function

$$\rho(z_1, \ldots, z_n) = |z_1|^2 + \cdots + |z_n|^2 - R^2.$$

In real coordinates,

$$[\nabla\rho]_{\mathbb{R}} = \begin{bmatrix}\frac{\partial\rho}{\partial x_1} & \frac{\partial\rho}{\partial y_1} & \cdots & \frac{\partial\rho}{\partial x_n} & \frac{\partial\rho}{\partial y_n}\end{bmatrix}^T$$

and

$$[i\nabla\rho]_{\mathbb{R}} = \begin{bmatrix}-\frac{\partial\rho}{\partial y_1} & \frac{\partial\rho}{\partial x_1} & \cdots & -\frac{\partial\rho}{\partial y_n} & \frac{\partial\rho}{\partial x_n}\end{bmatrix}^T.$$

Also, for each $p \in S^{2n-1}$, we have $\mathrm{Symm}_p(\rho) = [\rho_{jk}]_{j,k}$ is the zero matrix. Hence the polynomial defined by (III.42) is given by

$$P(t) = \det \begin{bmatrix} t\,|\nabla\rho(p)|\,I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}} \\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \end{bmatrix}.$$

Taking a common factor of t from first $2n$ columns, we get

$$P(t) = t^{2n} \det \begin{bmatrix} |\nabla\rho(p)|\,I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}} \\ (1/t)[\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \\ (1/t)[i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \end{bmatrix}$$

and then taking a common factor of $1/t$ from last 2 rows, we get

$$P(t) = t^{2n-2} \det \begin{bmatrix} |\nabla\rho(p)|\,I_{2n} & [\nabla\rho(p)]_{\mathbb{R}} & [i\nabla\rho(p)]_{\mathbb{R}} \\ [\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \\ [i\nabla\rho(p)]_{\mathbb{R}}^T & 0 & 0 \end{bmatrix}.$$

Hence all the roots of $P(t)$ are zero and hence all the complex-symmetric principal curvatures of the sphere S^{2n-1} are zero.

CHAPTER IV

REINHARDT HYPERSURFACES IN $\mathbb{C}^2$

A real hypersurface $M \subset \mathbb{C}^n$ is said to be *Reinhardt* if for each $p = (p_1, \ldots, p_n) \in M$ and for each $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, the point $(p_1 e^{i\theta_1}, \ldots, p_n e^{i\theta_n})$ is in M. Therefore there is a natural action of the torus group $\mathbb{T}^n = \{\lambda \in \mathbb{C}^n \mid |\lambda_j| = 1 \text{ for each } 1 \leq j \leq n\}$ on a Reinhardt hypersurface given by $\lambda \cdot (z_1, \ldots, z_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$. Reinhardt hypersurfaces arise naturally in complex analysis as boundaries of Reinhardt domains, that is, domain in $\mathbb{C}^n$ stable under the above action of the torus. Function theory on such domains is easier to study thanks to the presence of the large symmetry group (see [15]). Holomorphic function on Reinhardt domains admit expansions in Laurent series, a phenomenon well-known for annuli in $\mathbb{C}$. Examples of Reinhardt hypersurfaces include the sphere $S^{2n-1} = \{z \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$, the cylindrical hypersurface $\{z \in \mathbb{C}^n \mid |z_1| = 1\}$, etc.

In this chapter, we concentrate on Reinhardt hypersurface in $\mathbb{C}^2$. We compute the various curvatures associated with such a hypersurface and try to gain an understanding of the geometric significance of these curvatures by characterizing the hypersurfaces for which these curvatures vanish identically. As a preliminary, we will need to recall the curvature of curves in $\mathbb{R}^2$.

IV.1. Smooth curves in $\mathbb{R}^2$

Recall from Chapter 1, that a smooth curve in $\mathbb{R}^n$ is a smooth one dimensional submanifold of $\mathbb{R}^n$. Using Proposition I.2, curves admit a local parametrization over an open interval on $\mathbb{R}$. In fact, curves can be globally parametrized (see [10]) but we will have no use for this topological fact. Let I and J be two open intervals in $\mathbb{R}$ and let $h : J \to I$ be a differentiable function. If $\alpha : I \to \mathbb{R}^n$ is a local parametrization of the curve then the composite function $\beta = \alpha \circ h : J \to \mathbb{R}^n$ is called a *reparametrization* of α by h. The next proposition shows the existence of a unit speed parametrization for any curve α.

Proposition IV.1 (Unit-speed reparametrization)**.** *If $\alpha : I \to \mathbb{R}^n$ is a local parametrization of a curve, then there is a reparametrization $\widetilde{\alpha}$ of α such that $\widetilde{\alpha}$ has unit speed.*

Proof. Note that $\alpha'(t) \neq \mathbf{0}$ for all $t \in I$ since the differential of the parametrization α is injective (cf. Proposition I.2). Consider the arc length function $s(t) = \int_c^t |\alpha'(u)|\, du$, where $c \in I$. Then we have $s'(t) = |\alpha'(t)| \neq 0$ for each $t \in I$ and hence $ds/dt > 0$ on I. Then by the inverse function theorem, s has a differentiable inverse $t(s)$ with $dt/ds = 1/|\alpha'(t)|$. Define $\widetilde{\alpha}(s) = \alpha(t(s))$, a reparametrization of α. Then we get

$$\left|\widetilde{\alpha}'(s)\right| = \left|\alpha'(t(s))\frac{dt}{ds}\right| = \left|\alpha'(t(s))\right| \frac{1}{|\alpha'(t(s))|} = 1.$$

□

Let C be a curve in $\mathbb{R}^2$ with a local unit speed parametrization $\alpha : I \to \mathbb{R}^2$, let $\mathbf{t}(s) = (\alpha_1'(s), \alpha_2'(s)) \in \mathbb{R}^2$, and $\mathbf{n}(s) = (-\alpha_2'(s), \alpha_1'(s))$. Clearly $\mathbf{t}(s)$ is a tangent vector to C at $\alpha(s)$ and $\mathbf{n}(s)$ is normal to C at $\alpha(s)$. These vectors constitute the *Frenet frame* of C at $\alpha(s)$ and we have the following.

Proposition IV.2. *There is a smooth function $\kappa : I \to \mathbb{R}$, called the* signed curvature, *such that $\alpha''(s) = \kappa(s)\mathbf{n}(\mathbf{s})$ and κ is given by*

$$\kappa(s) = \alpha_1'(s)\alpha_2''(s) - \alpha_2'(s)\alpha_1''(s). \tag{IV.1}$$

If we orient C by the normal vector field $\mathbf{n}$*, then the (unique) principal curvature of C at $\alpha(s)$ is $-\kappa(s)$.*

Proof. Since α is a unit speed parametrization of C, for each $s \in I$, we have $\langle \alpha'(s), \alpha'(s)\rangle = 1$ and differentiating both sides with respect to s, we get $\langle \alpha''(s), \alpha'(s)\rangle = 0$. This tells us the vector $\alpha''(s)$ is normal to the curve and hence there exists $\kappa(s) \in \mathbb{R}$ such that $\alpha''(s) = \kappa(s)\mathbf{n}(\mathbf{s})$. Comparing the two vector components of $\alpha''(s)$ and $\kappa(s)\mathbf{n}(s)$, we get $\alpha_1''(s) = -\kappa(s)\alpha_2'(s)$ and

$\alpha_2''(s) = \kappa(s)\alpha_1'(s)$. Then

$$\alpha_1'(s)\alpha_2''(s) - \alpha_2'(s)\alpha_1''(s) = \kappa(s)(\alpha_1'(s))^2 + \kappa(s)(\alpha_2'(s))^2 = \kappa(s),$$

since $(\alpha_1'(s))^2 + (\alpha_2'(s))^2 = 1$ for each $s \in I$ which proves (IV.1).

Since C is a hypersurface in $\mathbb{R}^2$, the tangent space T_pC of C at $p = \alpha(s)$ is a one dimensional subspace of $\mathbb{R}^2$ spanned by the vector $\alpha'(s)$. Using Corollary II.3, the 1×1 matrix of the second fundamental form with respect to the basis $\{\alpha'(s)\}$ of T_pC is given by

$$h_{11} = \langle \mathbf{n}(s), \alpha''(s) \rangle = -\alpha_1''(s)\alpha_2'(s) + \alpha_2''(s)\alpha_1'(s) = -\kappa(s).$$

Since the matrix is diagonal with respect to the basis $\alpha'(s)$, it follows that the principal curvature of C at $\alpha(s)$ is $-\kappa(s)$. □

IV.2. Calculations for Reinhardt hypersurfaces

To each Reinhardt hypersurface M in $\mathbb{C}^n$, there is a corresponding hypersurface in $\mathbb{R}^n$, known as the *Reinhardt shadow* of M, given by

$$|M| = \{(|z_1|, \ldots, |z_n|) \in (\mathbb{R}_+)^n \mid (z_1, \ldots, z_n) \in M\}.$$

It is clear from the definition that the $|M| \subset M$, if we think of $\mathbb{R}^n \subset \mathbb{C}^n$. Also, the entire Reinhardt hypersurface M can be recovered starting with just its Reinhardt shadow. For a smooth Reinhardt hypersurfaces M in $\mathbb{C}^2$, the Reinhardt shadow $|M|$ is a smooth curve in $\mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2$, which we studied in the previous section. We will now compute the invariant quantities of a smooth Reinhardt hypersurface $M \subset \mathbb{C}^2$, from Theorem 2, in terms of the unit speed parametrization of the curve $|M|$.

Theorem 8. *Let M be a Reinhardt hypersurface in $\mathbb{C}^2$ and let $\alpha : I \to \mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2$ be a local unit speed parametrization of the smooth curve $|M|$. Then for each $p = (\alpha_1(t)e^{i\theta_1}, \alpha_2(t)e^{i\theta_2})$ in M, where $t \in I$ and $0 \le \theta_1, \theta_2 < 2\pi$, we have*

1. the principal curvatures of M at p are $-\kappa(t)$, $-\alpha_1(t)\alpha_2'(t)$, *and* $\alpha_1(t)\alpha_2'(t)$ *and the corresponding principal directions are* $(\alpha_1'(t)e^{i\theta_1}, \alpha_2'(t)e^{i\theta_2})$, $(ie^{i\theta_1}, 0)$, *and* $(0, ie^{i\theta_2})$,

2. the Levi principal curvature of M at p is

$$\lambda = \frac{1}{4}\left(\kappa(t) + \alpha_1'(t)\alpha_2'(t)\left(\frac{\alpha_1'(t)}{\alpha_1(t)} - \frac{\alpha_2'(t)}{\alpha_2(t)}\right)\right),$$

where $\kappa(t)$ *is the signed curvature of* $|M|$ *at* $\alpha(t)$ *and if* λ *vanishes identically, then M is one of the hypersurfaces*

$$|z_1| = c_1, \quad or \quad |z_2| = c_2, \quad or \quad |z_2| = c_4 |z_1|^{c_3}, \tag{IV.2}$$

where $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$, *and* $c_4 \geq 0$,

3. the complex-symmetric principal curvature of M at p is

$$\sigma = \frac{1}{4}\left(-\kappa(t) + \alpha_1'(t)\alpha_2'(t)\left(\frac{\alpha_1'(t)}{\alpha_1(t)} - \frac{\alpha_2'(t)}{\alpha_2(t)}\right)\right)$$

and if σ *vanishes identically, then M is one of the hypersurfaces*

$$|z_1| = c_1, \quad or \quad |z_2| = c_2, \quad or \quad |z_1|^2 - c_3 |z_2|^2 = c_4, \tag{IV.3}$$

where $c_1, c_2 > 0$ *and* $c_3, c_4 \in \mathbb{R}$,

4. the characteristic curvature of M at p is

$$K_T = \frac{(\alpha_1'(t))^3}{\alpha_2(t)} - \frac{(\alpha_2'(t))^3}{\alpha_1(t)}$$

and if K_T *vanishes identically, then M is the hypersurface*

$$|z_1|^{4/3} = |z_2|^{4/3} + c, \tag{IV.4}$$

where $c \in \mathbb{R}$,

5. *the skew curvature of M at p is*

$$K_{\text{skew}} = \left| \alpha_1'(t)\alpha_2'(t) \left(\frac{\alpha_1'(t)}{\alpha_2(t)} + \frac{\alpha_2'(t)}{\alpha_1(t)} \right) \right|$$

and if K_{skew} vanishes identically, then M is one of the hypersurfaces

$$|z_1| = c_1, \quad or \quad |z_2| = c_2, \quad or \quad |z_1|^2 + |z_2|^2 = c_3, \tag{IV.5}$$

where $c_1, c_2 > 0$ and $c_3 \geq 0$.

1. According to a theorem of E. Cartan (see [1]), a smooth hypersurface in $\mathbb{C}^n$ whose Levi form vanishes identically is foliated by complex hypersurfaces. In our case, the leaves of this foliation for the three hypersurfaces in (IV.2) (which are complex curves, that is, Riemann surfaces) are given by $z_1 = c_1 e^{i\theta}$, $z_2 = c_2 e^{i\theta}$, and $z_2 = c_4 e^{i\theta}(z_1)^{c_3}$, where $\theta \in \mathbb{R}$ parametrizes the leaves and the power $\zeta \mapsto \zeta^{c_3}$ is defined locally. The hypersurfaces of the form $|z_2| = c_4 |z_1|^{c_3}$ constitute part of the boundary of domains studied by Chakrabarti, Edholm, and McNeal (see [2] and [6]).
2. It has been shown (see [19]) that a compact Reinhardt surface of constant positive characteristic curvature is a sphere. The hypersurface (IV.4) cannot be extended to a compact hypersurface.
3. Thanks to (IV.5), the only complete Reinhardt hypersurfaces in $\mathbb{C}^2$ with zero skew curvature are spheres. It is interesting that the vanishing of this one quantity leads to such rigidity.

IV.3. Proof of Theorem 8

Consider the parametrization $\varphi(s,u,v) = \big(\alpha_1(s)e^{iu}, \alpha_2(s)e^{iv}\big)$, where $s \in I$ and $0 \leq u, v < 2\pi$, of M. Under the identification (I.3), we can view M as a hypersurface in $\mathbb{R}^4$ and by Corollary I.4, the vectors

$$\begin{aligned}
\varphi_s &= \frac{\partial \varphi}{\partial s}(t,\theta_1,\theta_2) = (\alpha_1'(t)e^{i\theta_1}, \alpha_2'(t)e^{i\theta_2}), \\
\varphi_u &= \frac{\partial \varphi}{\partial u}(t,\theta_1,\theta_2) = (i\alpha_1(t)e^{i\theta_1}, 0), \quad \text{and} \\
\varphi_v &= \frac{\partial \varphi}{\partial v}(t,\theta_1,\theta_2) = (0, i\alpha_2(t)e^{i\theta_2})
\end{aligned}$$

forms a (real) basis of the tangent space T_pM of M at p. For the sake of simplicity, we will omit the parameter t.

IV.3.1. The principal curvatures and the second fundamental form

By Proposition II.1, the vector

$$N(p) = \frac{\varphi_s \times \varphi_u \times \varphi_v}{|\varphi_s \times \varphi_u \times \varphi_v|}$$

is unit normal to M at p. Note that if $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ is the standard basis of $\mathbb{R}^4$, then by (II.3), we have

$$\varphi_s \times \varphi_u \times \varphi_v = \det \begin{bmatrix} \alpha_1' \cos\theta_1 & -\alpha_1 \sin\theta_1 & 0 & \mathbf{e_1} \\ \alpha_1' \sin\theta_1 & \alpha_1 \cos\theta_1 & 0 & \mathbf{e_2} \\ \alpha_2' \cos\theta_2 & 0 & -\alpha_2 \sin\theta_2 & \mathbf{e_3} \\ \alpha_2' \sin\theta_2 & 0 & \alpha_2 \cos\theta_2 & \mathbf{e_4} \end{bmatrix}.$$

After expanding and identification with $\mathbb{C}^2$, we have

$$\varphi_s \times \varphi_u \times \varphi_v = \alpha_1 \alpha_2 \left(\alpha_2' e^{i\theta_1}, -\alpha_1' e^{i\theta_2} \right).$$

Hence $N(p) = \left(\alpha_2' e^{i\theta_1}, -\alpha_1' e^{i\theta_2}\right)$. Since the normal $(\alpha_2', -\alpha_1')$ to the hypersurface $|M|$ at α is positively oriented, we get that $N(p)$ is a positively oriented normal to M at p. The second order derivatives of φ at p are

$$\varphi_{ss} = \left(\alpha_1'' e^{i\theta_1}, \alpha_2'' e^{i\theta_2}\right), \quad \varphi_{su} = \left(i\alpha_1' e^{i\theta_1}, 0\right), \quad \varphi_{sv} = \left(0, i\alpha_2' e^{i\theta_2}\right),$$

$$\varphi_{uu} = \left(-\alpha_1 e^{i\theta_1}, 0\right), \quad \varphi_{vv} = \left(0, -\alpha_2 e^{i\theta_2}\right), \quad \varphi_{uv} = (0,0).$$

Using Corollary II.3, the matrix of the second fundamental form with respect to the basis $\{\varphi_s, \varphi_u, \varphi_u\}$ of T_pM is given by

$$\begin{aligned} H &= \begin{bmatrix} \langle N(p), \varphi_{ss}\rangle & \langle N(p), \varphi_{su}\rangle & \langle N(p), \varphi_{sv}\rangle \\ \langle N(p), \varphi_{su}\rangle & \langle N(p), \varphi_{uu}\rangle & \langle N(p), \varphi_{uv}\rangle \\ \langle N(p), \varphi_{sv}\rangle & \langle N(p), \varphi_{uv}\rangle & \langle N(p), \varphi_{ss}\rangle \end{bmatrix} = \begin{bmatrix} \alpha_1''\alpha_2' - \alpha_1'\alpha_2'' & 0 & 0 \\ 0 & -\alpha_1\alpha_2' & 0 \\ 0 & 0 & \alpha_1'\alpha_2 \end{bmatrix} \\ &= \operatorname{diag}(-\kappa, -\alpha_1\alpha_2', \alpha_1'\alpha_2), \end{aligned}$$

where κ is the signed curvature of α. Hence the principal curvatures of M at p are $-\kappa(t)$, $-\alpha_1(t)\alpha_2'(t)$, and $\alpha_1(t)\alpha_2'(t)$. Since the matrix H is a diagonal with respect to the basis $\{\varphi_s, \varphi_u, \varphi_u\}$ of T_pM, the principal directions are the unit vectors along the basis elements, which are the vectors $(\alpha_1'(t)e^{i\theta_1}, \alpha_2'(t)e^{i\theta_2})$, $(ie^{i\theta_1}, 0)$, and $(0, ie^{i\theta_2})$.

IV.3.2. Levi principal curvature

If $\boldsymbol{\xi} = x\varphi_s + y(i\varphi_s) = x\varphi_s + y(\alpha_1'/\alpha_1)\varphi_u + y(\alpha_2'/\alpha_2)\varphi_v \in H_pM$, then we have

$$\mathsf{II}_p(\boldsymbol{\xi}) = -\kappa x^2 - \frac{(\alpha_1')^2\alpha_2'}{\alpha_1}y^2 + \frac{\alpha_1'(\alpha_2')^2}{\alpha_2}y^2.$$

We also have $i\boldsymbol{\xi} = x(i\varphi_s) - y(\varphi_s) = -y\varphi_s + x(\alpha_1'/\alpha_1)\varphi_u + x(\alpha_2'/\alpha_2)\varphi_v \in H_pM$ and

$$\mathsf{II}_p(i\boldsymbol{\xi}) = -\kappa y^2 - \frac{(\alpha_1')^2\alpha_2'}{\alpha_1}x^2 + \frac{\alpha_1'(\alpha_2')^2}{\alpha_2}x^2.$$

Then the Levi form of M at p is given by

$$\begin{aligned}\mathsf{L}(\boldsymbol{\xi}) &= \frac{-1}{4}\left[\mathsf{II}_p(\boldsymbol{\xi}) + \mathsf{II}_p(i\boldsymbol{\xi})\right] \\ &= \frac{-1}{4}\left[-\kappa - \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right](x^2+y^2) \\ &= \frac{1}{4}\left[\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right](x+iy)\overline{(x+iy)}.\end{aligned}$$

Since the Levi form is already diagonalized, the principal Levi curvature of M at p is

$$\lambda = \frac{1}{4}\left[\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right].$$

If the Levi principal curvature vanishes identically, then we have

$$\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right) = \alpha_1'\alpha_2'' - \alpha_1''\alpha_2' + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right) = 0.$$

Then for some real constants $\alpha_1 = |z_1| = c_1$, or $\alpha_2 = |z_2| = c_2$, or $\dfrac{\alpha_2''}{\alpha_2'} - \dfrac{\alpha_1''}{\alpha_1'} + \dfrac{\alpha_1'}{\alpha_1} - \dfrac{\alpha_2'}{\alpha_2} = 0$ for some positive constants c_1 and c_2. The third condition gives us

$$\begin{aligned}
0 &= \frac{\alpha_2''}{\alpha_2'} - \frac{\alpha_2'}{\alpha_2} - \frac{\alpha_1''}{\alpha_1'} + \frac{\alpha_1'}{\alpha_1} \\
&= \frac{\alpha_2}{\alpha_2'} \frac{\alpha_2\alpha_2'' - (\alpha_2')^2}{(\alpha_2)^2} - \frac{\alpha_1}{\alpha_1'} \frac{\alpha_1\alpha_1'' - (\alpha_1')^2}{(\alpha_1)^2} \\
&= \frac{\alpha_2}{\alpha_2'} \left(\frac{\alpha_2'}{\alpha_2}\right)' - \frac{\alpha_1}{\alpha_1'} \left(\frac{\alpha_1'}{\alpha_1}\right)' \\
&= \left(\log\left|\frac{\alpha_2'}{\alpha_2}\right|\right)' - \left(\log\left|\frac{\alpha_1'}{\alpha_1}\right|\right)' \\
&= \left(\log\left|\frac{\alpha_1\alpha_2'}{\alpha_1'\alpha_2}\right|\right)'.
\end{aligned}$$

Hence $\alpha_1\alpha_2' = c_3\alpha_1'\alpha_2$ for some $c_3 \geq 0$. Assuming α_1 or α_2 is not constant, we get $\alpha_2'/\alpha_2 = c_3\alpha_1'/\alpha_1$ and integrating both sides gives us $\alpha_2 = c_4\alpha_1^{c_3}$, for some non-negative constant of integration c_4. Hence we get $|z_2| = c_4 |z_1|^{c_3}$.

IV.3.3. Complex-symmetric principal curvature

We also have

$$\begin{aligned}
\mathsf{R}(\boldsymbol{\xi}) &= \frac{1}{4}\left[\mathsf{II}_p(\boldsymbol{\xi}) - \mathsf{II}_p(i\boldsymbol{\xi})\right] \\
&= \frac{1}{4}\left[-\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right](x^2 - y^2).
\end{aligned}$$

Hence $\mathsf{R}(\boldsymbol{\xi}) = \mathrm{Re}\mathsf{S}(\boldsymbol{\xi})$, where

$$\mathsf{S}(\boldsymbol{\xi}) = \frac{1}{4}\left[-\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right](x + iy)^2$$

is the complex-symmetric fundamental form of M at p and the complex-symmetric principal curvature of M at p is given by

$$\sigma = \frac{1}{4}\left[-\kappa + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right].$$

If the complex-symmetric principal curvature vanishes identically, then

$$-\kappa+\alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1}-\frac{\alpha_2'}{\alpha_2}\right)=\alpha_1''\alpha_2'-\alpha_1'\alpha_2''+\alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1}-\frac{\alpha_2'}{\alpha_2}\right)=0.$$

Then either $\alpha_1=|z_1|=c_1$, or $\alpha_2=|z_2|=c_2$, or $\frac{\alpha_1''}{\alpha_1'}-\frac{\alpha_2''}{\alpha_2'}+\frac{\alpha_1'}{\alpha_1}-\frac{\alpha_2'}{\alpha_2}=0$ for some positive constants c_1 and c_2. The third condition gives us

$$\begin{aligned}
0&=\frac{\alpha_1''}{\alpha_1'}-\frac{\alpha_2''}{\alpha_2'}+\frac{\alpha_1'}{\alpha_1}-\frac{\alpha_2'}{\alpha_2}\\
&=\frac{\alpha_2}{\alpha_1'}\frac{\alpha_2\alpha_1''-\alpha_1'\alpha_2'}{(\alpha_2)^2}-\frac{\alpha_1}{\alpha_2'}\frac{\alpha_1\alpha_2''-\alpha_1'\alpha_2'}{(\alpha_1)^2}\\
&=\frac{\alpha_2}{\alpha_1'}\left(\frac{\alpha_1'}{\alpha_2}\right)'-\frac{\alpha_1}{\alpha_2'}\left(\frac{\alpha_2'}{\alpha_1}\right)'\\
&=\left(\log\left|\frac{\alpha_1'}{\alpha_2}\right|\right)'-\left(\log\left|\frac{\alpha_2'}{\alpha_1}\right|\right)'\\
&=\left(\log\left|\frac{\alpha_1\alpha_1'}{\alpha_2\alpha_2'}\right|\right)'.
\end{aligned}$$

Hence $\alpha_1\alpha_1'=c_3\alpha_2\alpha_2'$ for some $c_3\in\mathbb{R}$. Integrating both sides gives us $(\alpha_1)^2-c_3(\alpha_2)^2=c_4$ for some constant of integration $c_4\in\mathbb{R}$. Hence we get $|z_1|^2-c_3|z_2|^2=c_4$.

IV.3.4. Characteristic curvature

The characteristic vector of M at p is given by

$$T(p)=JN(p)=\left(i\alpha_2'e^{i\theta_1},-i\alpha_1'e^{i\theta_2}\right)=\frac{\alpha_2'}{\alpha_1}\varphi_u-\frac{\alpha_1'}{\alpha_2}\varphi_v$$

and hence the characteristic curvature is

$$K_T=\mathrm{II}_p(T(p))=\langle H\cdot T(p),T(p)\rangle=\frac{(\alpha_1')^3}{\alpha_2}-\frac{(\alpha_2')^3}{\alpha_1}.$$

If the characteristic curvature vanishes identically, we get

$$\frac{(\alpha_1')^3}{\alpha_2}-\frac{(\alpha_2')^3}{\alpha_1}=0$$

and hence $\alpha_1(\alpha_1')^3 = \alpha_2(\alpha_2')^3$. Taking the cube root both sides gives us $(\alpha_1)^{1/3}\alpha_1' = (\alpha_2)^{1/3}\alpha_2'$. Integrating both sides gives us $(\alpha_1)^{4/3} = (\alpha_2)^{4/3} + c$, for some constant of integration $c \in \mathbb{R}$. Since $\alpha_1 = |z_1|$ and $\alpha_2 = |z_2|$, we get $|z_1|^{4/3} = |z_2|^{4/3} + c$.

IV.3.5. Skew curvature

Since φ_s and $N(p)$ are Hermitian orthogonal, the complex tangent space H_pM is given by

$$H_pM = \operatorname{span}_{\mathbb{C}}\{\varphi_s\} = \operatorname{span}_{\mathbb{R}}\{\varphi_s, i\varphi_s\} = \operatorname{span}_{\mathbb{R}}\left\{\varphi_s, \frac{\alpha_1'}{\alpha_1}\varphi_u + \frac{\alpha_2'}{\alpha_2}\varphi_v\right\},$$

where $\operatorname{span}_{\mathbb{K}}$ is the linear span over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The skew functional $W : H_pM \to \mathbb{R}$ is given by $W(\boldsymbol{\xi}) = \langle \boldsymbol{\xi}, H \cdot T(p)\rangle$, where $\boldsymbol{\xi} \in H_pM$. So for scalars $x, y \in \mathbb{R}$, we have

$$W(x\varphi_s + y(i\varphi_s)) = -\alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_2} + \frac{\alpha_2'}{\alpha_1}\right)y.$$

Then $K_{\text{skew}} = \|W\| = \sup_{|\boldsymbol{\xi}|=1} |W(\boldsymbol{\xi})|$ and the maximum is attained when $\boldsymbol{\xi} = i\varphi_s$ (that is $x = 0$ and $y = 1$). Hence the skew curvature is given by

$$K_{\text{skew}} = \left|\alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_2} + \frac{\alpha_2'}{\alpha_1}\right)\right|.$$

If the skew curvature vanishes identically, we get $\alpha_1'\alpha_2'(\alpha_1'/\alpha_2 + \alpha_2'/\alpha_1) = 0$ which implies either $\alpha_1 = |z_1| = c_1$, or $\alpha_2 = |z_2| = c_2$, or $\alpha_1\alpha_1' + \alpha_2\alpha_2' = 0$, for some positive constants c_1 and c_2. Integrating the third condition gives us $(\alpha_1)^2 + (\alpha_2)^2 = c_3$, for some $c_3 \in \mathbb{R}$. Hence we get $|z_1|^2 + |z_2|^2 = c_3$.

IV.4. The hypersurface $|z_2| = m|z_1|$

The subset of $\mathbb{C}^2$ given by $M = \{|z_2| = m|z_1|\}$ for $m > 0$ is a smooth hypersurface away from the origin, where it has a non-Lipschitz singularity. For $m = 1$, the hypersurface M is a part of the boundary of the classical Hartogs' triangle. The Reinhardt shadow of M has the unit speed parametrization $\alpha : (0,\infty) \to \mathbb{R}^2$ given by

$$\alpha(s) = \left(\frac{s}{\sqrt{1+m^2}}, \frac{ms}{\sqrt{1+m^2}} \right). \tag{IV.6}$$

Using the formulas developed above in Theorem 8, we see that

1. the principal curvatures of M are

$$\kappa(s) = 0, \quad -\alpha_1(s)\alpha_2'(s) = \frac{-ms}{1+m^2}, \quad \text{and} \quad \alpha_1'(s)\alpha_2(s) = \frac{ms}{1+m^2},$$

and hence the Gaussian curvature is 0. The respective principal directions are

$$\frac{1}{\sqrt{1+m^2}} \left(e^{i\theta_1}, \quad me^{i\theta_2} \right), \quad (ie^{i\theta_1}, 0), \quad \text{and} \quad (0, ie^{i\theta_2}).$$

Note that, even though M has a singularity at the origin, the principal curvatures extend smoothly to this singularity and therefore the singularity is not visible to the numerical curvature. However, the principal directions are discontinuous at the point $(0,0)$.

2. the Levi principal curvature of M is

$$\lambda = \frac{1}{4}\left[\kappa(t) + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right] = \frac{1}{4}\left[0 + \frac{m}{1+m^2}\left(\frac{1}{s} - \frac{1}{s}\right)\right] = 0,$$

3. the complex-symmetric principal curvature of M is

$$\sigma = \frac{1}{4}\left[-\kappa(t) + \alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_1} - \frac{\alpha_2'}{\alpha_2}\right)\right] = \frac{1}{4}\left[0 + \frac{m}{1+m^2}\left(\frac{1}{s} - \frac{1}{s}\right)\right] = 0.$$

4. The characteristic curvature of M is

$$K_T = \frac{(\alpha_1')^3}{\alpha_2} - \frac{(\alpha_2')^3}{\alpha_1} = \frac{1}{ms(1+m^2)} - \frac{m^3}{s(1+m^2)} = \frac{1-m^4}{ms(1+m^2)} = \frac{1-m^2}{ms}.$$

Note that for $m = 1$, the characteristic curvature vanishes identically. If $m \neq 1$, then the singularity is visible to the characteristic curvature.

5. The skew curvature of M is

$$K_{\text{skew}} = \left|\alpha_1'\alpha_2'\left(\frac{\alpha_1'}{\alpha_2} + \frac{\alpha_2'}{\alpha_1}\right)\right| = \left|\frac{m}{1+m^2}\left(\frac{1}{ms} + \frac{m}{s}\right)\right| = \left|\frac{m}{1+m^2} \cdot \frac{1+m^2}{ms}\right| = \frac{1}{s}.$$

Note that, the skew curvature is independent of m and as we approach the singularity at the origin, the skew curvature blows up and hence the singularity is visible to the skew curvature as well.

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