ON AN OBSERVATION OF SIBONY

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ABSTRACT. It is shown that if the boundary of a Reinhardt domain in $\mathbb{C}^n$ contains the origin, each holomorphic function on the domain which is infinitely many times differentiable up to the boundary extends holomorphically to a neighborhood of the origin.

In this note we prove the following result:

Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain such that the origin is a boundary point of Ω. Then each function in $\mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$, the space of functions holomorphic on Ω and smooth up to the boundary, extends holomorphically to a neighborhood of the origin.*

The special case of this result when Ω is the "Hartogs Triangle" $H = \{|z_1| < |z_2| < 1\} \subset \mathbb{C}^2$ was noted by Sibony in [9], which constitutes a refinement of the classical fact that the *Nebenhülle* of the Hartogs triangle is the bidisc (see [1]), i.e. the bidisc is the largest open set contained in each Stein neighborhood of the closure $\overline{H}$. Consequently, each function holomorphic in a neighborhood of $\overline{H}$ extends to the bidisc. As is well-known, it is possible for a smoothly bounded pseudoconvex domain to have nontrivial Nebenhülle (see [4]). On the other hand, each smoothly bounded pseudoconvex domain Ω admits a function in $\mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ which does not extend past any boundary point (see [2, 7]). Therefore, the Hartogs triangle displays a behavior which is specific to non-smooth domains, and it is interesting to understand precisely which geometric properties of the Hartogs triangle lead to its not being a "$\mathcal{C}^\infty$-domain of holomorphy." According to the theorem proved here, the answer is that the only things that matter are that H is Reinhardt and the origin belongs to the boundary of H.

Of course, the existence of a Nebenhülle or $\mathcal{C}^\infty$ envelope of holomorphy is interesting only when Ω is pseudoconvex, i.e., when there is no point outside Ω to which each function holomorphic on Ω extends. Using the theorem, it is easy to give examples of (nonsmooth) pseudoconvex domains with nontrivial $\mathcal{C}^\infty$ envelopes analogous to the Hartogs triangle: if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ is a multiindex, such that $\alpha \notin \mathbb{N}^n$ and $-\alpha \notin \mathbb{N}^n$, then let

$$H(\alpha) = \{z \in \mathbb{D}^n \mid |z^\alpha| < 1\},$$

where $\mathbb{D}^n$ is the unit polydisc. From the theorem, $H(\alpha)$ has a nontrivial $\mathcal{C}^\infty$-envelope of holomorphy, and noting the convexity of the image of the map $z \mapsto (\log|z_1|, \ldots, \log|z_n|)$ we see that $H(\alpha)$ is pseudoconvex (see [8, Section 3.8].) For $n = 2$, these domains are precisely the "fat" and "thin" generalized Hartogs triangles of rational exponent which have been studied extensively recently (see [6, 3, 5]), and shown to have unexpected properties as far as L^p regularity of the Bergman projection.

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Proof. We will use the usual multi-index notation in dealing with functions of several variables. Also, we will assume that Ω is bounded. This is no loss of generality, since we can always replace Ω by its intersection with a polydisc and prove the result for this bounded domain.

Let $f \in \mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Since f is holomorphic on the Reinhardt domain Ω, there is a Laurent expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \tag{1}$$

which is uniformly and absolutely convergent on compact subsets of Ω (see e.g. [8]). The coefficients $c_\alpha \in \mathbb{C}$ are represented by the well-known Cauchy formula: if $w \in \Omega$ with $w_j \neq 0$ for each j, we have the n-fold repeated line-integral representation

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|z_1|=|w_1|} \cdots \int_{|z_n|=|w_n|} \frac{f(z)}{z^\alpha} \cdot \frac{dz_n}{z_n} \cdots \frac{dz_1}{z_1}.$$

Parametrize the contours by $z_j = w_j e^{i\theta_j}$, where $0 \leqslant \theta_j \leqslant 2\pi$. Notice that then we have

$$dz_j = w_j \cdot i e^{i\theta_j} d\theta_j = i z_j d\theta_j,$$

so that

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \frac{f(w_1 e^{i\theta_1}, \ldots, w_n e^{i\theta_n})}{w^\alpha \exp\left(i(\alpha_1\theta_1 + \cdots + \alpha_n\theta_n)\right)} (i d\theta_n) \ldots (i d\theta_1).$$

Let $Z = \{w \in \mathbb{C}^n | w_j = 0 \text{ for some } j\}$. Therefore, using an obvious "vector-like" notation, we have for each $w \in \Omega \backslash Z$ that

$$c_\alpha w^\alpha = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \frac{f(w \cdot e^{i\theta})}{\exp\left(i\langle \alpha, \theta\rangle\right)} d\theta_n \ldots d\theta_1.$$

Write the multiindex $\alpha = \beta - \gamma$, where $\beta_j = \max(\alpha_j, 0)$ and $\gamma_j = (-\alpha_j, 0)$. Then $\beta, \gamma \in \mathbb{N}^n$ and we can rewrite

$$c_{\beta-\gamma} \cdot \frac{w^\beta}{w^\gamma} = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \frac{f(w \cdot e^{i\theta})}{\exp\left(i\langle \beta - \gamma, \theta\rangle\right)} d\theta_n \ldots d\theta_1.$$

Apply the differential operator $\left(\frac{\partial}{\partial w}\right)^\beta$ to both sides, which gives us

$$\frac{c_{\beta-\gamma} \cdot \beta!}{w^\gamma} = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \exp\left(i\langle \gamma, \theta\rangle\right) \frac{\partial^\beta f}{\partial w^\beta}(w \cdot e^{i\theta})\, d\theta_n \ldots d\theta_1.$$

Taking absolute values, and doing a simple sup norm estimate, we see that

$$\begin{aligned}
\left|\frac{c_{\beta-\gamma} \cdot \beta!}{w^\gamma}\right| &= \left|\frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \exp\left(i\langle \gamma, \theta\rangle\right) \frac{\partial^\beta f}{\partial w^\beta}(w \cdot e^{i\theta})\, d\theta_n \ldots d\theta_1\right| \\
&\leqslant \left\|\frac{\partial^\beta f}{\partial w^\beta}\right\|_\infty \\
&< \infty,
\end{aligned}$$

where the finiteness of the sup norm of the derivative follows since $f \in \mathcal{C}^\infty(\overline{\Omega})$ and Ω is assumed to be bounded. Therefore the function on $\Omega \backslash Z$ given by $w \mapsto c_\alpha w^{-\gamma}$ is bounded and therefore extends holomorphically across the analytic set Z to the function given by the same formula on Ω, and the extended function admits the same bound. Since the origin is a boundary point of Ω, this means that if $\gamma \not= 0$, then $c_\alpha = 0$, so that the Laurent series in (1) reduces to a *Taylor* series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha \tag{2}$$

i.e. there are no terms with negative powers of the coordinates, and represents the function f on Ω. Let $w \in \Omega$ so that the series (2) converges when $z = w$. It follows from Abel's lemma ([8, Lemma 1.15]) that the series (2) actually converges in the polydisc $P_w = \{z \in \mathbb{C}^n \,|\, |z_1| < |w_1|, \ldots, |z_n| < |w_n|\}$. Therefore, (2) converges on the open set $\Omega \cup P_w$ to a holomorphic function $\widetilde{f}$, and $\widetilde{f}|_\Omega = f$. The proof is complete.

□

An examination of the last step of the proof shows that we can be more specific in the conclusion of the theorem. Indeed, we have the following: *with notation and assumptions as in the theorem, each function in $\mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ extends to the smallest* complete *log-convex Reinhardt domain containing the domain Ω.*

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