Several Complex Variables are Better than Just One

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In this popular expository article, we discuss some important ways in which complex analysis in more than one variable is different from complex analysis in one variable. Analytic continuation in several variables is contrasted with that in one variable, and the notion of psuedoconvexity is defined. Hartogs phenomenon and the Levi problem are also discussed in an informal way.

1. Introduction

Complex analysis in one variable is one of the core areas of mathematics, that every student of mathematics, pure or applied, ought to learn. The subject finds applications in diverse areas – in number theory (e.g., the proof of the Prime Number Theorem), electrostatics (use of conformal mapping to study two-dimensional electrical fields), etc.

Less prominent in the curriculum is the study of holomorphic functions of several complex variables. This would be justified if the transition from one to several variables was merely a matter of juggling multi-indices and replacing ordinary derivatives by partial derivatives. However, this is far from being the case, and the aim of this article is to show with some examples the spectacular difference between functions of one and several complex variables. We have tried to make the exposition accessible to students with some knowledge of one variable complex analysis, hoping that it will provide inspiration to further explore this topic. We do not include any proofs, but give adequate references for the interested student.

Dedication

A few months after the author's visit to Pune to give the talk on which this article is based, in a horrible accident, his host Prof. Singh and his wife were hit by a lorry on a Pune road while riding a scooter. Prof. Singh, though seriously injured, survived, and recovered. His wife Isha was not so fortunate, and succumbed to her injuries. It is to her memory that this article is respectfully dedicated.

Keywords
Complex analysis, several complex variables, Hartogs phenomenon, domain of holomorphy, pseudoconvexity, Levi problem.

2. Holomorphic Functions

2.1 *Holomorphic Functions of One Variable*

Holopmorphic functions are much better behaved than differentiable functions of one variable.

The heroes of our story, the holomorphic functions, may be introduced in a number of ways, though the fact that all these definitions are equivalent is far from obvious. The classical definition is the following: let f be a complex-valued function on an open set $D \subset \mathbb{C}$. Then f is said to be *holomorphic* if it is complex-differentiable at each point of the domain, i.e.,

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1}$$

exists at each point $z \in D$. This definition, which is formally analogous to the definition of a differentiable function of one real variable has very different implications. In fact, holomorphic functions are much better-behaved when compared to differentiable functions of a real variable. This is the content of the celebrated 'Goursat's Theorem' [1, §8, pp. 100–102]. A holomorphic function has partial derivatives in x and y (where $z = x + iy$) of *every possible order*! Note the radical difference with real differentiability, where for each integer $k \geq 1$, it is possible to construct a function of a real variable, which has k derivatives, but the $(k+1)$-th derivative fails to exist at each point. This unexpected regularity of holomorphic functions is undoubtedly one of the most amazing things in all of mathematics.

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In fact much more is true: a function f on a domain $D \subset \mathbb{C}$ is holomorphic if and only if it is *complex analytic,* which means that for each $w \in D$ there is a sequence of complex numbers $\{a_\nu\}_{\nu=0}^{\infty}$ such that the power series representation

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - w)^\nu,$$

holds in a neighborhood of w in D. Let us recall an important consequence of complex analyticity: if D is

A function is holomorphic if and only if it is complex analytic.

a domain, and the holomorphic function f on D is not identically zero, then the zero-set $f^{-1}(0)$ of f, is a discrete subset of D. This means that for each point $w \in f^{-1}(0)$, there is a neighborhood U of w in D such that $f^{-1}(0) \cap U = \{w\}$.

2.2 *Holomorphic Functions of Several Variables*

If D is an open set in $\mathbb{C}^n$, a complex valued function f on D is defined to be holomorphic if it is holomorphic in each variable separately. This means that for each j, and each $(n-1)$-tuple $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$, the function $\zeta \mapsto f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)$ is holomorphic provided the open subset of the complex plane where it is defined is nonempty. Note that *no* assumption regarding continuity (let alone further regularity) of f has been made! The example of the real function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0), \end{cases}$$

(which is separately differentiable in both variables, but not even continuous at the origin) shows that we do not know a priori much about the regularity of f.

In several complex variables, the statement analogous to Goursat's theorem is the statement that if a function is holomorphic then it has continuous partial derivatives of all orders.

In several complex variables, the statement analogous to Goursat's theorem is 'Hartogs's Separate Analyticity Theorem' [5, Theorem 2.2.8, p. 28]. It states that if a function is holomorphic according to the above definition, then it has continuous partial derivatives of all orders. Its proof uses a subtle convexity argument that has resisted all attempts at simplification since the original appearance of the result in 1906. Like the theorem of Goursat, the result is remarkable for aesthetic rather than practical reasons, since almost any holomorphic function that arises in practice is usually seen to be continuous on inspection. Once a holomorphic function f is known to be continuous, it is not difficult, by considering each variable successively to show that f is complex analytic. In several variables, this means that for each $w \in D$, there is a power series representation of f around w:

$$f(z) = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} a_{\alpha_1,\ldots,\alpha_n}(z_1 - w_1)^{\alpha_1} \ldots (z_n - w_n)^{\alpha_n}, \quad (2)$$

valid in a neighborhood of w in D.

We denote by $\mathcal{O}(D)$ the space of holomorphic functions on the open set D.

3. Analytic Continuation in One Variable

Given a holomorphic function f on a domain D in $\mathbb{C}$, one is often interested in knowing if it is possible to extend f as a holomorphic function to a larger domain D'. For example, in number theory one considers the function on the half plane $D = \{\mathrm{Re}(z) > 1\}$ given by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$

which then extends to the punctured plane $D' = \mathbb{C} \setminus \{1\}$ as a holomorphic function. The extended function, known as the Riemann zeta function, plays an important role in the study of the distribution of primes among the integers.

Given a holomorphic function *f* on some domain, it is of interest to know if it is possible to extend *f* as a holomorphic function to a larger domain.

It is possible that the extension of a holomorphic function is 'multiple-valued'. A well-known example of this is the square root function $f(z) = \sqrt{|z|} \exp\left(\frac{i}{2}\theta\right)$ on the upper half plane $\{\mathrm{Im}(z) > 0\}$, where $0 < \theta < \pi$ is such that $z = |z|\, e^{i\theta}$. Consider the functions f_1 and f_2 which extend this function f to a larger domain, defined as follows: f_1 is defined on the plane slit along the positive real axis, i.e., the domain $D_1 = \mathbb{C} \setminus \{\mathrm{Im}(z) = 0, \mathrm{Re}(z) > 0\}$, and is again given by $f_1(z) = \sqrt{|z|} \exp\left(\frac{i}{2}\theta\right)$, where now $0 < \theta < 2\pi$, and f_2 is defined on the plane slit along the negative real axis, i.e., the domain $D_2 = \mathbb{C} \setminus \{\mathrm{Im}(z) = 0, \mathrm{Re}(z) < 0\}$, A better way to deal with this phenomenon is to think of the extension as being defined on a Riemann surface over $\mathbb{C}$.

and is given by the same formula $f_2(z) = \sqrt{|z|}\exp\left(\frac{i}{2}\theta\right)$, where we now take $-\pi < \theta < \pi$. It is clear that each of f_1 and f_2 extends the function f, but f_1 and f_2 do not agree on the lower half plane. Classically, one would say that the function f on D has an analytic extension to the whole of $\mathbb{C} = D_1 \cup D_2$, but the extension is multiple valued. It was realized by Riemann that a better way to deal with this phenomenon was to think of the extension as being defined on a 'Riemann surface' spread over $\mathbb{C}$.

It is natural to ask what the limits of such extendability are. More precisely, given a domain $D \subset \mathbb{C}$, we can ask first whether it is a *domain of existence* of some holomorphic function f on D. By definition a domain D is the domain of existence of a holomorphic function f on D, provided there is no larger domain $D' \subset \mathbb{C}$ such that f extends to a holomorphic function on D'. We can also ask whether it is a *domain of holomorphy* of some function g: by definition, the domain D is the domain of holomorphy of a function f holomorphic on D provided f does not admit an analytic extension to a larger domain $D' \supsetneq D$ *even as a multiple-valued function.* Equivalently, D is the domain of holomorphy of f, provided f does not admit an analytic extension to a Riemann surface spread over $\mathbb{C}$. Obviously, if D is the domain of holomorphy of a holomorphic function then D is also its domain of existence. The converse does not hold, as can be seen by considering a branch of the square root on the plane slit along the negative real axis, i.e., the function f_2 on the domain D_2 of the preceding paragraph.

It is clear that some domains are domains of holomorphy: e.g., $\mathbb{C}$, or $\mathbb{C}$ minus a finite set (why?). The unit disc $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$ is also a domain of holomorphy. Consider the holomorphic function on Δ represented by the series

$$f(z) = \sum_{\nu=0}^{\infty} z^{\nu!}$$

which is easily seen by the root test to converge on Δ (verify this!). By taking $z = r\exp\left(2\pi i\frac{p}{q}\right)$, where p, q are integers with $q \neq 0$, and letting $r \to 1^{-}$, we conclude that the function f blows up as we approach any point in a dense subset of the boundary in the radial direction. It easily follows that the function f cannot be extended even locally to a neighborhood of any point on the boundary of Δ, and Δ is the domain of holomorphy of f. In fact we have the following general result, due to Runge [4, Corollary 8.3.3,p. 270]:

Every convex domain is a domain of holomorphy of some function.

Theorem 1. *For every domain D in $\mathbb{C}$, there is a holomorphic function f on D, such that D is the domain of holomorphy of f.*

4. Analytic Continuation in Several Variables

It is easy to extend the notions of domain of existence and domain of holomorphy to several variables. Also, it is easy to show that if $D_1, \ldots, D_n$ are domains in $\mathbb{C}$, their product, i.e., the domain in $\mathbb{C}^n$ given as $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_j \in D_j\}$ is a domain of holomorphy. Moreover, one can adapt the proof of Theorem 1 to several variables to show that every *convex* domain is a domain of holomorphy.

In 1906, the German mathematician F Hartogs had the rare honor of inaugurating a new branch of mathematics in his doctoral thesis when he showed that if $n \geq 2$, not every domain D in $\mathbb{C}^n$ is the domain of holomorphy of a function $f \in \mathcal{O}(D)$. In fact, there exist pairs of domains D, D' in $\mathbb{C}^n$, $n \geq 2$ such that *every holomorphic function f on D extends holomorphically to D'.* We say in this case that the pair D, D' exhibits the *Hartogs phenomenon.*

There exist pairs of domains *D*, *D*' such that every holomorphic function *f* on *D* extends holomorphically to *D*'.

A famous example of Hartogs phenomenon is the following [5, Theorem 2.3.2, p. 30].

Theorem 2. *Let* Ω' *be a domain in* $\mathbb{C}^n$*, where* $n \geq 2$*, and let* K *be a compact subset of* Ω' *such that the complement*

$$\Omega = \Omega' \setminus K \tag{3}$$

is connected. Then every holomorphic function on Ω *extends to a holomorphic function on* Ω'.

The analog of Theorem 2 is false for $n = 1$. One simply chooses a point $\zeta \in K$, and considers the function

$$f(z) = \frac{1}{z - \zeta}, \tag{4}$$

which does not extend holomorphically to the point ζ. Recall that in one complex variable, an important topic is the classification of isolated singularities into removable singularities, poles and essential singularities (for example, the function in (4) has a pole at ζ.) Theorem 2 shows that the whole issue is very simple for functions of more than one variable, since every isolated singularity is removable!

A key question is: How does one determine whether a given domain *D* is a domain of holomorphy of some holomorphic function on *D*?

What we want is an intrinsic characterization of domains of holomorphy.

Note that we could construct the function f in (4) precisely because the zero-set of the function $z \mapsto z - \zeta$ is compact, in fact reduced to the point ζ. We leave it as an exercise for the reader to deduce from Theorem 2 that this is impossible for $n \geq 2$. (For help, see [7].)

5. The Levi Problem

We are now led to the question: how does one determine whether a given domain $D \subset \mathbb{C}^n$ is a domain of holomorphy of some holomorphic function on D? What we want is an intrinsic characterization of domains of holomorphy without constructing holomorphic functions explicitly on it. Surprisingly, it turns out that the answer to this question depends on a certain *convexity* property of the domain D!

To get a clue why this might be so, we consider the following simpler question: given a point $\zeta \in bD$ on the boundary of the domain D, construct a holomorphic function f on D, such that f cannot be extended holomorphically to any neighborhood of the point ζ. This is easy for one variable since we can take f as in (4). But in several variables, the problem might not have a solution! Let Ω be as in (3). Since every holomorphic function on Ω extends to Ω', if $\zeta \in K \cap (b\,\Omega)$, it is not possible to construct a function f not extendable holomorphically to ζ.

There is however a class of domains for which we can mimic the one-variable solution given by (4). Suppose that the boundary bD of a domain D is smooth, and the domain D is *strictly convex.* Let $T_\zeta(bD) \subset \mathbb{C}^n$ be the tangent hyperplane to the domain D at point ζ, consisting of all vectors in $\mathbb{C}^n$ tangent to bD at point ζ. The strict convexity of bD implies that for each $\zeta \in bD$, the tangent hyperplane $T_\zeta(bD)$ and the closure $\overline{D}$ meet only at ζ. Without loss of generality, after a translation, we can suppose that $\zeta = 0$, the origin. If we think of $\mathbb{C}^n$ as a $2n$-dimensional *real* vector space, the hyperplane $T_\zeta(bD)$ is a $(2n-1)$- dimensional real vector subspace. It is easy to see that any such $(2n-1)$-dimensional real vector subspace of $\mathbb{C}^n$ contains an $(n-1)$-complex dimensional complex linear subspace. In fact this subspace $H_\zeta(bD)$ is given by $T_\zeta(bD) \cap i(T_\zeta(bD))$, where $i : \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation which multiplies every vector by i. Since $H_\zeta(bD)$ is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ with codimension one, it follows that we can find a $\mathbb{C}$-linear map $\lambda : \mathbb{C}^n \to \mathbb{C}$ (a linear functional) such that the zero set $\lambda^{-1}(0) = H_\zeta(bD)$. Now consider the function

$$f(z) = \frac{1}{\lambda(z)}, \qquad (5)$$

which is holomorphic on $\mathbb{C}^n \setminus \lambda^{-1}(0) = \mathbb{C}^n \setminus H_\zeta(bD)$, and blows up along $H_\zeta(bD)$ (we say f has a pole along $H_\zeta(bD)$.) Since $H_\zeta(bD) \cap bD = \{\zeta\}$, it follows that

There is a class of domains for which we can mimic the one-variable solution.

$f \in \mathcal{O}(D)$, and f cannot be extended holomorphically to any neighborhood of ζ.

Note that in the case $n = 1$, the space $H_\zeta(bD)$ is zero-dimensional, and is reduced to the origin. Then the function given by (5) coincides, up to a non-zero factor, with the function f of (4).

Therefore, we can solve the problem of finding a holomorphic function not extendable to a point ζ in the boundary of a smoothly bounded strictly convex domain. If we try to state this result in a biholomorphically invariant and localized form, we are naturally led to the notion of *strong pseudoconvexity*, which we now explain.

Let D_1 and D_2 be domains in $\mathbb{C}^n$. A bijective map $\Phi = (\Phi_1, \ldots, \Phi_n) : D_1 \to D_2$ is said to be *biholomorphic*, if each component Φ_j of Φ is holomorphic, and so is each component of the inverse mapping $\Phi^{-1} : D_2 \to D_1$. Now let D be a domain in $\mathbb{C}^n$ with smooth (at least twice continuously differentiable) boundary bD, and let $\zeta \in bD$. We say that D is *strongly pseudoconvex* at the boundary point ζ, if there is a neighborhood ω of ζ in $\mathbb{C}^n$, and a biholomorphic map $\Phi : \omega \to \Phi(\omega) \subset \mathbb{C}^n$ such that the image $\Phi(bD)$ of the boundary is a strictly convex hypersurface in the open set $\Phi(\omega)$. We say that D is strongly pseudoconvex, if bD is smooth, and D is strongly pseudoconvex at each boundary point. This notion is illustrated in *Figure* 1.

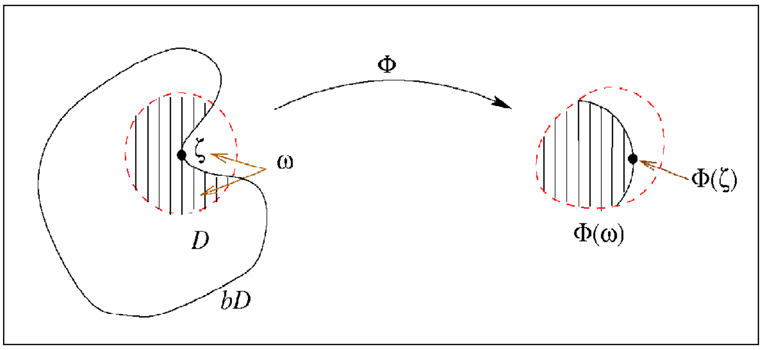

Figure 1. D is strongly pseudoconvex at the point $\zeta \in$ D, and Φ convexifies bD near ζ. The set $\omega \cap D$ is shaded .

Note that any domain $D \subset \mathbb{C}$ with smooth boundary is automatically strongly pseudoconvex. This statement may be taken to be a very weak form of the Riemann mapping theorem.

Any domain D in $\mathbb{C}$ with smooth boundary is automatically strongly pseudoconvex. This may be regarded as a weak form of the Riemann mapping theorem.

We urge the reader to verify the following immediate consequence of this definition: if a domain $D \subset \mathbb{C}^n$ is strongly pseudoconvex at a point $\zeta \in bD$, there is a neighborhood ω of ζ in $\mathbb{C}^n$, and a holomorphic $f \in \mathcal{O}(\omega \cap D)$ such that f does not extend holomorphically to any neighborhood of ζ. (This is essentially the same argument that produces the function f of (5).)

We are now in a position to define one of the central concepts of complex analysis. A domain $D \subset \mathbb{C}^n$ is said to be *pseudoconvex* if D is the union of a non-decreasing sequence of strongly pseudoconvex domains. It is easy to see that every domain in the complex plane is pseudoconvex (why?), and hence this notion is useless in one variable. Every strongly pseudoconvex domain is of course pseudoconvex.

Every domain in the complex plane is pseudoconvex.

It was observed very early in the history of complex analysis in several variables that *every domain of holomorphy is pseudoconvex.* The question arose whether every pseudoconvex domain was a domain of holomorphy. This is one of several equivalent versions of the famous *Levi Problem.* Note that if the Levi problem has an affirmative solution, it gives a geometric characterization of domains of holomorphy.

The reader may object that a solution to the Levi problem is useless as a characterization of domains of holomorphy, since the definition of a pseudoconvex domain does not give an effective way of checking whether a given domain is pseudoconvex. Fortunately, there are explicit ways to decide whether a given domain is pseudoconvex, based on the computation of certain quadratic forms associated to the domain [5, Theorem 2.6.7 and Theorem 2.6.12].

The question "Is every pseudoconvex domain a domain of holomorphy?" is one version of the famous Levi problem.

Levi proposed his problem in 1911. For about thirty years, it remained an open question. The Japanese mathematician Kiyoshi Oka published in 1942 a solution to the Levi problem for domains in $\mathbb{C}^2$. It took ten more years for the Levi problem to be settled in general: in 1953–54, Oka himself, and independently H Bremermann and F Norguet solved the general problem.

6. Weierstrass and Cartan Theorems

The solution of the Levi problem shows that if we want to generalize results that hold on domains $D \subset \mathbb{C}$ to domains in higher dimensions, we should consider pseudoconvex domains, rather than general domains in $\mathbb{C}^n$. As an example of this philosophy, we consider a result due to Weierstrass, which states [12, §15.13, p. 304]: If D is a domain in $\mathbb{C}$, and $\{p_\nu\}$ is a discrete sequence of points in D (i.e., it is has no limit point in D), and $\{a_\nu\}$ is any sequence of complex numbers, then there is a holomorphic function f on D such that $f(p_\nu) = a_\nu$ for each ν.

Now let Ω be as in (3), and let $\{p_\nu\}$ be a discrete sequence of points in Ω which accumulate on K (for definiteness, one may consider the situation, in which the points p_ν converge to a point $p \in K$.) If we now let $a_\nu = \nu$, Theorem 2 implies that there can be no holomorphic function f on Ω such that $f(p_\nu) = \nu$. Indeed, such a function f extends to a holomorphic F on Ω'. It follows that f is bounded near K, which is a contradiction to f being unbounded.

The reason why discrete subsets of domains are so important in complex analysis of one variable is that they can be locally represented as zero sets of holomorphic functions.

On the other hand, if we have a *pseudoconvex* $D \subset \mathbb{C}^n$, these pathologies disappear, and again, given any discrete subset $P \subset D$, and any map $a : P \to \mathbb{C}$, we can find an f holomorphic on D such that $f \equiv a$ on P. Stated in this form, the result is capable of an interesting generalization, which we now state.

The reason why discrete subsets of domains are so

important in complex analysis of one variable is that they can be locally represented as zero sets of holomorphic functions. In fact by another related result of Weierstrass [12, §15.11, p. 303], every discrete subset of a domain in the complex plain is the zero set of a holomorphic function on that domain. The generalization of this notion to higher dimensions is a complex analytic subset. A subset P of a domain $D \subset \mathbb{C}^n$ is said to be a complex analytic subset of D, if for each $p \in D$, there is a neighborhood ω of p in D, and holomorphic functions $f_1, f_2, \ldots, f_k$ on ω, such that

$$P \cap \omega = \{z \in \omega\colon f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}.$$

Every discrete subset of a domain in the complex plane is the zero set of a holomorphic function on that domain (Weierstrass).

A complex-valued function g defined on P is said to be *holomorphic*, if it is locally the restriction of a holomorphic function, i.e., for each $p \in P$, there is a neighborhood ω of p in D, and a holomorphic function g_p on ω such that the restriction of g_p to $P \cap \omega$ coincides with g. Note that according to this definition, for one variable, if P is an analytic subset of D (i.e., P is discrete in D) then *every* function $g : P \to \mathbb{C}$ is holomorphic! (why?) Further, in any dimension, a discrete subset of a domain D is an analytic subset (Why?), and again, every $\mathbb{C}$-valued function on such a set P is automatically holomorphic.

After these preliminaries we can state the following beautiful generalization of the Weierstrass theorem to higher dimensions due to Henri Cartan [2, p. 257, Theorem 1.9]: Let P be any analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^n$. Then every holomorphic function $g : P \to \mathbb{C}$ can be extended to a holomorphic function f on D.

7. Pseudoconvexity and Beyond

At this point we may raise the question why should there be such differences between domains in $\mathbb{C}$ and domains in $\mathbb{C}^n$, $n \geq 2$? At a philosophical level, the answer is simple: this is because $n = 1$ is a degenerate case, in

A discrete subset of a domain D is an analytic subset.

There are results about one complex variable that have no simple analog in several variables.

which due to the lack of room in $\mathbb{R}^2$ compared to $\mathbb{R}^4$, the full richness of the phenomena of complex analysis do not reveal themselves. On the other hand, since we are so used to this degenerate case, in several variables we single out the class of domains, the pseudoconvex domains, which have properties closest to those of domains in the plane, and work with them alone.

It would, however, be misleading to conclude that once pseudoconvex domains are introduced, all difficulties disappear, and the theory becomes completely analogous to that in one variable again. As an example of a result in one complex variable that has no simple analog in several variables (even restricting ourselves to pseudoconvex domains), we can consider the celebrated 'Riemann Mapping Theorem': a simply connected domain in the plane which is not the entire complex plane is biholomorphic to the unit disc. It was observed by Poincaré that this fails in higher dimensions, since, for example if $n \geq 2$, the unit ball

$$B_n = \{z \in \mathbb{C}^n \colon \ |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\},$$

is not biholomorphic to the unit polydisc

$$\Delta^n = \{z \in \mathbb{C}^n \colon \ |z_j| < 1, \text{for } j = 1, \ldots, n\},$$

although the two homeomorphic domains are both simply connected, bounded, and pseudoconvex ([6, Corollary 11.1.7, p. 433].) These and other related phenomena lead to a deeper study of domains and introduction of finer distinctions between them.

What about non-pseudoconvex domains? In general, very little can be said about general domains, although a few classes of non-pseudoconvex domains (e.g., the so-called 'q-convex domains') have been studied extensively. For some purposes, though not all, we can study functions on a non-pseudoconvex domain D by replacing D with its *envelope of holomorphy* $\mathcal{E}(D)$, the largest 'domain' to which each function f holomorphic on D extends holomorphically. For example, if Ω is as in (3) above, and Ω' is pseudoconvex, it is not difficult to see using Theorem 2 that $\mathcal{E}(\Omega) = \Omega'$. In general, the compulsory extension of holomorphic functions given by the Hartogs phenomenon is multiple-valued, and this leads to the notion of a *Riemann domain* spread over $\mathbb{C}^n$. Then the envelope of holomorphy of a domain in $\mathbb{C}^n$ is no longer a domain in $\mathbb{C}^n$, but a *Stein manifold* spread over $\mathbb{C}^n$ (the analog in several variables of open Riemann surfaces.)

8. Conclusion and Further Reading

It is clear that several complex variables are better than one. Just as the world is a more interesting place in having two genders, male and female, rather than only one, phenomena in several complex variables show a richness and complexity far greater than in one variable.

It is clear that several complex variables are better than one.

The reader interested in knowing more should begin by acquiring a thorough grounding in the most important special case of several complex variables, i.e., one complex variable. This subject is dealt in a large number of excellent texts including [1, 4], the second half of [12], and at a more sophisticated level [8]. After that a good idea would be to read the two expository articles [10, 7], and then proceed to the elementary texts [9, 13]. The next stage would be to acquire a working knowledge of the three main techniques used to study problems in several complex variables. The first and the most classical method is *sheaf theory*, which may be considered a generalization to several variables of the Weierstrass approach to one complex variable. This method leads quickly to the work of the founding fathers of several complex variables, Oka, Cartan, Serre, Grauert, Stein, etc. A modern account of this theory is in the three volumes of [3].

The second approach to the core results of the field is via partial differential equations, more precisely, through the theory of the inhomogeneous Cauchy–Riemann equations. This is a thoroughly modern method, depending on functional analysis and a priori estimates in the L^2-norm. A good place to begin is the masterly account [5] by one of the pioneers of this technique. A third more recent approach to the subject is via *integral representations*. In this technique formulas analogous to the *Cauchy integral formula* in one variable are developed for domains in many dimensions. The best source for this is [11].

The books [6, 2] represent eclectic syntheses by leading practitioners.

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