

# *CR Functions on Subanalytic Hypersurfaces*

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ABSTRACT. A general class of singular real hypersurfaces, called *subanalytic*, is defined. For a subanalytic hypersurface  $M$  in  $\mathbb{C}^n$ , Cauchy-Riemann (or simply CR) functions on  $M$  are defined, and certain properties of CR functions discussed. In particular, sufficient geometric conditions are given for a point  $p$  on a subanalytic hypersurface  $M$  to admit a germ at  $p$  of a smooth CR function  $f$  that cannot be holomorphically extended to either side of  $M$ . As a consequence it is shown that a well-known condition of the absence of complex hypersurfaces contained in a smooth real hypersurface  $M$ , which guarantees one-sided holomorphic extension of CR functions on  $M$ , is neither a necessary nor a sufficient condition for one-sided holomorphic extension in the singular case.

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## 1. MAIN RESULTS

In this paper we define a class of non-smooth real hypersurfaces in  $\mathbb{C}^n$ , which we call *subanalytic*, and study general properties of CR functions defined on them.

Precise definitions are given in Section 2.1 and Section 3.1, but roughly speaking, a subanalytic hypersurface  $M \subset \mathbb{C}^n$  is a real codimension one subanalytic set which divides any small enough neighbourhood  $U$  of any point  $p \in M$  into two one-sided neighbourhoods  $U^\pm$ . This gives a well-defined local orientation on the smooth part  $M^{\text{reg}}$  of  $M$ , which allows us to do integration. We define a locally integrable function  $f$  on a subanalytic hypersurface  $M$  to be CR if it satisfies

$$\int_{M^{\text{reg}}} f \bar{\partial} \varphi = 0,$$

where  $\varphi$  is a test form of bidegree  $(n, n - 2)$ . In Proposition 3.2, we show that, just as with smooth hypersurfaces, continuous boundary values of holomorphic functions on subanalytic hypersurfaces are CR.

It follows from the definition that the restriction of a CR function on a subanalytic set  $M$  to its regular part  $M^{\text{reg}}$  is CR in the usual sense. Although a function on  $M$  which is CR on  $M^{\text{reg}}$  may fail in general to be CR on  $M$ , our first theorem gives some sufficient conditions for a CR function on the smooth locus of a hypersurface  $M$  to be CR on all of  $M$ .

**Theorem 1.1.** *Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$ , and let  $E$  be a subanalytic subset of  $\mathbb{C}^n$  contained in  $M$  such that  $E$  is of codimension at least one in  $M$ . Suppose that  $f$  is a function on  $M$  which is CR on  $M \setminus E$ .*

- (i) *If  $f$  is continuous on  $M$  and vanishes on  $E$ , then  $f$  is CR on  $M$ .*
- (ii) *If  $E$  has codimension at least two in  $M$ , and for  $z$  near  $E$  we have  $|f(z)| = O(\text{dist}(z, E)^{-\alpha})$ , where  $0 \leq \alpha < 1$ , then  $f$  is CR on  $M$ . In particular, if  $f$  is bounded on  $M$ , then  $f$  is a CR function on  $M$ .*

Theorem 1.1 is proved in Section 3 using the classical technique of Bochner [2]. This method can be used to prove more general results of this type, but we only develop the topic to the extent needed for our purposes.

The main thrust of this paper is in determining the conditions under which there exist continuous CR functions on subanalytic hypersurfaces which are not (locally) boundary values of holomorphic functions. For a smooth hypersurface  $M$  in  $\mathbb{C}^n$ , the existence of such functions on  $M$  near a point  $p \in M$  is equivalent to the existence of the germ of a complex analytic hypersurface  $A$  through  $p$  contained in  $M$ . We say that a real hypersurface  $M$  (smooth or subanalytic) in a complex manifold is *non-minimal* at a point  $p \in M$ , if there is a germ of a complex analytic hypersurface  $A$  such that  $p \in A$  and  $A \subset M$ . If there is no such germ  $A$ , we say that  $M$  is *minimal* at  $p$ . Therefore, non-extendable CR functions exist near a point on a smooth hypersurface (or even a hypersurface that can be locally represented as a graph near a point) provided the hypersurface is non-minimal at that point.

For subanalytic hypersurfaces, the condition for the existence of non-extendable CR functions is more subtle. First of all, without an additional topological assumption, non-minimality by itself does not suffice. Further, there is another geometric condition, first introduced in [9], called *two-sided support*, that also

gives rise to non-extendable CR functions. We say that  $M$  has *proper* two-sided support at  $p \in M$  if there is an open neighbourhood  $\Omega$  of  $p$  such that  $M$  divides  $\Omega$  into two connected components  $\Omega^+$  and  $\Omega^-$ , and there exist germs at  $p$  of distinct complex analytic hypersurfaces  $A^\pm \subset \overline{\Omega^\pm}$  such that  $A^+ \cap M = A^- \cap M$ , see Section 4.3 for details. We have the following sufficient conditions for the existence of smooth non-extendable CR functions on subanalytic hypersurfaces.

**Theorem 1.2.** *Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$  and  $p \in M$ . Suppose that at least one of the following statements holds:*

- (i)  *$M$  is non-minimal at  $p$ , i.e., there exists a complex hypersurface  $Z \subset M$  such that  $p \in Z$ , and  $Z$  divides  $M$  locally into more than one component at  $p$ .*
- (ii)  *$M$  has proper two-sided support at  $p$ .*

*Then, for every integer  $m \geq 0$  there is a CR function on  $M$  near  $p$ , of class  $C^m$ , that does not extend as a holomorphic function to either side.*

By a  $C^m$ -smooth function on  $M$ , we mean a function which is the restriction to  $M$  of a  $C^m$ -smooth function on  $\mathbb{C}^n$ . It should be noted that the assumption that the hypersurface  $M$  is subanalytic plays a crucial role in the proof. While the notion of non-minimality and proper two-sided support makes sense for singular hypersurfaces of more general types, the proof of Theorem 1.2 would fail for more general singular hypersurfaces when  $m \geq 1$ .

Note that if a hypersurface  $M$  can be represented as a graph near a point  $p$ , and  $M$  is non-minimal at  $p$ , then the complex hypersurface contained in  $M$  must divide  $M$  into two components (see [7]). Hence, the assumption that  $Z$  locally divides  $M$  in (i) of Theorem 1.2 is automatically satisfied. However, in general, a subanalytic hypersurface cannot be locally represented as the graph of a function. At such a non-graph point  $p \in M$ , it is possible that  $M$  is non-minimal, but the complex hypersurface  $A \subset M$  through  $p$  does not locally divide  $M$  near  $p$ . In Sections 5 and 6 we consider some examples of such hypersurfaces, in particular, we prove the following.

**Theorem 1.3.**

- (i) *For  $n \geq 2$ , let*

$$M_1 = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(z_1 z_2) + |z_1|^2 = 0\}.$$

*Then  $M_1$  is non-minimal at the origin, but if  $f$  is a bounded CR function near the origin, then  $f$  extends holomorphically to one side of  $M_1$ . Further, if  $f$  is  $C^k$ -smooth on  $M$  near 0, then the extension is  $C^k$  up to the boundary.*

- (ii) *For  $n \geq 3$ , let*

$$N_1 = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(z_1 z_2 + z_1 \bar{z}_3) = 0\}.$$

*Then  $N_1$  is non-minimal at the origin, but every bounded CR function on  $N_1$  near the origin extends holomorphically to a full neighbourhood of the origin.*

Further, there are infinitely many biholomorphically inequivalent hypersurfaces  $M \subset \mathbb{C}^n$ ,  $N \subset \mathbb{C}^n$  with properties described in (i) and (ii).

Consequently, *minimality is not a necessary condition for the local one-sided holomorphic extension of all CR functions on a singular hypersurface*. This is in striking contrast with smooth or graph-like hypersurfaces.

In part (i), when  $f$  is merely bounded, the boundary value on  $M_1$  of the extension should be understood in the sense of distributions on the smooth part, see Section 3.3 for details. Part (ii) implies that every bounded CR function on  $N_1$  near 0 is actually real-analytic. The proof of Theorem 1.3 is based on the construction of the local envelope of holomorphy of an arbitrarily thin one-sided neighbourhood of  $M \setminus \{z_1 = 0\}$  (resp.  $N \setminus \{z_1 = 0\}$ ), followed by an application of the Lewy extension theorem. The envelopes are obtained by an explicit construction of analytic discs and the *Kontinuitätssatz*.

In the smooth (or even graph-like) situation, minimality is a necessary, as well as sufficient condition for local holomorphic extension of CR functions to one side. The sufficiency of minimality for  $C^2$  hypersurfaces is a celebrated result [20] of Trépreau. This has been generalized to graphs of continuous real-valued functions by Chirka [8]. It was shown in [9] that for singular hypersurfaces, minimality is no longer a sufficient condition for local one-sided holomorphic extension (see Section 4.3). Combining this with Theorem 1.3 above we conclude that *for singular hypersurfaces minimality is neither a necessary nor a sufficient condition for one-sided holomorphic extension of CR functions*.

## 2. SUBANALYTIC SETS AND HYPERSURFACES

**2.1. Definitions.** Recall that a subset  $E$  of a real analytic manifold  $X$  is called *semianalytic* if it is locally defined by finitely many real analytic equations and inequalities. More precisely, for each  $p \in X$ , there is a neighbourhood  $U$  of  $p$ , and real analytic in  $U$  functions  $f_i, g_{ij}$ , where  $i = 1, \dots, r, j = 1, \dots, s$ , such that

$$E \cap U = \bigcup_{i=1}^r \left( \bigcap_{j=1}^s \{x \in U : g_{ij}(x) > 0 \text{ and } f_i(x) = 0\} \right).$$

A real analytic set is clearly semianalytic. A *subanalytic* subset  $E$  of a real analytic manifold  $X$  is one which can be locally represented as the projection of a semianalytic set. More precisely, for every  $p \in X$ , there exist a neighbourhood  $U$  of  $p$  in  $X$ , a real analytic manifold  $Y$ , and a relatively compact semianalytic set  $Z \subset X \times Y$  such that  $E \cap U = \pi(Z)$ , where  $\pi : X \times Y \rightarrow X$  is the natural projection. In particular, semianalytic sets are subanalytic. An excellent reference on semi- and subanalytic sets is [4].

The *dimension* of a subanalytic set  $E$ ,  $\dim E$ , is the maximal dimension of the germ of a real analytic submanifold contained in  $E$ . If  $E$  is a subanalytic subset of a manifold  $X$ , by  $E^{\text{reg}}$  (the regular points of  $E$ ) we denote the set of points

$p \in E$  near which  $E$  is a (real-analytic) submanifold of  $X$  of dimension  $\dim E$ . Its complement  $E \setminus E^{\text{reg}}$  (the singular locus) is denoted by  $E^{\text{sing}}$ .

We now define the class of objects in which we are interested. Let  $\mathcal{X}$  be a topological space, let  $\mathcal{Y} \subset \mathcal{X}$  be a locally closed subset (i.e., for each  $q \in \mathcal{Y}$  there is a neighbourhood  $V$  of  $q$  in  $\mathcal{X}$  such that  $V \cap \mathcal{Y}$  is a closed subset of  $V$ ), and let  $p \in \mathcal{Y}$ . We will say that  $\mathcal{Y}$  *locally separates*  $\mathcal{X}$  at  $p$  if the following holds: for every neighbourhood  $V$  of  $p$  in  $\mathcal{X}$ , there is a *connected* open neighbourhood  $U$  of  $p$  contained in  $V$ , such that the set  $U \setminus \mathcal{Y}$  has exactly two open connected components  $U^\pm$ , and we have  $\overline{U^+} \cap \overline{U^-} = \mathcal{Y} \cap \overline{U}$ . A smooth hypersurface in  $\mathbb{R}^N$  locally separates  $\mathbb{R}^N$  at each point, which is part of our intuitive idea of a hypersurface. However, if  $\mathcal{X}$  is a manifold, the notion of being locally separating is much weaker than being a topological submanifold. For example, the set  $\mathcal{Y} = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^3 - x_4^2 = 0\}$  locally separates  $\mathbb{R}^4$  at each of its points, but is not a topological submanifold of  $\mathbb{R}^4$  near 0.

We can now make the following definition.

**Definition 2.1.** A locally closed subanalytic subset  $M$  of a real analytic manifold  $X$  is called a *subanalytic hypersurface* in  $X$  if  $M$  locally separates  $X$  at each point.

We note some elementary properties of subanalytic hypersurfaces.

(1) Let  $S$  be the set of smooth points of  $M$ , i.e., the set of points near which  $M$  is a smooth manifold. Then  $S$  is a submanifold (possibly non-closed and possibly non-connected) of codimension one in  $X$ . Indeed,  $S$  is a submanifold of  $X$  near each of its points. But no submanifold of codimension two or more can locally separate  $X$ .

(2) Let  $\Omega$  be a neighbourhood of  $p \in M$  such that  $\Omega \setminus M$  has two connected components  $\Omega^\pm$ . Then each component of  $\Omega \cap M^{\text{reg}}$  is orientable, and it is possible to assign an orientation to each component in such a way that the positive normal points into  $\Omega^+$  at each point.

To see this, consider the function  $\rho$  on  $X$  given

$$\rho(x) = \begin{cases} \text{dist}(x, M) & \text{for } x \in \Omega^+, \\ -\text{dist}(x, M) & \text{for } x \in \Omega^-. \end{cases}$$

Thanks to [4, Proposition 7.4], the function  $\rho^2$  is real analytic in a neighbourhood of  $M^{\text{reg}}$ . It is easy to see that  $\rho$  itself is smooth in a neighbourhood of  $M^{\text{reg}}$  and  $\nabla \rho$  is non-zero on  $M^{\text{reg}}$ . We orient each component of  $M^{\text{reg}}$  so that  $\nabla \rho$  is the positive normal.

(3) Here and in the sequel we denote by  $\mathcal{H}$  the Hausdorff measure of codimension 1 in  $\mathbb{R}^N$ . Let  $M \subset \mathbb{R}^N$  be a subanalytic hypersurface. We denote by  $L^1_{\text{loc}}(M)$  the space of functions on  $M$  that are locally integrable on  $M$  with respect to the

measure  $\mathcal{H}$  in the following sense:  $f \in L^1_{\text{loc}}(M)$ , if for every compact  $K$  contained in an open set  $U \subset X$ , where  $U$  has the property that  $M \cap U$  is closed in  $U$ , we have  $\int_{K \cap M} |f| \, d\mathcal{H} < \infty$  (since  $M$  is locally closed, every point has a neighbourhood in which  $M$  is closed). It will be clear from Lemma 2.3 that every bounded function on  $M$  is in  $L^1_{\text{loc}}(M)$ .

**2.2. Properties.** Subanalytic sets enjoy several closure properties: locally finite unions and intersections, set-theoretic differences, complements, topological closures and interiors, and proper projections onto linear subspaces of subanalytic sets are subanalytic. The sets of regular points and the singular locus of a subanalytic set are themselves subanalytic.

A fundamental property of subanalytic sets [10] is that they admit a stratification by real analytic manifolds. In fact, any subanalytic set  $X$  in an open set  $\Omega \subset \mathbb{R}^N$  can be represented as a locally finite disjoint union of subanalytic subsets  $A_i$  of  $\Omega$ , where each  $A_i$  is a (possibly non-closed) real analytic submanifold of  $\Omega$ , and the family  $\{A_i\}$  satisfies the *frontier* condition: if  $A_k \cap \overline{A_\ell}$  is nonempty, then  $A_k \subset \overline{A_\ell}$ , and  $\dim A_k < \dim A_\ell$ . For a subanalytic hypersurface, which is a bounded subanalytic subset of  $\mathbb{R}^N$ , the stratification will consist of a finite number of strata. Moreover, given a subanalytic set  $E \subset X$ , the stratification of  $X$  may be chosen to be compatible with  $E$ , i.e.,  $E$  (and therefore  $X \setminus E$ ) is a union of strata. The maximum dimension of a stratum in a stratification of a subanalytic set  $E$  equals  $\dim E$ , the dimension of  $E$ , and therefore, it is independent of the choice of stratification.

An important property of subanalytic sets is *Łojasiewicz’s inequality*. In a simple form that suffices for our purposes it can be stated as follows: let  $K$  be a subset of  $\mathbb{R}^N$ , and  $f : K \rightarrow \mathbb{R}$  be a function such that its graph is a compact subanalytic set in  $\mathbb{R}^{N+1}$ , and let  $X = f^{-1}(0)$ . Then there exist  $C, r > 0$  such that for any  $x \in K$ ,

$$(2.1) \quad |f(x)| \geq C \operatorname{dist}(x, X)^r.$$

Łojasiewicz’s inequality implies that any two subanalytic sets  $X$  and  $Y$  in  $\mathbb{R}^N$  are *regularly situated*, i.e., for any  $x_0 \in X \cap Y$  there exist a neighbourhood  $V$  of  $x_0$ , and  $C, r > 0$  such that for any  $x \in V$ ,

$$(2.2) \quad \operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \operatorname{dist}(x, X \cap Y)^r.$$

For more details see, e.g., [4]. These inequalities are crucial for our construction of smooth non-extendable CR functions.

Now we recall some metric properties of subanalytic sets. Let  $\Gamma \subset \mathbb{R}^N$  be an analytic subvariety (not necessarily closed.) Suppose there exist an open set  $U$  in a linear subspace  $E \subset \mathbb{R}^N$ , with  $\dim E = k$ , and an analytic map  $\varphi : U \rightarrow E^\perp$  with values in the orthogonal complement of  $E$ , such that  $\Gamma$  is the graph of  $\varphi$

in  $E \oplus E^\perp = \mathbb{R}^N$ . We say that  $\Gamma$  is an  $\varepsilon$ -analytic patch if the differential of  $\varphi$  is bounded by  $\varepsilon$ , i.e.,  $\|d\varphi(x)\| < \varepsilon$  for each  $x \in U$ .

Note that a smooth analytic submanifold of  $\mathbb{R}^N$  can be covered by a family of  $\varepsilon$ -analytic patches, thanks to the implicit function theorem. The following result shows that any bounded subanalytic set can be almost covered by finitely many  $\varepsilon$ -analytic patches.

**Result 2.2** ([13, Proposition 1.4; 17, Theorem 3.3]). *Let  $Y \subset \mathbb{R}^N$  be a bounded subanalytic set with  $\dim Y = k$ , and let  $\varepsilon > 0$ . Then there are disjoint subanalytic sets  $\Gamma_1^\varepsilon, \dots, \Gamma_k^\varepsilon$  contained in  $Y$  such that*

(i)  $\dim(Y \setminus \bigcup_{i=1}^k \Gamma_i^\varepsilon) < k$ ,

and

(ii) *the  $\Gamma_i^\varepsilon$  are  $\varepsilon$ -analytic patches.*

We now draw a corollary from this result. Since we are interested mainly in hypersurfaces, we assume that the subanalytic sets are of codimension one, although analogous results are easily proved for higher codimension.

**Lemma 2.3.** *Let  $M$  be a bounded subanalytic subset of  $\mathbb{R}^N$ ,  $\dim M = N - 1$ . There is a constant  $C > 0$  such that if  $A$  is an affine subspace of  $\mathbb{R}^N$  with  $A \cap \Omega \subset M$ , then*

$$\mathcal{H}(M \cap B(A, r)) \leq Cr^{N-\dim A-1},$$

where  $B(A, r) = \{x \in \mathbb{R}^N : \text{dist}(x, A) < r\}$ .

*Proof.* Let  $\mathbb{B}$  be a large enough ball in  $\mathbb{R}^N$  such that  $M \subset \mathbb{B}$ . Fix  $\varepsilon > 0$ , and using the theorem quoted before this lemma, write  $M$  as the disjoint union of a finite number  $K$  of  $\varepsilon$ -analytic patches  $\{\Gamma_i\}_{i=1}^K$  each of dimension  $N - 1$ , and a subanalytic set  $R$ , with  $\dim R < N - 1$ . Since  $\mathcal{H}(R) = 0$ , it follows that,

$$\mathcal{H}(B(A, r) \cap M) = \sum_{i=1}^K \mathcal{H}(B(A, r) \cap \Gamma_i).$$

Therefore, to prove the claim it is sufficient to show that for each  $i$  we have

(2.3) 
$$\mathcal{H}(B(A, r) \cap \Gamma_i) \leq C \cdot r^{N-\dim A-1},$$

for some constant  $C$  independent of  $A$ . We fix such an  $i$ .

There is an  $N - 1$  dimensional subspace  $E_i$  of  $\mathbb{R}^N$ , an open set  $U_i \subset E_i$ , and a real analytic map  $\varphi_i : U_i \rightarrow E_i^\perp$  such that  $\Gamma_i$  is the graph of  $\varphi_i$  in  $E_i \oplus E_i^\perp$ . Note that  $U_i \subset \mathbb{B}$  automatically holds. Denote by  $\pi_i$  the orthogonal projection from  $\mathbb{R}^N$  to  $E_i$ . Then,  $\pi_i(\Gamma_i) = U_i$ , and

$$\begin{aligned} \pi_i(B(A, r) \cap \Gamma_i) &\subset \pi_i(B(A, r)) \cap U_i \\ &\subset B(\pi_i(A), r) \cap U_i, \end{aligned}$$

since  $\pi_i$  does not increase distances. Note that  $\dim \pi_i(A) = \dim A$ , since  $\pi_i$  is injective on  $\Gamma_i$ . It is easy to see that

$$\mathcal{H}(B(\pi_i(A), r) \cap U_i) \leq C \cdot r^{N-\dim A-1},$$

where  $C$  depends only on the radius of the ball  $\mathbb{B}$  but is independent of  $A$  and  $r$ . Therefore,

$$\begin{aligned} \mathcal{H}(B(A, r) \cap \Gamma_i) &= \int_{\pi_i(B(A, r) \cap \Gamma_i)} \sqrt{1 + |d\varphi_i|^2} \, d\mathcal{H}(x) \\ &\leq \sqrt{1 + \varepsilon^2} \cdot \mathcal{H}(\pi_i(B(A, r) \cap \Gamma_i)) \\ &\leq C \sqrt{1 + \varepsilon^2} r^{N-\dim A-1}, \end{aligned}$$

which proves the result. □

### 3. CR FUNCTIONS AND REMOVABLE SINGULARITIES

**3.1. CR functions on singular hypersurfaces.** Recall that for a smooth real hypersurface  $M$  in a complex manifold  $X$ , a function  $f$  on  $M$  is called *CR* if it satisfies the tangential Cauchy-Riemann equations  $\bar{\partial}_b f = 0$ . More generally, we can consider distributions which satisfy this equation. If  $f \in L^1_{\text{loc}}(M)$  and  $M$  is orientable, the tangential Cauchy-Riemann equations can be rewritten in the adjoint form:  $f$  is CR if and only if for each  $\varphi \in \mathcal{D}^{n, n-2}(X)$  (the space of  $C^\infty$  forms with compact support in  $X$  of bidegree  $(n, n - 2)$ ) we have  $\int_M f \bar{\partial} \varphi = 0$ .

We wish to generalize the notion of CR functions to singular hypersurfaces. In [9], we considered the class of real-analytic hypersurfaces. The smooth part of such hypersurfaces is always locally orientable, and this allows us to define CR functions using the adjoint form of the Cauchy-Riemann equations.

For the more general class of subanalytic hypersurfaces considered here, we define CR functions in the same way. Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$ ,  $n \geq 2$ , and let  $f \in L^1_{\text{loc}}(M)$ . We say that  $f$  is *CR* at  $p \in M$ , if there exists an open neighbourhood  $\Omega$  of  $p$  in  $\mathbb{C}^n$  such that  $\Omega \setminus M$  has exactly two connected components,  $\Omega^+$  and  $\Omega^-$ , and for every  $\varphi \in \mathcal{D}^{n, n-2}(\Omega)$ , we have

$$(3.1) \quad \int_{M^{\text{reg}} \cap \Omega} f \bar{\partial} \varphi = 0,$$

where each component of  $M^{\text{reg}} \cap \Omega$  is oriented in such a way that the positive normal points into  $\Omega^+$ . We say that  $f$  is CR on  $M$  if it is CR at each point of  $M$ . When  $M$  and  $f$  are smooth, this is equivalent to the tangential Cauchy-Riemann equations  $\bar{\partial}_b f = 0$ .

It follows from the definition that being CR is a local property. If now  $\omega$  is an open set of a subanalytic hypersurface  $M$ , such that there exists a well-defined



orientation on  $\omega$ , then a simple argument involving partition of unity shows that if  $f$  is CR at every point of  $\omega$ , then  $f$  is CR on  $\omega$  in the sense that (3.1) holds for some open neighbourhood  $\Omega$  of  $\omega$  and all forms  $\varphi$  with compact support in  $\Omega$ .

**3.2. Proof of Theorem 1.1.** Let  $M$  be a bounded subanalytic hypersurface in  $\mathbb{C}^n$ . For a subset  $E$  of  $M$  and a function  $f$  on  $M$  define

$$S_f(E, r) = \operatorname{ess. sup}_{\mathcal{H}} \limits_{r/2 < \operatorname{dist}(z, E) < r} |f(z)|,$$

i.e.,  $S_f(E, r)$  is the essential supremum (with respect to the measure  $\mathcal{H}$ ) of  $|f|$  over the points on  $M$  which are at distance at least  $r/2$  and at most  $r$  from  $E$ . We let  $\operatorname{CR}(M)$  denote the space of CR functions on  $M$ .

Now let  $E$  be a closed subanalytic subset of  $M$  of dimension at most  $2n - 2$ . We fix a stratification of the subanalytic hypersurface  $M$  by finitely many disjoint subanalytic subsets of  $\mathbb{C}^n$  which are real analytic submanifolds of  $\mathbb{C}^n$  satisfying the frontier condition and compatible with  $E$ , i.e.,  $E$  and  $M \setminus E$  are the union of some strata. We have the following result:

**Proposition 3.1.** *Let  $f \in L^1_{\operatorname{loc}}(M) \cap \operatorname{CR}(M \setminus E)$ , let  $A(p)$  be the stratum through  $p \in M$ , and let  $k(p) = \dim A(p)$ . If at each  $p \in E$  we have*

$$(3.2) \quad \lim_{r \rightarrow 0^+} r^{2n - k(p) - 2} S_f(A(p), r) = 0,$$

then  $f$  is CR on  $M$ .

*Proof of Proposition 3.1.* For the given stratification (compatible with  $E$ ), denote by  $E^d$  the union of all strata contained in  $E$  of dimension greater than or equal to  $d$ . Then  $E^0 = E$ , the inclusion  $E^{d+1} \subset E^d$  holds, and  $E^{2n-1} = \emptyset$ . The proof will consist of an inductive process, in which we assume that  $f \in \operatorname{CR}((M \setminus E) \cup E^{d+1})$  and deduce that  $f \in \operatorname{CR}((M \setminus E) \cup E^d)$ . Since by hypothesis,  $f \in \operatorname{CR}((M \setminus E) \cup E^{2n-1})$  the proof will be completed in at most  $2n - 1$  iterations of this process.

For  $0 \leq d \leq 2n - 2$ , assume  $f \in \operatorname{CR}((M \setminus E) \cup E^{d+1})$ , and let  $p \in E^d \setminus E^{d+1}$ , so that  $k(p) = d$ . We fix a neighbourhood  $U$  of  $p$  in  $\mathbb{C}^n$ , with the following properties:

- (i)  $A(p) \cap U$  can be “flattened” by a real analytic diffeomorphism (possible since  $A(p)$  is locally a manifold),
- (ii)  $U$  does not intersect any stratum of dimension less than  $d$  (possible by the frontier condition),
- (iii)  $U \setminus M$  has exactly two components  $U^\pm$  (possible since  $M$  is locally separating).

For convenience, let  $A = A(p)$ , and let  $B(A, r) = \{x \in \mathbb{C}^n : \operatorname{dist}(x, A) < r\}$ . It is easy to see that for  $r > 0$  small, there is a cutoff function  $\psi_r$  supported in

$B(A, r)$ , such that  $\psi_r \equiv 1$  in  $B(A, r/2)$  and

$$|\bar{\partial}\psi_r| = O\left(\frac{1}{r}\right).$$

Note that  $\text{supp}(\bar{\partial}\psi) \subset B(A, r) \setminus B(A, r/2)$ .

We need to show the following: for every  $\varphi \in \mathcal{D}^{(n, n-2)}(U)$ , we have

$$\int_{M^{\text{reg}} \cap U} f \bar{\partial}\varphi = 0,$$

where  $M^{\text{reg}} \cap U$  is oriented as in equation (3.1). We write  $\varphi = (1 - \psi_r)\varphi + \psi_r\varphi$ . Then  $(1 - \psi_r)\varphi$  is an  $(n, n - 2)$  form with support in  $U \setminus A$ . Since  $(U \setminus A) \cap M \subset (M \setminus E) \cup E^{d+1}$ , and by hypothesis  $f$  is CR on  $(M \setminus E) \cup E^{d+1}$ , we have

$\int_{M^{\text{reg}}} f \bar{\partial}((1 - \psi_r)\varphi) = 0$ . Therefore,

$$\begin{aligned} \int_{M^{\text{reg}} \cap U} f \bar{\partial}\varphi &= \int_{M^{\text{reg}} \cap U} f \bar{\partial}(\psi_r\varphi) \\ &= \int_{M^{\text{reg}} \cap U} f \bar{\partial}\psi_r \wedge \varphi + \int_{M^{\text{reg}} \cap U} f \psi_r \bar{\partial}\varphi. \end{aligned}$$

Since  $\psi_r \rightarrow 0$  pointwise on  $M^{\text{reg}}$  and  $f \in L^1_{\text{loc}}(M)$ , it follows easily from the dominated convergence theorem that the second term approaches 0 as  $r \rightarrow 0+$ . Therefore, it remains to show that the first term approaches 0 as  $r \rightarrow 0+$ . Since  $\text{supp}(\bar{\partial}\psi_r) \subset B(A, r) \setminus B(A, r/2)$  and  $\text{supp}(\varphi) \subset U$ ,

$$\begin{aligned} &\left| \int_{M^{\text{reg}} \cap U} f \bar{\partial}\psi_r \wedge \varphi \right| \\ &\leq C \cdot \text{ess. sup}_{z \in (B(A, r) \setminus B(A, r/2)) \cap U} |f(z)| \cdot \frac{1}{r} \cdot \mathcal{H}(B(A, r) \cap M^{\text{reg}}) \\ &\leq C \left( \text{ess. sup}_{z \in (B(A, r) \setminus B(A, r/2)) \cap U} |f(z)| \right) \frac{1}{r} r^{2n-d-1} \quad \text{by Lemma 2.3} \\ &\leq CS_f(A, r)r^{2n-d-2}. \end{aligned}$$

Our result now follows from equation (3.2). □

*Proof of Theorem 1.1.*

(i) We fix some stratification of  $M$  compatible with  $E$ . Then for each  $p \in E$  we have  $S(A(p), r) \rightarrow 0$  as  $r \rightarrow 0$ . However, since  $k(p) \leq 2n - 2$  for each point  $p \in E$ , we have  $2n - k(p) - 2 \geq 0$ . Therefore, (3.2) is satisfied, and the result follows from Proposition 3.1.

(ii) We first verify that  $f \in L^1_{loc}(M)$  (this is true even if  $E$  has codimension one in  $M$ ). We fix a stratification of  $M$  compatible with  $E$ . Let  $p \in E$ , and let  $A$  be the stratum of  $M$  through  $p$ . We define for  $\nu \in \mathbb{N}$ ,

$$K_\nu = \{z \in M : 2^{-(\nu+1)} < \text{dist}(z, A) \leq 2^{-\nu}\}.$$

By Lemma 2.3,

$$\begin{aligned} \mathcal{H}(K_\nu) &\leq \mathcal{H}(B(A, 2^{-\nu}) \cap M) \\ &= O((2^{-\nu})^{2n-\dim A-1}) \\ &= O(2^{-\nu}). \end{aligned}$$

Note also that if  $z \in K_\nu$ ,  $|f(z)| = O(2^{\nu\alpha})$ . Therefore,

$$\begin{aligned} \int_{M^{\text{reg}} \cap B(A, 1)} |f(z)| \, d\mathcal{H}(z) &\leq C \sum_{\nu=0}^{\infty} 2^{\nu\alpha} \cdot 2^{-\nu} \\ &= \frac{C}{1 - 2^{\alpha-1}} < \infty. \end{aligned}$$

To verify that  $f$  is CR, we note that for any  $p \in E$ , we have  $S(A(p), r) \leq Cr^{-\alpha}$  and  $2n - k(p) - 2 \geq 1$ . Therefore, near  $p$ ,  $S(A(p), r)r^{2n-k(p)-2} \leq Cr^{1-\alpha} \rightarrow 0$  as  $r \rightarrow 0+$ , and the result again follows from Proposition 3.1.  $\square$

Let  $M$  be a smooth hypersurface in  $\mathbb{C}^n$ , minimal at a point  $p \in M$ . Let  $E$  be a smooth real submanifold of  $\mathbb{C}^n$  of codimension two, and  $E \subset M$ . Then, after shrinking  $M$ ,  $M \setminus E$  has two components  $M^\pm$ . Let  $f$  be the function on  $M$  which is 1 on  $M^+$  and 0 on  $M^-$ . Then  $f \in L^\infty_{loc}(M) \subset L^1_{loc}(M)$ , and  $f \in \text{CR}(M \setminus E)$ . We claim that  $f$  is *not* CR. Indeed, if  $f$  were CR, it would extend holomorphically to a one-sided neighbourhood of  $p$  by Trépreau’s theorem. Then the extension had to be both identically 1 and identically 0, which is a contradiction. Therefore,  $f$  is not CR, and *singularities of codimension 1 are not in general removable for locally bounded functions*.

**3.3. CR functions as boundary values of holomorphic functions.** Let  $M$  be a smooth real hypersurface in  $\mathbb{C}^n$ . Let  $f$  be a bounded holomorphic function defined on one side of  $M$  (more generally we may assume that  $f$  is of polynomial growth near  $M$ ). Then there is a well-defined CR distribution  $\text{bv } f$  on  $M$  acting on a test function  $\chi$  on  $M$  by

$$\text{bv } f(\chi) = \lim_{t \rightarrow 0+} \int_M f(z + t\nu(z))\chi(z) \, d\mathcal{H},$$

where  $\nu$  is the unit normal vector to  $M$  with a suitable orientation. The distribution  $\text{bv } f$  is called the boundary value of  $f$  on  $M$ . In particular, if  $f$  is continuous

up to  $M$ , then simply by  $f = f|_M$ . For more details, see [3] or [14]. Conversely, given a distribution  $u$  on  $M$ , we say that  $u$  extends to a holomorphic function  $f$  on one side of  $M$ , if  $\text{bv } f = u$ .

For a subanalytic hypersurface  $M$  and a one-sided connected neighbourhood  $\omega$  of  $M$  we say that a holomorphic function  $F$  on  $\omega$  is the holomorphic extension of a CR function  $f$  on  $M$  if  $f|_{M^{\text{reg}}}$  is the boundary value of  $F$  on the smooth hypersurface  $M^{\text{reg}}$  in the sense of the previous paragraph.

Just as for smooth hypersurfaces, continuous boundary values of holomorphic functions on subanalytic hypersurfaces are CR. More precisely, the following holds.

**Proposition 3.2.** *Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$  and let  $\Omega$  be a domain in  $\mathbb{C}^n$  such that  $\Omega \setminus M$  has two components  $\Omega^+$  and  $\Omega^-$ . Let  $f$  be a continuous function on  $\Omega^+ \cup (M \cap \Omega)$  such that  $f|_{\Omega^+}$  is holomorphic. Then  $f|_{M \cap \Omega}$  is CR.*

*Proof.* By Theorem 1.1 (ii), we only need to consider the case when  $M^{\text{sing}}$  has codimension one in  $M$ . We fix a stratification of  $M$  compatible with  $M^{\text{sing}}$ . Thanks to Theorem 1.1 (ii) again, it is sufficient to show that  $f$  is CR near every point  $p$  of any stratum  $A$  of dimension  $2n - 2$  contained in  $M^{\text{sing}}$ . Fix  $p$ , and shrink  $\Omega$  around  $p$  so that  $\Omega \cap M$  is the disjoint union of  $A \cap \Omega$  and  $M^{\text{reg}} \cap \Omega$  (this is possible by the frontier condition). To prove that  $f|_{M \cap \Omega}$  is CR we need to show that

$$(3.3) \quad \int_{M^{\text{reg}} \cap \Omega} f \bar{\partial} \varphi = 0$$

for any  $\varphi \in \mathcal{D}^{n, n-2}(\Omega)$ .

By Sard’s theorem, we can find a sequence  $\varepsilon_j \searrow 0$  such that the smooth hypersurfaces  $A_j = \{z \in \Omega : \text{dist}(z, A) = \varepsilon_j\}$  meet  $M^{\text{reg}}$  transversely. We set  $A_j^+ = A_j \cap \overline{\Omega^+}$ ,  $M_j = \{z \in M^{\text{reg}} : \text{dist}(z, A) \geq \varepsilon_j\}$ , and  $M_j^+ = M_j \cup A_j^+$ . Note that  $M_j^+$  divides  $\Omega$  into two open sets (not necessarily connected)

$$\Omega_j^+ = \{z \in \Omega^+ : \text{dist}(z, A) > \varepsilon_j\}$$

and

$$\Omega_j^- = \Omega \setminus \overline{\Omega_j^+}.$$

We orient  $(M_j^+)^{\text{reg}}$  by declaring that at each point the normal into the set  $\Omega_j^+$  is the positively directed normal. This allows us to define CR functions on  $M_j^+$  in the usual way. Fixing  $j$  for now, we claim that  $f|_{M_j^+}$  is CR. Indeed, since  $M_j$  is smooth, the restriction of  $f$  to the interior of  $M_j$  is CR (boundary value of a holomorphic function on a smooth boundary). Moreover,  $f|_{A_j^+}$  is also CR (restriction of a holomorphic function). Thus, it remains to show that  $f$  is CR near any point  $q \in A_j \cap M_j$ . For this we adapt the argument in [11, Proposition 4].

It suffices to show that there exists a small neighbourhood  $\omega$  of  $q$  in  $\mathbb{C}^n$  such that for any form  $\psi \in \mathcal{D}^{n,n-2}(\omega)$ ,

$$(3.4) \quad \int_{(M_j^+ \cap \omega)^{\text{reg}}} f \bar{\partial} \psi = \int_{\text{int}(M_j) \cap \omega} f \bar{\partial} \psi + \int_{\text{int}(A_j^+) \cap \omega} f \bar{\partial} \psi = 0.$$

Let  $\omega$  be a small enough ball with centre at  $q$  in  $\mathbb{C}^n$ , so that, by the Baouendi-Tréves theorem [3, Theorem 2.4.1] the CR function  $f$  can be approximated uniformly on  $M^{\text{reg}} \cap \omega$  by a sequence of holomorphic polynomials  $f_\nu$  on  $\mathbb{C}^n$ . Further, by Sard’s theorem, after shrinking  $\omega$ , we can assume that the sphere  $\partial\omega$  meets  $M_j$  transversely, so that Stokes’ theorem applies to the domain  $\text{int}(M_j) \cap \omega$ . Then

$$(3.5) \quad \begin{aligned} \int_{\text{int}(M_j) \cap \omega} f \bar{\partial} \psi &= \lim_{\nu \rightarrow \infty} \int_{\text{int}(M_j) \cap \omega} f_\nu \bar{\partial} \psi \\ &= \lim_{\nu \rightarrow \infty} \int_{\text{int}(M_j) \cap \omega} d(f_\nu \psi) \\ &= \int_{M_j \cap A_j \cap \omega} f \psi, \end{aligned}$$

where the last equality holds by Stokes’ theorem.

Now we let  $\{V_k\}$  be an exhaustion of  $\text{int}(A_j^+) \cap \omega$  by smoothly bounded relatively compact subdomains. Note that  $f$  is actually holomorphic in a neighbourhood of each  $V_k$ . A parallel computation gives

$$\int_{V_k} f \bar{\partial} \psi = \int_{V_k} \bar{\partial}(f \psi) = \int_{V_k} d(f \psi) = \int_{\partial V_k} f \psi.$$

Letting  $k$  go to infinity, we obtain, using the continuity of  $f$ :

$$(3.6) \quad \int_{\text{int}(A_j^+) \cap \omega} f \bar{\partial} \psi = \int_{M_j \cap A_j \cap \omega} f \psi.$$

Since in equations (3.5) and (3.6) the integrals on the right are taken with the opposite orientation, (3.4) follows. Hence,  $f$  is CR on  $M_j^+ \cap \omega$ , and therefore, on  $M_j^+$ .

Since  $f \in \text{CR}(M_j^+)$ , it follows that for any  $\varphi \in \mathcal{D}^{n,n-2}(\Omega)$ ,

$$\int_{\text{int}(M_j)} f \bar{\partial} \varphi = - \int_{\text{int}(A_j)} f \bar{\partial} \varphi.$$

Therefore,

$$\int_{M^{\text{reg}} \cap \Omega} f \bar{\partial} \varphi = \lim_{j \rightarrow \infty} \int_{\text{int}(M_j)} f \bar{\partial} \varphi = - \lim_{j \rightarrow \infty} \int_{\text{int}(A_j)} f \bar{\partial} \varphi = 0,$$

since  $\text{Vol}(A_j) \rightarrow 0$ . This proves the proposition. □

If in Proposition 3.2 we assume more about  $M$ , then we can relax the condition that  $f$  is continuous up to the boundary. For example, Theorem 1.1 (ii) implies that if  $M^{\text{sing}}$  has codimension at least two, and  $f$  is a bounded holomorphic function on  $\Omega^+$ , then  $\text{bv } f$  is a CR function. This gives examples of non-continuous CR functions on  $M$  as boundary values of holomorphic functions on  $\Omega^+$ .

**3.4. Jump formula.** We recall some facts regarding the jump representation of CR functions, which in the smooth case goes back to the work of Andreotti-Hill [1] and Chirka [5] (see also [14] for a detailed account). Let  $\Omega$  be a domain in  $\mathbb{C}^n$  such that  $H^{0,1}(\Omega) = 0$  (for example we can take  $\Omega$  to be pseudoconvex). Let  $M$  be a subanalytic hypersurface, which is closed in  $\Omega$  such that  $\Omega \setminus M$  consists of two connected components  $\Omega^\pm$ . Orient  $M^{\text{reg}}$  such that the positive normal points into  $\Omega^+$  at each point. Denote by  $[M]$  the current of integration of degree one in  $\Omega$  defined by  $[M]\varphi = \int_{M^{\text{reg}}} \varphi$  for compactly supported smooth  $(2n - 1)$  forms  $\varphi$  in  $\Omega$  (this is well defined since  $M$  has locally finite  $\mathcal{H}$ -measure.) Let  $[M] = [M]^{0,1} + [M]^{1,0}$  be the natural splitting of  $[M]$  into currents of bidegree  $(0, 1)$  and  $(1, 0)$  respectively. Let  $f$  be a CR function on  $M$ . Then the fact that  $f$  is CR (i.e., equation (3.1) holds) can be expressed in the language of currents by the equation  $\bar{\partial}(f[M]^{0,1}) = 0$ .

Since  $H^{0,1}(\Omega) = 0$ , the equation

$$(3.7) \quad \bar{\partial}u = f[M]^{0,1}$$

can be solved for a distribution  $u$  on  $\Omega$ . We set  $f^\pm = u|_{\Omega^\pm}$ . Then  $f^\pm$  are holomorphic on  $\Omega^\pm$ , and a study of the local behavior of  $f^\pm$  near  $M^{\text{reg}}$  using the Bochner-Martinelli transform (see, e.g., [14]) shows that the following *Jump formula* holds in the sense of distributions on  $M^{\text{reg}}$ :

$$(3.8) \quad f = \text{bv } f^+ - \text{bv } f^-.$$

Since  $f \in L^1_{\text{loc}}(M)$ , we have, in fact, a stronger result that for every compact  $K \subset \Omega$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{M^{\text{reg}} \cap K} |f^+(\zeta + \varepsilon\nu(\zeta)) - f^-(\zeta - \varepsilon\nu(\zeta)) - f(\zeta)| d\mathcal{H}(\zeta) = 0,$$

where  $\nu(\zeta)$  is the unit normal vector to  $M^{\text{reg}}$  at  $\zeta \in M$ .

#### 4. NON-EXTENDABLE CR FUNCTIONS: PROOF OF THEOREM 1.2

**4.1. Preliminaries.** In this section we give two sufficient conditions for the existence of CR functions on a subanalytic hypersurface which do not admit local holomorphic extensions to either side. The first condition is non-minimality together with an additional topological assumption. The second situation when non-extendable CR functions arise is due to two-sided support.

Throughout the section  $\mu$  will denote the multi-index  $\mu = (\mu_1, \dots, \mu_n)$ , where the  $\mu_j$  are non-negative integers, and  $|\mu| = \mu_1 + \dots + \mu_n$ , further,

$$D^\mu := \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \dots \partial x_n^{\mu_n}}$$

will denote the operator of partial differentiation of order  $\mu$ . With this notation the *Leibniz formula* takes the form

$$(4.1) \quad D^\mu(fg) = \sum_{k \leq \mu} \frac{\mu!}{k!(\mu - k)!} D^k f d^{\mu-k} g,$$

where  $\mu! = \mu_1! \mu_2! \dots \mu_n!$  and  $k \leq \mu$  means that  $k_j \leq \mu_j$  for each  $j = 1, \dots, n$ .

For an open set  $\Omega \subset \mathbb{R}^n$ , and for an integer  $m \geq 0$ , we denote as usual by  $C^m(\Omega)$  the space of functions on  $\Omega$  whose partial derivatives of order  $\leq m$  exist and are continuous. By  $C_b^m(\Omega)$  we denote the subspace of  $C^m(\Omega)$  consisting of functions with *bounded* partial derivatives of order  $\leq m$ . For  $f \in C^m(\Omega)$  and  $x \in \Omega$ , let

$$|f(x)|_{C^m} = \sum_{|k| \leq m} |D^k f(x)|.$$

Then, using the Leibniz rule, it is easy to prove by induction that for  $|\mu| = m$ ,

$$(4.2) \quad \left| D^\mu \left( \frac{1}{g(x)} \right) \right| \leq C_m \frac{(|g(x)|_{C^m})^m}{|g(x)|^{1+m}}.$$

The following two lemmas will be used in the proof of Theorem 1.2.

**Lemma 4.1.** *Let  $\Omega \Subset \mathbb{R}^n$  be a bounded domain, and  $E$  be a closed subset of  $\Omega$ . Let  $m \geq 0$ . Suppose that  $f \in C^m(\Omega \setminus E)$ , and  $g \in C_b^m(\Omega)$  is such that  $|g(x)| \leq C \text{dist}(x, E)$  for some  $C > 0$ , and for every multi-index  $\mu$ ,  $|\mu| \leq m$ , there are constants  $B_\mu > 0$ ,  $p(\mu) \geq 0$  such that*

$$|D^\mu f(x)| \leq B_\mu \text{dist}(x, E)^{-p(\mu)}.$$

Then there exists an integer  $L$  such that the function

$$(4.3) \quad h = \begin{cases} f g^L & \text{on } \Omega \setminus E, \\ 0 & \text{on } E, \end{cases}$$

is in  $C_b^m(\Omega)$ .

**Lemma 4.2.** *Let  $E_1, E_2$  be closed subanalytic subsets of  $\mathbb{R}^n$ , let  $Z = E_1 \cap E_2 \neq \emptyset$  and let  $z \in Z$ . Then there exist a neighbourhood  $\Omega$  of  $z$  in  $\mathbb{C}^n$ , a function  $\chi \in C^\infty(\Omega \setminus Z)$  with  $0 \leq \chi \leq 1$ , and a constant  $r > 0$  such that:*

- (1)  $\chi \equiv 1$  in a neighbourhood  $U_1$  of  $E_1 \setminus Z$  in  $\Omega \setminus Z$ ; and  $\chi \equiv 0$  in a neighbourhood  $U_2$  of  $E_2 \setminus Z$  in  $\Omega \setminus Z$ .
- (2) Neighbourhoods  $U_1$  and  $U_2$  can be chosen in such a way that there exists a constant  $c$  such that if  $x$  is not in  $U_1$  (resp.  $U_2$ ), then

$$\text{dist}(x, E_1) \geq c \text{dist}(x, Z)^r \quad (\text{resp. } \text{dist}(x, E_2) \geq c \text{dist}(x, Z)^r).$$

- (3) For every integer  $m \geq 0$ , there is a constant  $C > 0$  such that for any  $\mu$  with  $|\mu| = m$ ,

$$(4.4) \quad |D^\mu(\chi(x))| \leq C \text{dist}(x, Z)^{-rm^3}.$$

In order not to interrupt the flow of the proof, we postpone the proofs of Lemmas 4.1 and 4.2 to Section 4.4.

**4.2. Non-minimality.** Recall that a subanalytic hypersurface  $M$  is said to be non-minimal at a point  $p \in M$ , if there is a complex hypersurface  $A$  in a neighbourhood  $\omega$  of  $p \in \mathbb{C}^n$  which passes through the point  $p$  and is contained in  $M$ . In general, as shown by Theorem 1.3, unlike the smooth case, non-minimality does not directly imply the existence of non-extendable CR functions. The following proposition proves Theorem 1.2 if condition (1) holds.

**Proposition 4.3.** *Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$  and  $p \in M$ . Suppose there is a germ of a complex hypersurface  $Z \subset M$  such that  $p \in Z$ ,  $Z$  divides  $M$  locally into more than one component at  $p$ . Then, for every integer  $m \geq 0$  there is a CR function on  $M$  near  $p$ , of class  $C^m$ , which does not extend as a holomorphic function to either side.*

*Proof.* Let  $\Omega$  be a ball centred at  $p$  such that  $\Omega$  is divided by  $M$  into two components  $\Omega^+$ , and such that there exists  $\varphi \in \mathcal{O}(\Omega)$  with  $Z \cap \Omega = \varphi^{-1}(0)$ . After shrinking  $\Omega$ , we may assume that  $\omega := \Omega \cap M$  is divided by  $Z$  into more than one component. We can therefore write  $\omega \setminus Z$  as the disjoint union of two non-empty open sets  $\omega^+$  and  $\omega^-$ , which have  $Z$  as their common boundary. By Lemma 4.2, after shrinking the ball  $\Omega$  if required, there is a  $\chi \in C^\infty(\Omega \setminus Z)$  such that  $\chi \equiv 1$  on  $\overline{\omega^-} \setminus Z$ , and  $\chi \equiv 0$  on  $\omega^+ \setminus Z$ , and for some  $r > 0$ , we have  $|D^\mu \chi(z)| \leq C \text{dist}(z, Z)^{-r|\mu|^3}$ . Therefore, by Lemma 4.1, for a fixed  $m$ , there is an integer  $L \geq 0$ , such that  $f = \varphi^L \chi \in C^m(\Omega)$ .

We claim that  $f|_\omega$  is a CR function on  $M$  near  $p$ , and  $f$  does not extend holomorphically near  $p$  to either  $\Omega^+$  or  $\Omega^-$ . Indeed, on  $\omega$  the function  $f$  is given by

$$f(z) = \begin{cases} 0 & \text{for } z \in \omega^+ \cup (Z \cap \omega), \\ \varphi(z)^L & \text{for } z \in \omega^-. \end{cases}$$

Since at each point on  $\omega \setminus Z$ ,  $f$  is the restriction of a holomorphic function defined in a neighbourhood of that point,  $f$  is CR on  $\omega \setminus Z$ , continuous, and vanishes



on  $Z$ . Therefore, by Theorem 1.1 (i) it is CR on  $\omega$ . Clearly, by the boundary uniqueness theorem,  $f$  cannot extend as a holomorphic function to either of the open sets  $\Omega^\pm$ .  $\square$

We remark here that it is well known that if  $M$  is a  $C^\infty$ -smooth hypersurface, then near non-minimal points there are CR functions of class  $C^\infty$  that do not extend holomorphically to either side of  $M$ . Under precisely what hypotheses there exist non-extendable  $C^\infty$ -smooth CR functions on (singular) subanalytic hypersurfaces is an open question.

**4.3. Proper two-sided support.** We first recall a notion that was introduced in [9].

**Definition 4.4.** Let  $M$  be a subanalytic hypersurface in  $\Omega \subset \mathbb{C}^n$ , and let  $p \in M$ . We say that  $M$  has *two-sided support* at  $p$  if there are germs of complex analytic hypersurfaces  $A^\pm \subset \overline{\Omega^\pm}$  which pass through  $p$ . We say that it has *proper two-sided support* at  $p$  if  $A^\pm$  may be taken to be different, and such that

$$(4.5) \quad A^+ \cap M = A^- \cap M.$$

Note that according to this definition, non-minimality is a special case of (non-proper) two-sided support, namely, when  $A^+$  and  $A^-$  coincide. However, unlike non-minimality, proper two-sided support cannot occur at smooth points:

**Proposition 4.5.** *If a point  $p \in M$  admits two-sided support by distinct complex hypersurfaces on the two sides, then  $M$  cannot be represented in holomorphic coordinates near  $p$  as a graph over a real hyperplane. Hence,  $A^+ \cap A^- \subset M^{\text{sing}}$ .*

*Proof.* Suppose that  $M$  is represented near  $p = 0$  as a graph over a real hyperplane  $H$ , and let  $L$  be any complex two-dimensional linear subspace of  $\mathbb{C}^n$  transverse to  $H$ . Then,  $M \cap L$  is represented as a graph over  $H \cap L$  and has proper two-sided support by  $A^\pm \cap L$ . Assume without loss of generality that  $A^+$  is situated above the graph  $M$  and  $A^-$  below it. Let  $v$  be a vector in  $L$  orthogonal to  $H \cap L$ . We set  $B_t = \{z + tv : z \in A^\pm \cap L\}$ . Then  $B_t$  is a complex curve in  $L$ , and for  $t > 0$ , we have

$$(4.6) \quad B_t \cap C = \emptyset,$$

where  $C = A^- \cap L$ . On the other hand,  $B_0 \cap C$  contains the point 0 and as  $t \rightarrow 0+$ ,  $B_t \rightarrow B_0$ . We claim that this situation is not possible. This can be deduced from general properties of intersection of analytic varieties (see e.g. [6]), but we give a simple proof. Let  $U$  be a neighbourhood of 0 in  $\mathbb{C}^2$ , and let  $f_t$  be a family of holomorphic functions on  $U$ , depending continuously on  $t$  such that  $B_t \cap U = f_t^{-1}(0)$ . Let  $\Delta$  be the unit disc in  $\mathbb{C}$ , and let  $\varphi : \Delta \rightarrow C \cap U$  be a Puiseux parametrization of  $C$  near 0 such that  $\varphi(0) = 0$ . Let  $g_t = f_t \circ \varphi$ . Then

$g_0(0) = 0$ , and  $g_0$  is holomorphic on the closed unit disc. It follows that there exists  $\varepsilon > 0$  such that for  $|t| < \varepsilon$ , we have

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{g_t'(\zeta)}{g_t(\zeta)} d\zeta = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{g_0'(\zeta)}{g_0(\zeta)} d\zeta,$$

since the integral on the left assumes only integer values (the number of zeros of  $g_t$  in  $\Delta$ ) and depends continuously on  $t$ . Therefore,  $g_t(z) = 0$  for some  $z \in \Delta$ , and thus both  $B_t$  and  $C$  pass through  $\varphi(z)$ , which contradicts (4.6).  $\square$

We note that two-sided support occurs frequently in nature. In fact, if  $A^\pm$  are distinct complex hypersurfaces in an open set  $\Omega$  in  $\mathbb{C}^n$  such that  $E = A^+ \cap A^-$  is non-empty with  $p \in E$ , and each of  $A^\pm \setminus E$  is connected (this happens when  $A^\pm$  are irreducible), then

$$M = \{z \in \Omega : \text{dist}(z, A^+) = \text{dist}(z, A^-)\}$$

has proper two-sided support at  $p$ . It is easy to verify that  $\Omega^\pm = \{z \in \Omega : \pm(\text{dist}(z, A^+) - \text{dist}(z, A^-)) > 0\}$  are connected, and it follows from [4, Remarks 3.11] that  $M$  is subanalytic (and real-analytic if  $A^\pm$  are smooth.) If  $M$  is not minimal at  $p$ , after a small perturbation, we get a hypersurface  $\tilde{M}$  which has two-sided support at  $p$  by  $A^\pm$ , and which is minimal at  $p$ . In [9], the quadratic cones (zero-sets of real quadratic forms in  $\mathbb{C}^n$ ) with two-sided support were classified.

We now prove the other half of Theorem 1.2.

**Proposition 4.6.** *Let  $M$  be a subanalytic hypersurface in  $\mathbb{C}^n$  and  $p \in M$ . Suppose that  $M$  has proper two-sided support at  $p$ . Then, for every integer  $m \geq 0$  there is a CR function on  $M$  near  $p$  of class  $C^m$  that does not extend as a holomorphic function to either side.*

*Proof.* Let  $A^\pm$  be the two supports of  $M$  on the opposite sides at  $p$ . Let  $\Omega$  be a neighbourhood of  $p$  in  $\mathbb{C}^n$  such that  $\Omega \setminus M$  has two components  $\Omega^\pm$ , and  $A^\pm \subset \overline{\Omega^\pm}$ . After shrinking  $\Omega$ , we may assume that there are holomorphic functions  $\varphi_\pm$  on  $\Omega$  such that  $A^\pm = \varphi_\pm^{-1}(0)$ . We set  $Z = A^+ \cap A^-$ . It follows that  $A^+ \cap M = A^- \cap M = Z$ .

We now construct a  $C^m$ -smooth CR function on  $M$  near  $p$  which does not admit a local holomorphic extension to either of  $\Omega^\pm$ .

By Lemma 4.2, there exist a function  $\chi \in C^\infty(\Omega \setminus Z)$  such that  $\chi \equiv 0$  in a neighbourhood  $U^-$  of  $A^- \setminus Z$ ,  $\chi \equiv 1$  in a neighbourhood  $V$  of  $M \setminus Z$ , and  $r > 0$  such that  $|D^\mu \chi(z)| \leq C \text{dist}(z, Z)^{-r|\mu|^3}$  for any multi-index  $\mu$ . Further,

$$(4.7) \quad \begin{aligned} |\varphi_-(z)| &\geq C \text{dist}(z, A^-)^s && \text{\L}ojasiewicz's inequality (2.1) \\ &\geq C \text{dist}(z, Z)^{rs} && \text{by conclusion (2) of Lemma 4.2.} \end{aligned}$$

Define

$$g = \begin{cases} \frac{\chi}{\varphi_-} & \text{on } \Omega \setminus A^-, \\ 0 & \text{on } A^- \setminus Z. \end{cases}$$

Then  $g$  is smooth on  $\Omega \setminus Z$  and vanishes on  $U^-$ . Therefore, using the Leibniz rule and (4.2), we conclude that for any multi-index  $\mu$ ,

$$\begin{aligned} |D^\mu g(z)| &\leq C \sum_{k \leq \mu} |D^k \varphi_-(z)^{-1}| |D^{\mu-k} \chi(z)| \\ &\leq C \sum_{k \leq \mu} |\varphi_-(z)^{-|k|-1}| |D^{\mu-k} \chi(z)| \\ &\leq C \text{dist}(z, Z)^{-2|\mu|rs} \text{dist}(z, Z)^{-r|\mu|^3} \\ &\leq C \text{dist}(z, Z)^{-p(\mu)}, \end{aligned}$$

where the constants are independent of  $z$  (but may depend on  $\varphi_-$ ). For a fixed  $m$ , by Lemma 4.1, there exists  $L \geq 0$  such that the function

$$h^- = \begin{cases} \varphi_+^L g & \text{on } \Omega \setminus Z, \\ 0 & \text{on } Z, \end{cases}$$

is  $C^m$ -smooth on  $\Omega$ . By interchanging the role of  $\varphi_+$  and  $\varphi_-$  we may construct the same way a  $C^m$ -smooth function  $h^+$ . Since the restriction of the function  $h^+ - h^-$  to  $M^{\text{reg}}$  is CR, and the function vanishes on  $M^{\text{sing}}$ , it follows from Theorem 1.1 (i), that it is CR on  $M$ . By construction,  $h^+ - h^-$  cannot have holomorphic extension to either side of  $M$ . Indeed, by the boundary uniqueness theorem, such an extension must coincide with the meromorphic function

$$\tilde{f} = \frac{\varphi_+^L}{\varphi_-} - \frac{\varphi_-^L}{\varphi_+}$$

on  $\Omega \setminus (A^+ \cup A^-)$  and therefore cannot be defined on a one-sided neighbourhood of  $p$  on either side. □

**4.4. Proofs of Lemma 4.1 and Lemma 4.2**

*Proof of Lemma 4.1.* Suppose first that  $f \in C_b^m(\Omega \setminus E)$ . Then  $fg$  becomes continuous on  $\Omega$  if it is extended by 0 on  $E$ , which proves the result for  $m = 0$ . For  $m = 1$ , we can use the definition of partial derivatives to check that  $h = fg^2$  (again extended by 0 on  $E$ ) is in  $C^1(\Omega)$ . For  $m \geq 2$ , we may take  $h = fg^{m+1}$ ; the proof is an easy induction using the Leibniz formula.

By the last paragraph, to prove the general case, it suffices to show that for a given  $f \in C^m(\Omega \setminus E)$ , there is an  $L$  such that  $f g^L \in C_b^m(\Omega \setminus E)$ . Let  $q(m) = \max_{|\mu| \leq m} p(\mu)$ . For every  $m$ , then there exists  $C$  such that

$$|D^\mu f| \leq C \operatorname{dist}(\cdot, E)^{-q(m)},$$

and  $q(m)$  is increasing in  $m$ . For a fixed  $m$ , we let  $L$  be an integer such that  $L \geq q(m) + m$ . If  $\mu$  is any multi-index such that  $|\mu| \leq m$ , then

$$\begin{aligned} |D^\mu(g^L f)| &\leq C \sum_{k \leq \mu} |D^{\mu-k} g^L| |D^k f| \\ &\leq C \sum_{k \leq \mu} |g|^{L-|k|} \operatorname{dist}(\cdot, E)^{-q(|k|)} \\ &\leq C \operatorname{dist}(\cdot, E)^{L-|\mu|} \operatorname{dist}(\cdot, E)^{-q(|\mu|)} \\ &\leq C \operatorname{dist}(\cdot, E)^{L-m-q(m)}. \end{aligned}$$

By the choice of  $L$ , we have  $f g^L \in C_b^m(\Omega \setminus E)$ . □

For the proof of Lemma 4.2, we will need to use the following fact regarding the existence of a regularized distance function in  $\mathbb{R}^n$ :

**Result 4.7** ([18, Chapter VI, Theorem 2]). *For any closed subset  $E \subset \mathbb{R}^n$ , there is a  $C^\infty$  function  $\delta$  on  $\mathbb{R}^n \setminus E$  such that,*

- (1)  $C_1 \operatorname{dist}(x, E) \leq \delta(x) \leq C_2 \operatorname{dist}(x, E)$ , for  $x \in \mathbb{R}^n \setminus E$ , and
- (2) for every multi-index  $\mu$ , we have

$$(4.8) \quad |D^\mu \delta(x)| \leq \frac{B_\mu}{\delta(x)^{|\mu|-1}},$$

where the constants  $C_1, C_2$  and  $B_\mu$  are independent of  $E$ .

*Proof of Lemma 4.2.* In this proof we denote by  $C$  any constant which is independent of the point  $x \in \mathbb{R}^n \setminus Z$ .

Let  $\lambda$  be a  $C^\infty$ -smooth function on  $\mathbb{R}$  with values in the interval  $[0, 1]$  such that

$$\lambda(t) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases}$$

For  $j = 1, 2$ , let  $\delta_j$  be a regularization of  $\operatorname{dist}(z, E_j)$  as given by the result quoted above. We define  $\chi \in C^\infty(\mathbb{R}^n \setminus Z)$  by

$$\chi(x) = \begin{cases} \lambda\left(\frac{\delta_1(x)}{\delta_2(x)}\right) & \text{if } x \notin E_2, \\ 0 & \text{if } x \in E_2 \setminus Z. \end{cases}$$

Then conclusion (1) of the lemma holds for  $\chi$ , if we take

$$U_2 = \left\{ \frac{\delta_1(x)}{\delta_2(x)} \geq 1 \right\} \quad \text{and} \quad U_1 = \left\{ \frac{\delta_1(x)}{\delta_2(x)} \leq \frac{1}{2} \right\}.$$

Therefore, we need to consider only  $x \in U$ , where

$$U = \left\{ x \in \mathbb{R}^n \setminus Z : \frac{1}{2} < \frac{\delta_1(x)}{\delta_2(x)} < 1 \right\}.$$

Since  $E_1$  and  $E_2$  are regularly situated (see (2.2)), it follows that there exist a bounded neighbourhood  $\Omega$  of  $z$  in  $\mathbb{C}^n$  and  $r > 0$  such that for  $x \in \Omega$ ,

$$\text{dist}(x, E_1) + \text{dist}(x, E_2) \geq C \text{dist}(x, Z)^r.$$

After shrinking  $\Omega$ , we may assume that for  $x \in \Omega$ , we have  $\text{dist}(x, Z) < 1$ ,  $\delta_1(x) < 1$ ,  $\delta_2(x) < 1$ . Thanks to the comparability of  $\delta_j$  and  $\text{dist}(x, E_j)$ , we have

$$\delta_1(x) + \delta_2(x) \geq C \text{dist}(x, Z)^r.$$

If  $x \notin U_1$ , then  $\delta_1(x) > \frac{1}{2}\delta_2(x)$ , and therefore,  $\delta_1(x) \geq C \text{dist}(x, Z)^r$ . By the comparability of  $\delta_1$  with  $\text{dist}(\cdot, E_1)$ , conclusion (2) follows. The estimate for  $x \notin U_2$  follows exactly the same way. Consequently, if  $x \in U \cap \Omega$ , then for  $j = 1, 2$ ,

$$(4.9) \quad \delta_j(x) \geq C \text{dist}(x, Z)^r.$$

For the last conclusion, note that it holds for  $\mu = 0$  if  $C > 1$ . Now, for  $x \in U_1 \cup U_2$ , the function  $\chi$  is locally constant. Therefore, we only need to estimate  $D^\mu(\chi(x))$  for  $x \in (U \cap \Omega) \setminus Z$ .

First, for any multi-index  $k$ ,

$$\begin{aligned} (4.10) \quad |D^k(\delta_2(x)^{-1})| &\leq \frac{C}{\delta_2(x)^{|k|+1}} \cdot (|\delta_2(x)|_{C^{|k|}})^{|k|} && \text{from (4.2)} \\ &\leq \frac{C}{\delta_2(x)^{|k|+1}} \cdot \left( \frac{1}{\delta_2(x)^{|k|-1}} \right)^{|k|} && \text{from (4.8)} \\ &= \frac{C}{(\delta_2(x))^{|k|^2+1}} \\ &\leq \frac{C}{\text{dist}(x, Z)^{r(|k|^2+1)}} && \text{from (4.9)}. \end{aligned}$$

By the Leibniz rule, for any multi-index  $\mu$ ,

$$\begin{aligned}
 (4.11) \quad & |D^\mu(\delta_1(x)\delta_2(x)^{-1})| \\
 & \leq C \sum_{k \leq \mu} |D^{\mu-k}\delta_1(x)| |D^k(\delta_2(x)^{-1})| \\
 & \leq \sum_{k \leq \mu} \frac{1}{(\text{dist}(x, Z))^{r(|k|^2 - |k| + |\mu|)}} \quad \text{using (4.8), (4.9), (4.10)} \\
 & \leq \frac{C}{\text{dist}(x, Z)^{r|\mu|^2}}
 \end{aligned}$$

Finally, by (4.2),

$$\begin{aligned}
 |D^\mu \chi| & \leq C |\lambda|_{C^m} |\delta_1 \delta_2^{-1}|_{C^m}^m \\
 & \leq C \left( \frac{1}{\text{dist}(\cdot, Z)^{r|\mu|^2}} \right)^{|\mu|} \quad \text{by (4.11)} \\
 & = C \text{dist}(\cdot, Z)^{-r|\mu|^3},
 \end{aligned}$$

which completes the proof of (4.4). □

**4.5. Global non-extendable CR functions.** While in this paper we confine ourselves mainly to the question of local extension, we make a few observations regarding global analogs of the constructions of non-extendable CR functions, i.e., we consider the question whether there can be a CR function on the boundary  $\partial\Omega$  of a domain  $\Omega$  that does not have holomorphic extension into a global one-sided neighbourhood of  $M$  in  $\Omega$  or its complement. If the ambient manifold is  $\mathbb{C}^n$  with  $n \geq 2$ , or a Stein manifold of dimension at least 2, by the Bochner-Hartogs Theorem, such non-extendable CR functions do not exist, as long as  $\partial\Omega$  is smooth and connected; in fact, every CR function on  $\partial\Omega$  extends to all of  $\Omega$ . The analogous result continues to hold if  $\Omega$  is a subanalytic domain in  $\mathbb{C}^n$ . More precisely, the following holds.

**Proposition 4.8** (Bochner-Hartogs Theorem). *Let  $\Omega \Subset \mathbb{C}^n$ ,  $n \geq 2$ , be a bounded domain, such that  $M = \partial\Omega$  is a connected subanalytic hypersurface. Let  $f$  be a CR function on  $M$ , which is continuous on  $M^{\text{reg}}$ . Then there exists a function  $F$  holomorphic on  $\Omega$  which is a holomorphic extension of  $f$ , i.e.,  $\text{bv} F = f$  on  $M^{\text{reg}}$ . If for  $k \geq 0$ , the function  $f$  is  $C^k$ -smooth on  $M^{\text{reg}}$ , then  $F$  extends as a  $C^k$  function to  $M^{\text{reg}}$ .*

*Proof.* By the Jump formula of Section 3.4,  $\text{bv} f^+ - \text{bv} f^- = f$ , where  $f^+$  is holomorphic on  $\mathbb{C}^n \setminus \bar{\Omega}$  and  $f^-$  on  $\Omega$  such that on  $M^{\text{reg}}$ , equation (3.8) holds in the sense of distributions. Since  $M = \partial\Omega$  is connected, so is  $\mathbb{C}^n \setminus \bar{\Omega}$ . Therefore, by Hartogs' theorem,  $f^+$  extends to an entire function  $\tilde{f}^+$  on  $\mathbb{C}^n$ . We take  $F = \tilde{f}^+ - f^-$ . Then  $F$  has distributional boundary values  $f$  on  $M^{\text{reg}}$ , and the statement in the last sentence follows from [3, Theorem 7.2.6 and Theorem 7.5.1]. □

Non-extendable CR functions cannot be constructed on boundaries of bounded domains in Stein manifolds because Stein manifolds do not contain compact complex hypersurfaces, and therefore global analogs of non-minimality and two-sided support cannot occur.

We now consider an example (cf. Section 12 of [12], pointed out to us by M.C. Shaw), where global non-minimality and global two-sided support lead to the existence of non-extendable global CR functions. Let  $M \subset \mathbb{C}\mathbb{P}^2$  be the compact connected real analytic hypersurface

$$M = \{[z_0, z_1, z_2] : |z_1| = |z_2|\}.$$

$M$  is smooth except at the point  $[1, 0, 0]$ , and  $\mathbb{C}\mathbb{P}^2 \setminus M$  is the disjoint union of  $\Omega^+$  and  $\Omega^-$ , where

$$\Omega^\pm = \{[z_0, z_1, z_2] : \pm(|z_2| - |z_1|) > 0\},$$

are “Projective Hartogs Triangles”. The domains  $\Omega^\pm$  are biholomorphic to each other and pseudoconvex.

$M$  is both (globally) non-minimal and has proper (global) two-sided support at the singular point  $[1, 0, 0]$ . This allows us to construct non-extendable CR functions on  $M$  in two different ways, each showing that the Bochner-Hartogs theorem does not hold for  $\Omega^\pm$ .

Non-minimality: note that  $M$  is Levi-flat (in the sense that the smooth part is Levi-flat). It is “foliated” by projective lines  $\{z_1 = e^{i\theta} z_2\}$ ,  $\theta \in \mathbb{R}$  (although all these “leaves” pass through the singular point  $[1 : 0 : 0]$ ). Let  $Z$  be the union of two of these leaves. For definiteness assume that

$$Z = \{z_1 = z_2\} \cup \{z_1 = -z_2\} = \{z_1^2 - z_2^2 = 0\}.$$

Then  $M \setminus Z$  consists of two components:

$$M^+ = \{[z_0, z_1, z_2] : z_1 = e^{i\theta} z_2 \text{ for } 0 < \theta < \pi\}$$

$$M^- = \{[z_0, z_1, z_2] : z_1 = e^{i\theta} z_2 \text{ for } \pi < \theta < 2\pi\}.$$

Let  $f$  be the function on  $M$  defined by  $f \equiv 1$  on  $M^+$  and  $f \equiv 0$  on  $M^-$ . We claim that  $f$  is a bounded CR function on  $M$  but  $f$  does not extend to either  $\Omega^+$  or  $\Omega^-$ . Indeed, on  $M^{\text{reg}}$ , the function  $f$  is CR except along the complex hypersurface  $Z$  which is tangent to the Cauchy-Riemann vector fields  $T^{0,1}(M^{\text{reg}})$ , and therefore is removable for  $L^1_{\text{loc}}$  CR functions on  $M^{\text{reg}}$  (see [15, Proposition 1]). It follows that  $f$  is CR on  $M^{\text{reg}}$ . Therefore, by Theorem 1.1 (ii),  $f$  is CR on  $M$ . Clearly,  $f$  does not extend holomorphically to  $\Omega^+$  or  $\Omega^-$ .

Two-sided support:  $M$  has global proper two-sided support. In effect,  $\{z_2 = 0\} \subset \overline{\Omega^+}$  and  $\{z_1 = 0\} \subset \overline{\Omega^-}$ . The function

$$g(z) = \frac{z_1}{z_2} - \frac{z_2}{z_1},$$

is bounded on  $M$  (each of the two terms has absolute value 1) and is CR on  $M^{\text{reg}}$ . It follows from Theorem 1.1 (ii) that  $g$  is an  $L^\infty$  CR function on  $M$ . However, as in the proof of Proposition 4.6,  $g$  cannot extend to either of  $\Omega^\pm$ , since such an extension must blow up along  $\{z_1 = 0\}$  and  $\{z_2 = 0\}$ . It follows that  $g$  is a non-extendable CR function.

Note, however, that there are no non-constant *continuous* CR functions on  $M$ . For any  $\theta \in \mathbb{R}$ , the restriction of any such function to the compact leaf  $\{z_1 = e^{i\theta} z_2\} \subset M$  is holomorphic, and therefore constant. Since these leaves all pass through the point  $[1, 0, 0]$ , the values of these constants are the same for all leaves. This shows that the statement in Proposition 4.3 that we can construct CR functions of arbitrary smoothness is purely local. On  $M$  there are non-extendable bounded CR functions defined globally, but there are no continuous CR functions that do not extend.

### 5. THE HYPERSURFACE $M$

**5.1. Definition of  $M$  and precise statement of the extension result.** First we give a more precise form of Theorem 1.3, part (i). For a point  $z \in \mathbb{C}^n$ ,  $n \geq 2$ , we will write the coordinates as  $z = (z_1, z_2, \bar{z})$ , where  $z_1, z_2 \in \mathbb{C}$  and  $\bar{z} \in \mathbb{C}^{n-2}$  (where, as usual, if  $n = 2$ ,  $\mathbb{C}^0$  is taken to be the one-point space  $\{0\}$ ). Let  $\ell \geq 2$  be an integer,  $\ell = \infty$  or  $\ell = \omega$ . Let  $\Omega$  be a neighbourhood of the origin in  $\mathbb{C}^n$ , and let  $\gamma$  be a  $C^\ell$ -smooth subanalytic (i.e., its graph is a subanalytic set) function on some neighbourhood of  $\bar{\Omega}$  in  $\mathbb{C}^2$ . Assume that  $\gamma(0) \neq 0$ . We let

$$(5.1) \quad \rho(z) = \text{Re}(z_1 z_2) + |z_1|^2 \gamma(z),$$

and define

$$(5.2) \quad M = \{z \in \Omega : \rho(z) = 0\}.$$

Then  $M$  is a subanalytic hypersurface in the sense of Definition 2.1. After making the linear change of variables  $(z_1, z_2, \bar{z}) \mapsto (z_1, -z_2, \bar{z})$ , if necessary, we will further assume that  $\gamma(0) > 0$ .

Let  $\gamma_\ell(z) = 1 + x_2^\ell |x_2|$ . Then  $\gamma_\ell(z)$  is a subanalytic function which is  $C^\ell$ -smooth but not  $C^{\ell+1}$ -smooth, and the same is true of  $\rho$ . Since  $\nabla \rho(z) \neq 0$  for  $z \in M^* = M \setminus \{z_1 = z_2 = 0\}$ , it follows from the implicit function theorem that  $M^*$  is a hypersurface of smoothness at least  $C^\ell$ . Representing  $M^*$  as a graph over  $\mathcal{Y}_2 \neq 0$ , we see that  $M^*$  is  $C^\ell$ -smooth but not  $C^{\ell+1}$ -smooth. Since any biholomorphic map near 0 sending  $M$  onto another  $M$  for a different  $\gamma_\ell$  must preserve the smoothness class of  $M^*$ , it follows that the  $M$ 's are in general not biholomorphic for different  $\gamma$ 's.

**Theorem 5.1.** *Let  $M$  be defined as in (5.2), and let  $U \subset \Omega$  be a neighbourhood of the origin in  $\mathbb{C}^n$ . Then there exists a neighbourhood  $V$  of the origin such that any bounded CR function  $f$  on  $M \cap U$  extends to a holomorphic function  $F$  in*



$V^- = \{\rho < 0\} \cap V$ . Further, for  $k \geq 0$ , if  $f$  is  $C^k$ -smooth on  $M$ , then  $F$  extends to a  $C^k$ -smooth function on  $V^- \cup (M \cap V)$ , and  $F|_{M \cap V} = f$ .

Observe that  $\{z_1 = 0\} \cap \Omega \subset M$ , so that  $M$  is non-minimal. For general  $\ell$ ,  $M^* = M \setminus \{z_1 = z_2 = 0\}$  is a  $C^\ell$ -smooth hypersurface with a quadratic singularity of codimension 3, in particular, the singularity is isolated if  $n = 2$ . Indeed, for  $c > 0$ , denote by  $\rho_c$  the real quadratic form

$$(5.3) \quad \rho_c(z) = \operatorname{Re}(z_1 z_2) + c|z_1|^2$$

on  $\mathbb{C}^n$ , and let

$$(5.4) \quad M_c = \{z \in \mathbb{C}^n : \rho_c(z) = 0\}.$$

Then the defining function  $\rho$  of  $M$  near 0 is of the form  $\rho = \rho_{y(0)} + h$ , where  $\rho_{y(0)}$  is as in (5.3), and  $h(z) = |z_1|^2(y(z) - y(0)) = O(|z|^3)$ . If  $n = 2$ , the real quadratic form  $\rho_{y(0)}$  is non-degenerate, and has two positive and two negative eigenvalues. Therefore, by the Morse Lemma, there is a  $C^\ell$ -diffeomorphism of a neighbourhood of 0 onto another neighbourhood of 0 in  $\mathbb{C}^2$  that maps  $M$  onto the real quadratic cone  $M_{y(0)}$ . The latter is, in fact, the tangent cone of  $M$  at the origin. If  $n \geq 3$ , the quadratic form  $\rho_{y(0)}$  is degenerate, but still there is a  $C^\ell$ -diffeomorphism  $\Phi$  in a neighbourhood of 0 in  $\mathbb{C}^n$  which maps  $M$  onto the real quadratic cone  $M_1 = \{\rho_1 = 0\}$ . The map  $\Phi$  can be given explicitly by

$$(5.5) \quad \Phi(z) = \left( \sqrt{y(z)} z_1, \frac{1}{\sqrt{y(z)}} z_2, \tilde{z} \right),$$

(valid also for  $n = 2$ ). Observe that  $\Phi$  maps the complex hypersurface  $\{z_1 = 0\}$ , which makes  $M$  non-minimal, onto itself.

One can verify that the set  $M_1 \setminus \{z_1 = 0\}$  is connected, and therefore,  $M \setminus \{z_1 = 0\}$  is also connected. If  $y(z) \equiv c$ , then  $\Phi$  is a  $\mathbb{C}$ -linear isomorphism between  $M_c$ , given by (5.4), and  $M_1$ . A simple computation shows that there is a neighbourhood  $U \subset \Omega$  of 0 such that the set  $U^- = \{z \in U : \rho(z) < 0\}$  is pseudoconvex. In fact the Levi form has one positive eigenvalue at each boundary point in  $(M \setminus \{z_1 = 0\}) \cap U$  (which is therefore strongly pseudoconvex if  $n = 2$ ), so the hypothesis of the Lewy extension theorem holds.

The proof of Theorem 5.1 can be outlined as follows. First we construct explicitly a family of analytic discs attached to  $M_1$  and use the *Kontinuitätssatz* to prove that holomorphic functions defined in some thin neighbourhood  $\omega$  of  $M_1 \setminus \{z_1 = 0\}$  extend analytically along any path in a bigger one-sided neighbourhood of the origin, the size of which is independent of  $\omega$ . This is done in Section 5.2. Then we show in Section 5.3 that the analytic continuation from  $\omega$  does not yield multiple-valued functions. In the terminology of [16], this means that the complex hypersurface  $\{z_1 = 0\}$  is (locally) *W-removable* at the origin. Finally,

in Section 5.4 we conclude the proof by showing that every CR function on  $M \setminus \{z_1 = 0\}$  near the origin extends to a holomorphic function on a one-sided neighbourhood of the origin, and that the extension has the required boundary regularity on  $M$ .

**5.2. Construction of the extension.**

**Proposition 5.2.** *Let*

$$M_1 = \{z \in \mathbb{C}^n : \rho_1(z) = 0\},$$

where

$$\rho_1(z) = \operatorname{Re}(z_1 z_2) + |z_1|^2,$$

and let  $U$  be a neighbourhood of 0 in  $\mathbb{C}^n$ . Let  $\omega \subset U^- := U \cap \{\rho_1 < 0\}$  be a connected one-sided neighbourhood of

$$S_1 = (M_1 \setminus \{z_1 = 0\}) \cap U.$$

Then there is a neighbourhood  $V$  of the origin in  $\mathbb{C}^n$ , such that given any  $p \in V^- := V \cap \{\rho_1 < 0\}$ , there is a path  $\tau \subset V^-$  starting in  $\omega \cap V^-$  and terminating at  $p$ , along which any holomorphic function in  $\omega$  admits analytic continuation.

The proof of the proposition relies on the following lemma which will be also used later.

**Lemma 5.3.** *Let  $V = B(0, \frac{1}{3})$ , and let  $V^- = V \cap \{\rho_1 < 0\}$ . There exists a continuous family of analytic discs  $\{D_w\}_{w \in V^-}$  in  $\mathbb{C}^n$ , with the following properties.*

- (1)  $D_w \subset B(0, 1)$ .
- (2)  $w \in D_w$ .
- (3)  $\partial D_w \subset S_1$ .
- (4) Let  $p_0 = (\frac{1}{8}, -\frac{1}{8}, \tilde{0}) \in S_1 \cap V$ , where  $\tilde{0} \in \mathbb{C}^{n-2}$ . Then the discs  $\{D_w\}$  shrink to  $\{p_0\}$  as  $V^- \ni w \rightarrow p_0$ .

*Proof.* Given the point  $w = (w_1, w_2, \tilde{w}) \in \mathbb{C}^n$ , let  $\alpha = \frac{1}{2}(w_1 - \overline{w_2})$ . (We suppress the dependence of  $\alpha$  on  $w$  for notational clarity.) Consider the subset of  $\mathbb{C}$  given by

$$(5.6) \quad \Sigma_w = \{\zeta \in \mathbb{C} : |\zeta - \alpha|^2 \leq |\alpha|^2 - |w_1|^2\},$$

which, depending on the right hand side, may be a closed disc, a point, or empty. If  $\rho_1(w) < 0$ , it is easily verified that  $|w_1 - \alpha|^2 < |\alpha|^2 - |w_1|^2$ . Therefore, if  $w \in V^-$ , the set  $\Sigma_w$  contains the point  $w_1$ , and thus, it is a disc of positive radius  $\sqrt{|\alpha|^2 - |w_1|^2}$ . It also follows from the definition of  $\Sigma_w$  that if  $w_1 \neq 0$ , then  $0 \notin \Sigma_w$ .

For  $w \in V^-$ , we consider the map  $\varphi : \Sigma_w \rightarrow \mathbb{C}^n$  given by

$$\varphi_w : \zeta \mapsto \left( \zeta, \frac{|w_1|^2}{\zeta} - 2\bar{\alpha}, \tilde{w} \right),$$

and let  $D_w = \varphi_w(\Sigma_w)$ . Note that this is well defined since  $w_1 \neq 0$ . A computation shows that  $\varphi_w(w_1) = w$ , and  $\rho_1(\varphi_w(\zeta)) = 0$  if  $\zeta \in \partial\Sigma_w$ . Therefore,  $D_w$  is an analytic disc contained in  $\{\rho_1 < 0\}$ . It passes through point  $w$ , and its boundary is attached to  $M_1$ . Furthermore,  $D_w \subset B(0, 1)$ . Indeed, to see this, it is sufficient to show that  $\partial D_w = \varphi(\partial\Sigma_w) \subset B(0, 1)$ , and then apply the maximum principle. If  $\zeta \in \partial\Sigma_w$ , we have  $|\zeta - \alpha|^2 = |\alpha|^2 - |w_1|^2$ , or

$$(5.7) \quad |w_1|^2 = 2 \operatorname{Re} \bar{\alpha} \zeta - |\zeta|^2.$$

Now, with  $\zeta$  as above, we have,

$$\begin{aligned} |\varphi(\zeta)| &= \left( |\zeta|^2 + \left| \frac{|w_1|^2}{\zeta} - 2\bar{\alpha} \right|^2 + |\tilde{w}|^2 \right)^{1/2} \\ &\leq |\zeta| + \left| \frac{|w_1|^2}{\zeta} - 2\bar{\alpha} \right| + |\tilde{w}| \\ &\leq |\zeta| + \frac{2 \operatorname{Re} \bar{\alpha} \zeta - |\zeta|^2}{|\zeta|} + 2|\alpha| + |\tilde{w}| \quad (\text{using (5.7)}) \\ &= 2|\alpha| + 2 \frac{\operatorname{Re} \bar{\alpha} \zeta}{|\zeta|} + |\tilde{w}| \\ &\leq 4|\alpha| + |\tilde{w}| \quad (\text{using Cauchy-Schwarz}) \\ &\leq 2(|w_1| + |w_2|) + |\tilde{w}| \\ &\leq 3|w| \quad (\text{using Cauchy-Schwarz again}) \\ &< 1. \end{aligned}$$

Also observe that  $D_w \rightarrow \{p_0\}$  as  $w \rightarrow p_0$ ,  $w \in V^-$ . This follows directly by taking the limit in (5.6) and noting that  $\Sigma_w$  shrinks to a point as  $w \rightarrow p_0$ .  $\square$

*Proof of Proposition 5.2.* Since  $M_1$  is invariant under dilations, we may assume without loss of generality that  $U$  is the unit ball. Let  $V, V^-$  and  $p_0$  be as in Lemma 5.3. The continuous family of analytic discs  $D_w$  constructed in Lemma 5.3 can be used to prove analytic continuation of holomorphic functions from  $\omega$  to  $V^-$ . Indeed, since  $V^-$  is connected, and  $p_0 \in \partial V^-$ , there is a path

$\tau : [0, 1] \rightarrow \mathbb{C}^2$ , such that  $\tau(0) = p_0$ ,  $\tau(1) = p$  and  $\tau((0, 1]) \subset V^-$ . There exists  $\eta > 0$  so small that  $\tau^{-1}(\omega)$  contains the interval  $(0, 2\eta)$ , and such that the interior of the disc  $D_{\tau(\eta)}$  is completely contained in  $\omega$ . Then the restriction of  $\tau$  to the interval  $[\eta, 1]$  is a path which starts in  $\omega$  and ends at  $p$ . We claim that any holomorphic function on  $\omega$  admits a holomorphic extension along this path  $\tau$ . To see this observe that the discs  $D_{\tau(t)}$  are attached to  $S$ , so after shrinking them slightly, we obtain discs  $\Delta_t \subset D_{\tau(t)}$  such that  $\partial\Delta_t \subset \omega$  for each  $t$ ,  $\eta \leq t \leq 1$ . Since  $\Delta_\eta \subset \omega$ , the result follows from the Kontinuitätssatz.  $\square$

**5.3. Schlichtness of the envelope of holomorphy of  $\omega$ .** It follows from Proposition 5.2 that every holomorphic function on  $\omega$  extends to a possibly multiple-valued holomorphic function on  $V^-$ . The next proposition shows that the extension is, in fact, single-valued.

**Proposition 5.4.** *If in Proposition 5.2, the set  $U$  is a ball, the envelope of holomorphy  $\mathcal{E}(\omega)$  of  $\omega$  is schlicht.*

*Proof.* Without loss of generality,  $U$  is the unit ball. We first show that  $\mathcal{E}(U^+)$  is schlicht. While it is possible to prove this using a direct monodromy argument, we will deduce this from a general result due to Trapani. Suppose that  $\Omega \subset D$  are domains in a Stein manifold. Define a *complex retraction* of  $D$  into  $\Omega$  to be a homotopy  $F_t : D \rightarrow D$ ,  $0 \leq t \leq 1$ , of holomorphic maps, such that

- (1)  $F_0 = \mathbb{1}_D$ ,
- (2)  $F_1(D) \subset \Omega$ ,
- (3) for each  $t$ ,  $F_t(\Omega) \subset \Omega$ .

We then have the following result [19, Theorem 1]: *Let  $D$  be a domain of holomorphy in a Stein manifold, and  $\Omega \subset D$ . If there is a complex retraction of  $D$  into  $\Omega$ , then,  $\Omega$  has a schlicht envelope of holomorphy.* To apply this, we take  $D = U \setminus \{z_1 = 0\}$ ,  $\Omega = U^+$ , and  $F_t(z) = (z_1, (1 - t)z_2, (1 - t)\bar{z})$ . We claim that  $F_t$  is a complex retraction of  $D$  into  $\Omega$ . Condition (1) is clear. Since  $F_1(D) = \{z \in \mathbb{C}^n : 0 < |z_1| < 1, z_2 = 0, \bar{z} = 0\} \subset \Omega$ , condition (2) follows. For condition (3) we note that  $F_t(D) \subset D$  for each  $t$ , and if  $z \in \Omega$ , we have  $\rho_1(F_t(z)) = (1 - t) \operatorname{Re}(z_1 z_2) + |z_1|^2 = (1 - t)\rho_1(z) + t|z_1|^2 > 0$ , so that  $F_t(\Omega) \subset \Omega$ . This shows that the envelope of  $U^+$  is schlicht.

Now we deduce that  $\mathcal{E}(\omega)$  is schlicht. By the Lewy extension theorem, there is a one-sided neighbourhood  $\tilde{\omega}$  of the hypersurface  $S_1$  (whose Levi form has one positive eigenvalue) on the  $U^-$  side to which every function in  $\mathcal{O}(U^+)$  extends holomorphically. After shrinking  $\tilde{\omega}$  we may assume that  $\tilde{\omega} \subset \omega$ . Set  $U^\# = U^+ \cup S_1 \cup \tilde{\omega}$ . Then the restriction map induces an isomorphism  $\mathcal{O}(U^\#) \cong \mathcal{O}(U^+)$ . In particular,  $\mathcal{E}(U^\#) = \mathcal{E}(U^+)$ .

Seeking a contradiction, assume now that the envelope  $\pi : \mathcal{E}(\omega) \rightarrow \mathbb{C}^n$  is multiple sheeted. Then there exist a function  $f \in \mathcal{O}(\omega)$ , a point  $p \in \mathbb{C}^2$  and two paths  $\tau_1$  and  $\tau_2$ , starting in  $\omega$  and ending in  $p$ , along which  $f$  has holomorphic extensions  $f_1$  and  $f_2$  such that  $f_1(p) \neq f_2(p)$ . Note that  $U^-$  is a pseudoconvex

domain containing  $\omega$ , and therefore,  $\tau_1, \tau_2 \subset U^- \subset \pi(\mathcal{E}(\omega))$ . Further, without loss of generality, we can assume that the paths  $\tau_1$  and  $\tau_2$  start in  $\tilde{\omega}$ .

Now let  $U_1 = U \setminus \{z_1 = 0\}$ . Then  $U_1$  is pseudoconvex. Note that  $U^\# \cup U^- = U_1$  and  $U^\# \cap U^- = \tilde{\omega}$ . It is possible to solve a Cousin Problem in  $U_1$  to obtain functions  $F^\# \in \mathcal{O}(U^\#)$  and  $F^- \in \mathcal{O}(U^-)$  such that  $f|_{\tilde{\omega}} = F^\# - F^-$ .

For  $j = 1, 2$ , set  $F_j = F^- + f_j$ . Then  $F_j$  is a holomorphic function along the path  $\tau_j$  which extends the function  $F^\# \in \mathcal{O}(U^\#)$ . But then we have  $F_1(p) \neq F_2(p)$  which contradicts the fact that  $\mathcal{E}(U^\#) = \mathcal{E}(U^+)$  is schlicht. □

**5.4. Proof of Theorem 5.1.** First we note that Propositions 5.2 and 5.4 also hold for  $M = \{\rho = 0\}$ , where the function  $\rho$  is as in Theorem 5.1. The crucial observation is that for a small enough neighbourhood  $U$  of 0 in  $\mathbb{C}^n$ , as in the proof of Proposition 5.2, we can obtain discs  $\Delta_w$  which pass through any specified point in  $U^- = \{z \in U : \rho < 0\}$ , remain inside  $U$ , shrink as one approaches certain boundary points, and whose boundaries are contained in the one-sided neighbourhood  $\omega$  of  $S = (M \setminus \{z_1 = 0\}) \cap U$  to which every CR function on  $S$  admits the Lewy holomorphic extension. To see this, let  $c > 0$  be such that  $\gamma > c$  on  $U$ . As before, set  $M_c = \{z \in \mathbb{C}^n : \rho_c(z) = \text{Re}(z_1 z_2) + c|z_1|^2 = 0\}$ , and let  $U_c^- = \{z \in U : \rho_c(z) < 0\}$ . Then  $U^- \subset U_c^-$ , and  $M_c$  is biholomorphic to  $M_1$  by a complex linear map. After applying the linear biholomorphism, and a dilation, we may assume that  $U$  is the unit ball and  $c = 1$ . For  $w \in U^-$ , we can clearly choose  $\Delta_w$  to be a subset of a connected component of  $\overline{D_w} \cap \overline{U^-}$  (where  $D_w$  is as in Lemma 5.3) such that the properties claimed are verified. This provides the generalization of Proposition 5.2 to general  $M$ . It is easy to verify that the proof of Proposition 5.4 also carries over, *mutatis mutandis*, to this general case.

Suppose now that  $f$  is a CR function on  $S$ . As was observed in the previous paragraph, by the Lewy extension theorem,  $f$  extends to a holomorphic function  $\tilde{f}$  on a one-sided neighbourhood  $\omega \subset \{\rho < 0\}$ , and by the previous steps,  $\tilde{f}$  extends further to a holomorphic function  $F$  on  $V^-$ . The extension  $F$  assumes the boundary values  $f$  on  $S$  in the same way in which the Lewy extension assumes the value  $f$  on  $S$ . To complete the proof, we need to understand the behaviour of  $F$  as one approaches  $\{z_1 = 0\}$ .

First assume that  $f$  is in  $L^\infty$ . Then, by the paragraph above,  $F$  has distributional boundary values  $f$  on  $S \cap V$ . We need to show that  $F$  has distributional boundary values equal to  $f$  on  $M^{\text{reg}} \cap V$ .

Let  $M^* = M \setminus \{z_1 = z_2 = 0\}$ . Then the fact that (5.5) is a  $C^\ell$ -diffeomorphism shows that  $M^*$  is a  $C^\ell$ -smooth hypersurface (where  $\gamma$  in (5.1) is  $C^\ell$ -smooth.) Clearly  $M^{\text{reg}} \subset M^*$ . We will show that  $F$  has distributional boundary values  $f$  on  $M^* \cap V$ .

Let  $\nu(z)$  denote the unit normal to  $M^*$  directed towards  $\{\rho < 0\}$ . For  $t > 0$ , define a function  $F_t$  on  $M^*$  by  $F_t(z) = F(z + t\nu(z))$ . Since  $|F| \leq \sup_S |f|$ , we have  $|F_t(z)| \leq \sup_S |f|$  for each  $t > 0$  and  $z \in M^*$ . Moreover, thanks to [14, Theorem 3.1],  $F_t(z) \rightarrow f(z)$  at each Lebesgue point  $z$  of  $f|_S$ , and since

$\{z_1 = 0\}$  has measure 0, we have on  $M^* \cap V$ ,

$$\lim_{t \rightarrow 0^+} F_t = f \quad \text{almost everywhere.}$$

If  $\chi \in C_c^\infty(M^* \cap V)$ , then by the dominated convergence theorem,

$$\lim_{t \rightarrow 0^+} \int_{M^* \cap V} F(z + t\nu(z))\chi(z) \, d\mathcal{H} = \int_{M^* \cap V} f(z)\chi(z) \, d\mathcal{H}.$$

It follows that  $f$  is the boundary value of  $F$  in the sense of distributions.

If  $f \in C^k(M)$ ,  $k \geq 0$ , then the function  $F$  obtained above extends as a  $C^k$ -smooth function to  $S \cap V = (M \setminus \{z_1 = 0\}) \cap V$ , and has boundary values equal to  $f$ . Since  $f$  is continuous on  $M^* \cap V$ , and  $F$  has distributional boundary values  $f$  on  $M^* \cap V$ , it follows from [3, Theorems 7.2.6 and 7.5.1] that  $F$  extends as a  $C^k$ -smooth function to  $M^* \cap V$ .

It remains to prove that  $F$  extends as a  $C^k$  function also to

$$\Sigma = \{z_1 = z_2 = 0\} \cap V.$$

First assume that  $k = 0$ . Let  $w \in \Sigma$ . We need to show that as  $z \rightarrow w$  through points in  $V^-$ , we have that  $F(z) \rightarrow f(w)$ . Suppose that for each  $z \in V^-$ , there exists a disc  $\Delta(z)$  in  $\overline{V^-}$  passing through  $z$  such that the boundary  $\partial\Delta(z) \subset M^* \cap V$ , and such that  $\Delta(z)$  shrinks to the point  $w$  as  $V^- \ni z \rightarrow w$ . Applying the maximum principle to the holomorphic function  $F(z) - f(w)$  on the disc  $\Delta(z)$ , we have

$$\begin{aligned} |F(z) - f(w)| &\leq \sup_{\zeta \in \partial\Delta(z)} |F(\zeta) - f(w)| \\ &= \sup_{\zeta \in \partial\Delta(z)} |f(\zeta) - f(w)|. \end{aligned}$$

Since  $\Delta(z)$  shrinks to  $w$  as  $z \rightarrow w$ , it follows that

$$\lim_{V^- \ni z \rightarrow w} \sup_{\zeta \in \partial\Delta(z)} |f(\zeta) - f(w)| = 0.$$

To complete the proof, we specify the discs  $\Delta(z)$ . We can take, for example,

$$\Delta(z) = \left\{ \zeta \in \overline{V^-} : \zeta_2 = z_2, \zeta_1 = \bar{z}_1 \right\}.$$

For small  $z$ , both required properties are easily verified.

Now let  $k \geq 1$ . By the previous paragraph, every partial derivative of  $F$  of order  $k$  extends continuously to 0. It follows that  $F$  extends as a  $C^k$ -smooth function to the origin.

6. THE HYPERSURFACE  $N$

**6.1. Definition of  $N$  and precise statement.** We now define  $N$  and state the general form of Theorem 1.3 (ii). Let  $(z_1, z_2, z_3, \bar{z}) \in \mathbb{C}^3 \times \mathbb{C}^{n-3}$  be the coordinates in  $\mathbb{C}^n$ ,  $n \geq 3$ . Let  $U$  be a neighbourhood of 0 in  $\mathbb{C}^n$ , and let  $h$  be a  $C^\ell$ -smooth real-valued function on a neighbourhood of  $\bar{U}$  such that  $h(0) = 0$ , where  $\ell \geq 2$ . We let

$$(6.1) \quad \sigma(z) = \operatorname{Re}(z_1 z_2 + z_1 \bar{z}_3) + |z_1|^2 h(z),$$

and set

$$(6.2) \quad N = \{z \in U : \sigma(z) = 0\}.$$

As with  $M$ , we set  $U^\pm = \{z \in U : \pm\sigma(z) > 0\}$ .

**Theorem 6.1.** *Let  $N$  be given as in (6.2). For any neighbourhood  $U$  of the origin, there exists a neighbourhood  $V \subset \mathbb{C}^n$  of 0 such that any bounded CR function  $f$  on  $N \cap U$  extends to a holomorphic function  $F$  on  $V$  with  $F|_{N \cap V} = f$ .*

We note that the singularity of  $N$  is degenerate, i.e., the real Hessian of  $\sigma$  at 0 is not invertible. In fact, the Hessian has two positive and two negative eigenvalues, and the remaining eigenvalues vanish. Therefore, the Morse Lemma does not apply. However, there is a diffeomorphism  $\Psi$  from a neighbourhood of the origin in  $\mathbb{C}^n$  into  $\mathbb{C}^n$ , which maps  $N$  onto a neighbourhood 0 in the cone

$$(6.3) \quad N_1 = \{z \in \mathbb{C}^n : \sigma_1(z) = \operatorname{Re}(z_1 z_2 + z_1 \bar{z}_3) = 0\}.$$

If  $y(z) = 1 + h(z)$ , this map is given explicitly by

$$(6.4) \quad \Psi(z) = \left( \sqrt{y(z)} z_1, \frac{z_2}{\sqrt{y(z)}}, \frac{y(z) - 1}{\sqrt{y(z)}} z_1 + \frac{z_3}{\sqrt{y(z)}}, \bar{z} \right).$$

Note that  $\Psi$  maps the hypersurface  $\{z_1 = 0\} \subset N$  onto itself. A computation shows that  $N_1^{\text{sing}} = \{z \in \mathbb{C}^n : z_1 = 0, z_2 + \bar{z}_3 = 0\}$ . It follows that  $N \setminus \{z_1 = 0\}$  is smooth.

The following notation will be used in the proof. For  $\delta > 0$ , denote by  $B(\delta) = \{z \in \mathbb{C}^n : |z| < \delta\}$  the ball radius  $\delta$  in  $\mathbb{C}^n$  centred at 0, and set  $N^\pm(\delta) = B(\delta) \cap U^\pm$ , where  $U^\pm = \{z \in U : \pm\sigma(z) > 0\}$  and  $\sigma$  is the defining function of  $N$  as in (6.1). Note that  $N^\pm(\delta)$  are one-sided neighbourhoods of 0 with respect to  $N$ . We use the notation  $N_1^\pm(\delta)$  for similarly defined one-sided neighbourhoods of  $N_1$ . The proof of Theorem 6.1 follows a similar pattern. After proving some results concerning envelopes of one-sided neighbourhoods of 0 with respect to  $N$ , in Proposition 6.3 we characterize local envelope of holomorphy of arbitrarily thin neighbourhoods of  $N \setminus \{z_1 = 0\}$  at 0.

**6.2. Envelope of one-sided neighbourhoods of  $N$ .**

**Proposition 6.2.** *Given any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that every  $g \in \mathcal{O}(N^+(\varepsilon))$  extends to a single-valued holomorphic function in  $B(\delta) \setminus \{z_1 = 0\}$ .*

*Proof.* We first prove the proposition for  $N_1$ . Let  $L = \{z \in \mathbb{C}^n : z_1 = z_3\}$ . Then

$$N_1 \cap L = \left\{ \operatorname{Re}(z_1 z_2) + |z_1|^2 = 0, z_1 = z_3 \right\}.$$

Thus,  $N_1 \cap L$  (considered as a singular hypersurface in  $L \cong \mathbb{C}^{n-1}$ ) is equivalent to the hypersurface  $M_1$  of the previous section. By Lemma 5.3, there exists a continuous family of discs  $\{D_w\}$  attached to  $(N_1 \cap L) \setminus \{z_1 = 0\}$  and passing through  $w$ , where  $w = (w_1, w_2, w_1, \bar{w}) \in L$ , and  $\operatorname{Re}(w_1 w_2) + |w_1|^2 < 0$ . Further, the discs  $D_w$  shrink to a point  $p_1 = (a, -a, a, \bar{0})$ , as  $w \rightarrow p_1$ , where  $a > 0$  is sufficiently small.

Consider the translations

$$T_\tau(z_1, z_2, z_3, \bar{z}) = (z_1, z_2 + \tau, z_3 - \bar{\tau}, \bar{z}).$$

Note that  $N_1$  is invariant under  $T_\tau$ , and  $T_\tau(\{z_1 = 0\}) = \{z_1 = 0\}$  for any  $\tau \in \mathbb{C}$ . It follows that for any  $\tau \in \mathbb{C}$ , the discs  $T_\tau(D_w)$  are also attached to  $(N_1 \cap L) \setminus \{z_1 = 0\}$ .

Let  $\delta = \frac{1}{10}$ , and let  $w = (w_1, w_2, w_3, \bar{w})$  be an arbitrary point in  $N_1^-(\delta)$ . We set  $\tau = \bar{w}_1 - \bar{w}_3$ , and  $w' = (w_1, w_2 - \tau, w_1, \bar{w})$ . Note that  $w' \in L$ , and

$$\operatorname{Re}(w_1(w_2 - \tau)) + |w_1|^2 = \operatorname{Re}(w_1 w_2 + w_1 \bar{w}_3) < 0.$$

Therefore,  $T_\tau(D_{w'})$  passes through  $w$  and is attached to  $(N_1 \cap L) \setminus \{z_1 = 0\}$ . The disc  $T_\tau(D_{w'})$  can be given explicitly by

$$\varphi_w(\zeta) = \left( \zeta, \frac{|w_1|^2}{\zeta} + w_2 - \bar{w}_1, \zeta - \bar{\tau}, \bar{w} \right),$$

where  $\zeta$  belongs to the set  $\Sigma_{(w_1, w_2 - \tau)}$  defined in (5.6). Further, repeating the calculations of Lemma 5.3, for  $w \in N^-(\delta)$  we have

$$|\varphi(\zeta)| < \sqrt{|\zeta|^2 + \left| \frac{|w_1|^2}{\zeta} - (\bar{w}_1 - w_2) \right|^2} + |\zeta|^2 + |\tau|^2 < 2\sqrt{2}|w| + 3|w| < 1.$$

This shows that the discs are contained in the unit ball.

A computation shows that the Levi form of  $N_1$  has one positive eigenvalue at each point of  $N_1 \setminus \{z_1 = 0\}$ . Therefore, by the Lewy extension theorem, every holomorphic function on  $N_1^+(1)$  extends to each point of  $(N_1 \cap B(1)) \setminus \{z_1 = 0\}$ , in particular, to the boundaries of the discs constructed above. As before, by the



Kontinuitätssatz, any function holomorphic in  $N_1^+(1)$  admits analytic continuation along any path starting at  $p_1$  and ending at any point  $p$  in  $N_1^-(\delta)$ .

Now we show that the extension so obtained is single-valued. For this it is sufficient to show that any loop in  $B(\delta) \setminus \{z_1 = 0\}$  can be deformed into a path in  $N_1^+(\delta)$ . Let  $\varphi : [0, \delta) \rightarrow [0, \infty)$  be a diffeomorphism. Then the diffeomorphism  $z \mapsto \varphi(|z|)z$  maps  $B(\delta) \setminus \{z_1 = 0\}$  to  $\mathbb{C}^n \setminus \{z_1 = 0\}$  and  $N_1^+(\delta)$  to  $N_1^+ = \{z \in \mathbb{C}^n : \sigma_1(z) > 0\}$ . It is therefore sufficient to show that every path in  $\mathbb{C}^n \setminus \{z_1 = 0\}$  can be deformed to a path in  $N_1^+$ .

For  $\lambda$  real, let

$$(6.5) \quad N_\lambda^+ = \{z \in \mathbb{C}^n : \sigma_1(z) + (\lambda - 1)|z_1|^2 > 0\}.$$

Then

$$\bigcup_{\lambda \geq 1} N_\lambda^+ = \mathbb{C}^n \setminus \{z_1 = 0\}.$$

Hence, if  $\alpha$  is any bounded path in  $\mathbb{C}^n \setminus \{z_1 = 0\}$ , then there exists a  $\mu \geq 1$  such that  $\alpha \subset N_\mu^+$ . Now as  $s$  increases from 1 to  $\sqrt{\mu}$ , a map given by

$$z \mapsto \left( sz_1, \frac{1}{s}z_2, \left(s - \frac{1}{s}\right)z_1 + \frac{1}{s}z_3, \bar{z} \right)$$

continuously deforms  $N_\mu^+$  into  $N_1^+$ . (cf. equation (6.4).) This proves the proposition for  $N_1$ .

For the general case of a hypersurface  $N$ , we shrink  $U$  such that there exists  $\lambda$ ,  $0 < \lambda < 1$ , with the property that  $h(z) \geq (\lambda - 1)$  for  $z \in U$ . Then,  $N_\lambda^+(\varepsilon) \subset N^+(\varepsilon)$  for every  $\varepsilon > 0$ . The linear biholomorphism of (6.4) with  $\gamma \equiv \lambda$  maps  $N_\lambda^+$  onto  $N_1^+$  while fixing  $\{z_1 = 0\}$ . It follows that there is  $\delta > 0$  such that every holomorphic function on  $N_\lambda^+(\varepsilon)$  extends to  $B(\delta) \setminus \{z_1 = 0\}$ . Thus, Proposition 6.2 follows by restricting  $g$  to  $N_\lambda^+(\varepsilon)$ . □

**6.3. Envelope of holomorphy of a neighbourhood of  $N \setminus \{z_1 = 0\}$ .** We now deduce the following consequence of Proposition 6.2.

**Proposition 6.3.** *Given a neighbourhood  $U$  of  $0$  in  $\mathbb{C}^n$ , there is a neighbourhood  $V$  of  $0$  in  $\mathbb{C}^n$  with the following property. If  $\omega \subset U$  is a neighbourhood of  $(N \setminus \{z_1 = 0\}) \cap U$ , the envelope  $\mathcal{E}(\omega)$  contains the set  $V \setminus \{z_1 = 0\}$ .*

Note that the neighbourhood  $\omega$  can be arbitrarily thin.

*Proof.* Let  $\varepsilon > 0$  be such that  $B(\varepsilon) \subset U$ . Note that the open set  $B(\varepsilon) \setminus \{z_1 = 0\}$  is connected and pseudoconvex, and is divided into two connected components  $N^\pm(\varepsilon)$  by the smooth hypersurface  $(N \cap B(\varepsilon)) \setminus \{z_1 = 0\}$ . Moreover, shrinking  $\varepsilon$  if required, we can assume that the Levi-form of  $(N \cap B(\varepsilon)) \setminus \{z_1 = 0\}$  has one positive and one negative eigenvalue at each point.

We set  $N^\sharp = N^+(\varepsilon) \cup \omega$  and  $N^\flat = N^-(\varepsilon) \cup \omega$ . Then

$$N^\sharp \cup N^\flat = B(\varepsilon) \setminus \{z_1 = 0\} \quad \text{and} \quad N^\sharp \cap N^\flat = \omega.$$

Now let  $f \in \mathcal{O}(\omega)$ . Since  $B(\varepsilon) \setminus \{z_1 = 0\}$  is pseudoconvex, after solving the Cousin problem, we can write,  $f = f^+ - f^-$ , where  $f^+ \in \mathcal{O}(N^\sharp)$  and  $f^- \in \mathcal{O}(N^\flat)$ . By Proposition 6.2 above, there is  $\delta^+ > 0$  such that  $f^+|_{N^+(\varepsilon)}$  (and therefore,  $f^+$ ) extends to a holomorphic function  $\tilde{f}^+$  on  $B(\delta^+) \setminus \{z_1 = 0\}$ .

The linear map  $(z_1, z_2, z_3, \bar{z}) \mapsto (-z_1, z_2, z_3, \bar{z})$  maps  $N$  onto the singular hypersurface  $\tilde{N} = \{z \in U : \tilde{\sigma}(z) = 0\}$ , where

$$\tilde{\sigma}(z) = \operatorname{Re}(z_1 z_2 + z_1 \bar{z}_3) + |z_1|^2 \tilde{h}(z),$$

and  $\tilde{h}(z_1, z_2, z_3, \bar{z}) = -h(-z_1, z_2, z_3, \bar{z})$ . Since this map sends  $N^\pm(\varepsilon)$  to  $\tilde{N}^\mp(\varepsilon)$ , by Proposition 6.2 again, there is  $\delta^- > 0$  such that  $f^-|_{N^-(\varepsilon)}$  (and therefore,  $f^-$ ) extends to a holomorphic function  $\tilde{f}^-$  on  $B(\delta^-) \setminus \{z_1 = 0\}$ .

We set  $\delta = \min(\delta^+, \delta^-)$ , and

$$F = \tilde{f}^+ - \tilde{f}^-.$$

Then  $F$  is holomorphic on  $B(\delta) \setminus \{z_1 = 0\}$ , and  $F|_\omega = f$ . This completes the proof with  $V = B(\delta)$ .  $\square$

We now complete the proof of Theorem 6.1. Let  $U$  be a neighbourhood of 0 in  $\mathbb{C}^n$  and let  $f \in \operatorname{CR}(N \cap U)$  be a bounded CR function. By the Lewy extension theorem,  $f$  extends to a neighbourhood  $\omega$  of  $(N \setminus \{z_1 = 0\}) \cap U$ . Then, by Proposition 6.3,  $f$  extends to the set of the form  $V \setminus \{z_1 = 0\}$ , where  $V$  is a neighbourhood of 0. The constructed extension remains a bounded function on the complement of  $\{z_1 = 0\}$ , and therefore, by the removable singularity theorem, it admits holomorphic extension to a neighbourhood of the origin. This completes the proof.

**Acknowledgments.** The first author would like to thank J.-P. Rosay, E. Bedford and M.-C. Shaw for helpful discussions, and the organizers of the SCV semester at Institut Mittag-Leffler, Djursholm for an invitation to their program, where part of this work was done; very special thanks are due to Meera and Tanmay for all their support. Research of the second author is supported in part by the Natural Sciences and Engineering Research Council of Canada.

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KEY WORDS AND PHRASES: CR functions; holomorphic extension; singular hypersurfaces.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 32V10; 32B20; 32V25.

*Received: July 28th, 2009.*

*Article electronically published on June 25th, 2010.*