

AUTOMORPHISM GROUPS OF NILPOTENT LIE ALGEBRAS ASSOCIATED TO CERTAIN GRAPHS

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ABSTRACT. We consider a family of 2-step nilpotent Lie algebras associated to uniform complete graphs on odd number of vertices. We prove that the symmetry group of such a graph is the holomorph of the additive cyclic group \mathbb{Z}_n . Moreover, we prove that the (Lie) automorphism group of the corresponding nilpotent Lie algebra contains the dihedral group of order $2n$ as a subgroup.

1. INTRODUCTION

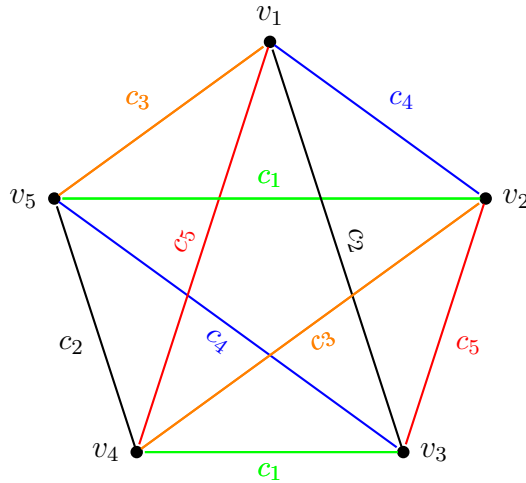
Many classes of 2-step nilpotent Lie algebras associated with various types of graphs have been studied recently from different points of view, see, e.g., [DM, DDM, GGI, F, FJ, M, N, PS, PT, R, LW]. A *2-step nilpotent Lie algebra* is a Lie algebra where each 3-fold Lie bracket $[X, [Y, Z]]$ of elements X, Y, Z of the Lie algebra is 0. The 3-dimensional Heisenberg Lie algebra is well-studied example of a 2-step nilpotent Lie algebra. In [DM], the authors studied the automorphism group of a 2-step nilpotent Lie algebra associated with a simple graph and then classified the graphs which correspond to the 2-step nilpotent Anosov Lie algebras. These Lie algebras give rise to interesting hyperbolic dynamics on nilmanifolds.

In this paper, we consider a similar problem for an interesting class of edge-colored directed simple graphs H_n where n is an odd integer. We begin by considering the underlying *undirected* edge-colored graph G_n of H_n . The example G_5 occurred in the recent paper [PS, Example 5.7], in connection with uniform Lie algebras. The graphs G_n are remarkable for having a large amount of symmetry which can be used for constructing other objects associated with it with nontrivial symmetry, e.g.

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Einstein solvmanifolds [D], infranilmanifolds etc. Interestingly, we found that, G_n arises naturally when we consider the cyclic group \mathbb{Z}_n of n elements as a space on which \mathbb{Z}_n acts by translations, i.e. as a *torsor* without a distinguished identity element. In section 3 we will give the algebraic definition of G_n but now we introduce it geometrically. For every odd integer n we construct the edge-colored simple graph G_n by beginning with the complete graph on n vertices v_1, \dots, v_n and thinking of these vertices as the vertices of a regular n -gon in the plane. We color the edges with n colors c_1, \dots, c_n in such a way that for every vertex v_k the $\frac{n-1}{2}$ edges which are perpendicular to the axis of symmetry of the n -gon passing through v_k are colored with the same color c_k . Below we illustrate this for $n = 5$.



In our first result, we compute explicitly the group of symmetries $\text{CPA}(G_n)$ of G_n preserving the coloring structure which we call the *color permuting automorphisms*, see Definition 2.1.

Theorem 1.1. $\text{CPA}(G_n) \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n)$.

Here $\text{Aut}(\mathbb{Z}_n)$ is the group of group-automorphisms of the cyclic group \mathbb{Z}_n . The semidirect product $\mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n)$ is known as the *holomorph* of \mathbb{Z}_n .

This result is interesting because the graph G_n is constructed out of the cyclic group \mathbb{Z}_n and therefore the result expresses an aspect of the combinatorics of this familiar object.

As already mentioned, the real motivation for considering these graphs comes from the theory of 2-step nilpotent Lie algebras. This idea goes back to [DM] for simple graphs and has been extended by many authors [DM, DDM, GGI, F, FJ, M, N, PS, PT, R, LW]. In [R, PS], with each directed edge-colored graph G , a 2-step nilpotent Lie algebra \mathcal{N}_G was associated and its properties were studied. This construction is recalled in Section 4.1 below. These Lie algebras can be thought as a quotient of the 2-step nilpotent Lie algebras associated with simple graphs as in [DM]. In order to obtain a Lie algebra from an edge-colored graph, we will further need that the edges are directed. We assign a certain natural orientation of the edges to G_n to obtain the directed edge-colored simple graphs H_n in Section 4.7. We are interested in understanding the group $\text{Aut}(\mathcal{N}_{H_n})$ of Lie algebra automorphisms of the corresponding Lie algebra \mathcal{N}_{H_n} . In the situation considered in [DM], each graph automorphism gives rise to a Lie algebra automorphism of the corresponding 2-step nilpotent Lie algebra. However, if we allow the repetition of the edge-colors, then only certain type of graph automorphisms or symmetries can be extended to the automorphisms of the associated Lie algebra. We call those automorphisms as *graph Lie automorphisms* and we denote the group of all such automorphisms of a graph G by $\text{GLA}(G)$. We compute explicitly the group $\text{GLA}(G_n)$ and prove the following theorem.

Theorem 1.2. $\text{GLA}(H_n) \cong D_n$, dihedral group of order $2n$. Consequently $\text{Aut}(\mathcal{N}_{H_n})$ contains a subgroup isomorphic to the dihedral group of order $2n$.

If H_n is thought as above to be a regular n -gon in the plane along with all the diagonals which are colored and directed in a certain way, then $\text{GLA}(H_n)$ can be thought of the Euclidean group of symmetries of this polygon, which is well-known to be the dihedral group of order $2n$.

The automorphism group of a nilpotent Lie algebra plays an important role in studying certain Einstein homogeneous spaces and infranilmanifolds (see, e.g. [DD, DV, LW]). It would be very interesting to find a complete description of the automorphism group of \mathcal{N}_{H_n} .

2. EDGE-COLORED GRAPHS AND THEIR AUTOMORPHISMS

In this section we recall some definitions (see [PS] for example). Let (S, E) denote a finite simple graph where S is the set of vertices and E is the set of edges. We denote an edge by a 2-set $\{\alpha, \beta\}$. Let \mathcal{C} denote a finite set of *colors*. An *edge-coloring* is a surjective function $c : E \rightarrow \mathcal{C}$. We call a graph $G = (S, E, c : E \rightarrow \mathcal{C})$ an *edge-colored graph*.

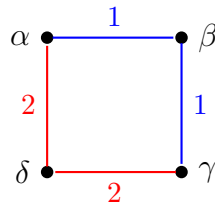
Recall that a bijection $\sigma : S \rightarrow S$ is a *graph automorphism of (S, E)* if the following holds: For all $\alpha, \beta \in S$, $\{\sigma(\alpha), \sigma(\beta)\} \in E$ if and only if $\{\alpha, \beta\} \in E$. In this case, we extend σ on the set E by defining $\sigma(\{\alpha, \beta\}) = \{\sigma(\alpha), \sigma(\beta)\}$.

Definition 2.1. Let $G = (S, E, c : E \rightarrow \mathcal{C})$ be an edge-colored graph. A graph automorphism χ of (S, E) is called a *color permuting automorphism of G* if there exists a permutation ϕ of the set of colors \mathcal{C} such that $\phi \circ c = c \circ \chi$ on E . The set of all color permuting automorphisms form a group which we denote by $\text{CPA}(G)$.

Example 2.2. Let C_4 denote a cycle graph on 4 vertices where the vertex set $S = \{\alpha, \beta, \gamma, \delta\}$ and $E = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \delta\}, \{\alpha, \delta\}\}$. Let $\mathcal{C} = \{1, 2\}$. We define the edge-coloring $c : E \rightarrow \mathcal{C}$ by

$$c(\{\alpha, \beta\}) = c(\{\beta, \gamma\}) = 1$$

$$c(\{\gamma, \delta\}) = c(\{\delta, \alpha\}) = 2.$$



Let χ be the permutation of S given by $\chi = (\alpha \ \gamma)(\beta \ \delta)$, which is a graph automorphism of (S, E) . Then the permutation of colors $\phi = (1 \ 2)$ satisfies $\phi \circ c = c \circ \chi$ on E and hence χ is a color permuting automorphism of C_4 with the above coloring c .

Note that $\tau = (\alpha \beta \gamma \delta)$ is not a color permuting automorphism of C_4 because

$$c(\{\tau(\alpha), \tau(\beta)\}) = 1 \neq c(\{\tau(\beta), \tau(\gamma)\}).$$

It can be seen that $\text{CPA}(C_4) = \{\text{id}, (\alpha \gamma)(\beta \delta)\}$. □

A uniform graph is a special type of an edge-colored graph.

Definition 2.3. We say that an edge-colored graph $(S, E, c : E \rightarrow \mathcal{C})$ is a *uniform graph* if it satisfies the following properties.

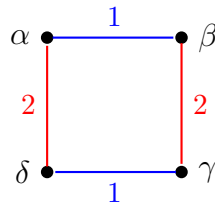
- (1) No two edges incident on the same vertex have the same color, i.e. $c(\{\alpha, \beta\}) \neq c(\{\alpha, \gamma\})$ if $\beta \neq \gamma$.
- (2) Each color occurs the same number of times, i.e. $|c^{-1}(\{c_i\})| = |c^{-1}(\{c_j\})|$ for all $c_i, c_j \in \mathcal{C}$.

Example 2.4. Consider the same uncolored graph (S, E) as in Example 2.2 and the same set of colors $\mathcal{C} = \{1, 2\}$. We define a new edge-coloring $c : E \rightarrow \mathcal{C}$ by

$$c(\{\alpha, \beta\}) = c(\{\gamma, \delta\}) = 1$$

$$c(\{\beta, \gamma\}) = c(\{\delta, \alpha\}) = 2.$$

We can see that $G = (S, E, c : E \rightarrow \mathcal{C})$ is a uniform graph and $\text{CPA}(G) \cong D_8$, the dihedral group with 8 elements.



The edge-colored graph in Example 2.2 is not uniform.

3. A UNIFORM GRAPH ASSOCIATED TO \mathbb{Z}_n

In this section we associate an edge-colored graph to a cyclic group of odd order and compute its symmetries. We note however, that this construction and the computation are very general and can be done for any abelian group of odd order.

Throughout we assume that n is an odd integer. We let \mathbb{Z}_n denote the cyclic group of order n written additively and denote the elements of \mathbb{Z}_n as $\{0, \dots, n-1\}$. Let G_n be an edge-colored graph where the underlying uncolored graph is a complete graph with vertex set \mathbb{Z}_n and where the set of colors \mathcal{C} is also \mathbb{Z}_n . We let the color of an edge $\{i, j\}$ be $(i + j) \in \mathbb{Z}_n$ where of course the addition $+$ means addition modulo n in \mathbb{Z}_n . More precisely, the edge-coloring in G_n is given by $c : E \rightarrow \mathcal{C}$ is defined by $c(\{i, j\}) = i + j$ for all $i, j \in \mathbb{Z}_n$.

Proposition 3.1. The edge-colored graph G_n is a uniform graph.

Proof. Let $i, j, k \in \mathbb{Z}_n$ be distinct. Then if $c(\{i, j\}) = c(\{i, k\})$, then $i + j = i + k$. Hence $j = k$. This shows that no two edges incident on the same vertex have the same color.

Consider the group homomorphism $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ defined by $f(i) = i + i = 2i$. Since n is odd, f is injective. For, if $i \in \mathbb{Z}_n$ with $2i = 0$, then the order of i is either 1 or 2 and divides the odd number n . Hence $i = 0$ and f is a group isomorphism.

For $m \in \mathbb{Z}_n$, we denote the set of all edges with color m by A_m . Equivalently,

$$A_m = \{\{i, j\} : i, j \in \mathbb{Z}_n, m = i + j, i \neq j\}.$$

We will prove that $|A_m| = \frac{n-1}{2}$. Since f is a bijection, there is a unique $l \in \mathbb{Z}_n$ such that $2l = m$. Hence $A_m = \{\{i, m - i\} : i \in \mathbb{Z}_n, i \neq l\}$. Note that for each $i \in \mathbb{Z}_n$, we have $\{i, m - i\} = \{m - i, i\}$. Therefore, $|A_m| = \frac{n-1}{2}$. In other words, the number of edges with color m is constant for all m . This proves that the edge-colored graph G_n is uniform. \square

3.2. Color Permuting Automorphism Group of G_n . In this section, we study the structure of the color permuting automorphism group $\text{CPA}(G_n)$ of the edge-colored graph G_n and prove Theorem 1.1.

Definition 3.3. We call a bijection $\tau : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ *special* if for all $a, b, c, d \in \mathbb{Z}_n$ with $a \neq b, c \neq d$, and $a + b = c + d$, we have $\tau(a) + \tau(b) = \tau(c) + \tau(d)$.

We first observe the following.

Proposition 3.4. The following statements are equivalent for a bijection $\tau : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$.

- (1) τ is special.
- (2) For all $a, b, c, d \in \mathbb{Z}_n$ with $c \neq d$, and $a + b = c + d$, we have $\tau(a) + \tau(b) = \tau(c) + \tau(d)$.
- (3) For all $a, b, c, d \in \mathbb{Z}_n$ with $a - c = d - b$, we have $\tau(a) - \tau(c) = \tau(d) - \tau(b)$.
- (4) $\tau \in \text{CPA}(G_n)$.

Proof. Assume (1). Let $a, c, d \in \mathbb{Z}_n$ and assume that $a + a = c + d$. Let $l = \tau(c) + \tau(d)$. We will prove that $\tau(a) + \tau(a) = l$. Let $B = \mathbb{Z}_n \setminus \{a\}$. If $x \in B$, then $x \neq 2a - x$ as n is odd, and $x + (2a - x) = c + d$. By our assumption (1), we have $\tau(x) + \tau(2a - x) = \tau(c) + \tau(d) = l$. From this, we can conclude that $\tau(B) = \{l - \tau(2a - x) : x \neq a\} = \mathbb{Z}_n \setminus \{l - \tau(a)\}$. Since τ is a bijection, this implies that $\tau(a) = l - \tau(a)$ and hence $\tau(a) + \tau(a) = l$ which proves (2).

Assume (2). Then the statement (3) is clear for elements $a, b, c, d \in \mathbb{Z}_n$ with $c \neq d$. The case $a \neq b$ follows similarly. If $a = b$ and $c = d$ and $a - c = d - b$, then $a = b = c = d$ as n is odd. Hence $\tau(a) - \tau(c) = \tau(c) - \tau(a) = 0$. This proves (3).

It is clear that (3) \implies (1).

Assume (1). We define $\phi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ as $\phi(m) = \sigma(i) + \sigma(j)$ where $m = i + j$ and $i \neq j$. We note that ϕ is well-defined function because τ is special. Let $l \in \mathbb{Z}_n$. We write $l = a + b$ where $a \neq b$. Then $\phi(\tau^{-1}(a) + \tau^{-1}(b)) = a + b = l$. Hence ϕ is surjective and hence it is a bijection. Also the color of an edge $\{\tau(i), \tau(j)\}$ is the

same as $\phi(i + j)$, i.e. $\phi(\text{color of the edge } \{i, j\})$. This proves that $\tau \in \text{CPA}(G_n)$. Hence (1) \implies (4).

Suppose now $\tau \in \text{CPA}(G_n)$. Then there exists a permutation ϕ of \mathbb{Z}_n such that the color of the edge $\{\tau(a), \tau(b)\}$ is the same as $\phi(\text{color of the edge } \{a, b\})$. Equivalently, $\tau(a) + \tau(b) = \phi(a + b)$. In particular, if $a + b = c + d$, then $\tau(a) + \tau(b) = \tau(c) + \tau(d)$. Hence τ is special. Hence (4) \implies (1). \square

Next we define a notion of an affine bijection on an abelian group.

Definition 3.5. Let $(A, +)$ denote an abelian group. A bijection $f : A \rightarrow A$ is called *affine* if there exists a group automorphism γ_f of A such that

$$f(a + x) = \gamma_f(a) + f(x)$$

for all $a, x \in A$. We denote the set of all affine bijections on A by $\text{Aff}(A)$.

It is not difficult to check that $\text{Aff}(A)$ is a group under composition.

Proposition 3.6. Let $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a bijection. Then the following statements are equivalent.

- (1) $f \in \text{CPA}(G_n)$.
- (2) f is special.
- (3) $f \in \text{Aff}(\mathbb{Z}_n)$.

Proof. (1) \iff (2) by Proposition 3.4.

Assume that f is special. We define $\gamma_f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ by $\gamma_f(a) = f(a) - f(0)$ for all $a \in \mathbb{Z}_n$. First we prove that γ_f is a group homomorphism. Let $i, j \in \mathbb{Z}_n$. As f is special, by Proposition 3.4, for all $i, j \in \mathbb{Z}_n$, we have $f(i + j) - f(j) = f(i) - f(0)$.

Hence for all $i, j \in \mathbb{Z}_n$,

$$\begin{aligned}
\gamma_f(i+j) &= f(i+j) - f(0) \\
&= f(i+j) - f(j) + f(j) - f(0) \\
&= f(i) - f(0) + f(j) - f(0) \\
&= \gamma_f(i) + \gamma_f(j).
\end{aligned}$$

Suppose that $\gamma_f(i) = 0$. This implies that $f(i) - f(0) = 0 \implies f(i) = f(0)$. As f is one-to-one, $i = 0$. Hence $\ker \gamma_f = \{0\}$. This proves that γ_f is a bijection and hence a group automorphism of \mathbb{Z}_n .

Also for $a, x \in \mathbb{Z}_n$, we have $f(a+x) - f(x) = f(a) - f(0)$ by Proposition 3.4. Hence $f(a+x) - f(x) = \gamma_f(a)$ and $f(a+x) = \gamma_f(a) + f(x)$. This proves that $f \in \text{Aff}(\mathbb{Z}_n)$ and (2) \implies (3).

We will prove that (3) \implies (2). For, we assume that $f \in \text{Aff}(\mathbb{Z}_n)$ and let $\gamma_f \in \text{Aut}(\mathbb{Z}_n)$ such that $f(a+x) = \gamma_f(a) + f(x)$ for all $a, x \in \mathbb{Z}_n$. Let $i, j, k, l \in \mathbb{Z}_n$ with $i - j = k - l$. We will prove that $f(i) - f(j) = f(k) - f(l)$.

$$\begin{aligned}
f(i) - f(j) &= f((i-j) + j) - f(j) \\
&= \gamma_f(i-j) \\
&= \gamma_f(k-l) \\
&= f((k-l) + l) - f(l) \\
&= f(k) - f(l).
\end{aligned}$$

By Proposition 3.4, f is special. □

Proposition 3.7. $\text{Aff}(\mathbb{Z}_n) \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n)$.

Proof. We define $\phi : \text{Aff}(\mathbb{Z}_n) \rightarrow \text{Aut}(\mathbb{Z}_n)$ by $\phi(f) = \gamma_f$ where $\gamma_f \in \text{Aut}(\mathbb{Z}_n)$ such that $f(a+x) = \gamma_f(a) + f(x)$ for all $a, x \in \mathbb{Z}_n$. We note that if $f \in \text{Aff}(\mathbb{Z}_n)$, then for all $x \in \mathbb{Z}_n$, $\gamma_f(x) = f(x) - f(0)$. We first prove that ϕ is a group homomorphism. Let

$f, g \in \text{Aff}(\mathbb{Z}_n)$. We need to prove $\gamma_{f \circ g} = \gamma_f \circ \gamma_g$. Let $a \in \mathbb{Z}_n$. Then

$$\begin{aligned}
\gamma_f(\gamma_g(a)) &= \gamma_f(g(a) - g(0)) \\
&= \gamma_f(g(a)) - \gamma_f(g(0)) \text{ as } \gamma_f \in \text{Aut}(\mathbb{Z}_n) \\
&= f(g(a)) - f(0) - (f(g(0)) - f(0)) \\
&= f \circ g(a) - f \circ g(0) \\
&= \gamma_{f \circ g}(a).
\end{aligned}$$

This shows that $\phi(f \circ g) = \phi(f) \circ \phi(g)$ and hence ϕ is a group homomorphism.

It is clear that $\ker \phi = \{f \in \text{Aff}(\mathbb{Z}_n) : f(x) = x + f(0) \text{ for all } x \in \mathbb{Z}_n\}$. Equivalently, $\ker \phi = \{T_a : a \in \mathbb{Z}_n\}$ where $T_a : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is a translation by a given by $T_a(x) = x + a$. For, if $f \in \ker \phi$, then $f = T_{f(0)}$. Also given $a \in \mathbb{Z}_n$, we have $T_a \in \text{Aff}(\mathbb{Z}_n)$ as $T_a(x + y) = x + y + a = x + T_a(y)$ and hence $T_a \in \ker \phi$. We note that $\ker \phi \cong \mathbb{Z}_n$.

If $\gamma \in \text{Aut}(\mathbb{Z}_n)$, then $\gamma(a + x) = \gamma(a) + \gamma(x)$ for all $a, x \in \mathbb{Z}_n \implies \gamma \in \text{Aff}(\mathbb{Z}_n)$ and $\phi(\gamma) = \gamma$. Hence ϕ is surjective and $\phi \circ \iota = \text{id}$ where $\iota : \text{Aut}(\mathbb{Z}_n) \rightarrow \text{Aff}(\mathbb{Z}_n)$ is the inclusion and $\text{id} : \text{Aut}(\mathbb{Z}_n) \rightarrow \text{Aut}(\mathbb{Z}_n)$ is the identity map. This means that the following exact sequence splits.

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow \text{Aff}(\mathbb{Z}_n) \xrightarrow{\phi} \text{Aut}(\mathbb{Z}_n) \longrightarrow 1$$

This proves that $\text{Aff}(\mathbb{Z}_n) \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n)$. □

Proof of Theorem 1.1. By Proposition 3.6, $\text{CPA}(G_n) = \text{Aff}(\mathbb{Z}_n)$ and by Proposition 3.7, $\text{Aff}(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n)$. □

Remark 1. We note that $\text{CPA}(G_n)$ acts transitively on the set of vertices \mathbb{Z}_n as all rotations are the color permuting automorphisms of G_n . The stabilizer of 0 under this action is precisely the automorphisms group of \mathbb{Z}_n , $\text{Aut}(\mathbb{Z}_n)$. For if $f \in \text{CPA}(G_n)$ and $f(0) = 0$, then there exists $\gamma_f \in \text{Aut}(\mathbb{Z}_n)$ such that $f(a + x) = \gamma_f(a) + f(x)$ for all $a, x \in \mathbb{Z}_n$. Then for $a \in \mathbb{Z}_n$, we have $f(a) = f(a + 0) = \gamma_f(a) + f(0) = \gamma_f(a)$. This shows that $f = \gamma_f \in \text{Aut}(\mathbb{Z}_n)$.

4. 2-STEP NILPOTENT LIE ALGEBRAS

4.1. Associating a Lie algebra with a graph. In this section we recall the construction of a 2-step nilpotent Lie algebra associated with an edge-colored *directed* graph (see [R] and also [PS]). Consider an edge-colored directed simple graph $H = (S, E, c : E \rightarrow \mathcal{C})$ where S is the set of vertices, E is the set of directed edges, and c is a surjective edge-coloring function from the set of (directed) edges to the set of colors \mathcal{C} . We will denote a directed edge from α to β by an ordered pair (α, β) . By abuse of notation, we will denote the color of the directed edge $(\alpha, \beta) \in E$, by simply $c(\alpha, \beta)$ rather than the more accurate $c((\alpha, \beta))$.

We associate with H a 2-step nilpotent Lie algebra \mathcal{N}_H over \mathbb{R} in the following way. The underlying vector space of \mathcal{N}_H is $V \oplus W$ where V is the \mathbb{R} -vector space consisting of formal \mathbb{R} -linear combinations of elements of S (so that S is a basis of V), and W is the \mathbb{R} -vector space consisting of formal \mathbb{R} -linear combinations of elements of \mathcal{C} . The Lie bracket structure on \mathcal{N}_H is given by the following

- (1) If $(\alpha, \beta) \in E$ and $c(\alpha, \beta) = Z$, then $[\alpha, \beta] = -[\beta, \alpha] = Z$.
- (2) If $(\alpha, \beta) \notin E$, then $[\alpha, \beta] = [\beta, \alpha] = 0$.
- (3) $[Y, Z] = 0$ for all $Y \in \mathcal{N}_H$ and $Z \in W$.

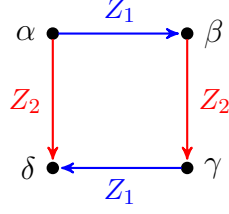
We say that \mathcal{N}_H is the *2-step nilpotent Lie algebra* associated with the graph H . Note that the derived Lie algebra $[\mathcal{N}_H, \mathcal{N}_H]$ is the span of \mathcal{C} and the dimension of \mathcal{N}_H is $|S| + |\mathcal{C}|$.

The above construction is a generalization of the construction of 2-step nilpotent Lie algebras associated with simple graphs as in [DM, M] where the edge-coloring c is a bijection.

Example 4.2. Consider the following directed edge-colored graph $H = (S, E, c : E \rightarrow \mathcal{C})$, where $S = \{\alpha, \beta, \gamma, \delta\}$, $E = \{(\alpha, \beta), (\beta, \gamma), (\gamma, \delta), (\delta, \alpha)\}$, $\mathcal{C} = \{Z_1, Z_2\}$ and edge-coloring $c : E \rightarrow \mathcal{C}$ is given by

$$c(\alpha, \beta) = c(\gamma, \delta) = Z_1,$$

$$c(\beta, \gamma) = c(\alpha, \delta) = Z_2.$$



Then the associated 2-step nilpotent Lie algebra \mathcal{N}_H is of dimension 8. The only non-zero Lie brackets among the basis vectors of \mathcal{N}_H are given by

$$[\alpha, \beta] = [\gamma, \delta] = Z_1 = -[\beta, \alpha] = -[\delta, \gamma],$$

$$[\beta, \gamma] = [\alpha, \delta] = Z_2 = -[\gamma, \beta] = -[\delta, \alpha].$$

4.3. Automorphism group. Recall that a linear isomorphism $\tau : \mathcal{N} \rightarrow \mathcal{N}$ is called a *Lie automorphism* of the Lie algebra \mathcal{N} if for all $X, Y \in \mathcal{N}$,

$$\tau[X, Y] = [\tau(X), \tau(Y)].$$

The group of all automorphisms of the Lie algebra \mathcal{N} is called the *automorphism group of \mathcal{N}* and is denoted by $\text{Aut}(\mathcal{N})$.

Given an edge-colored directed graph $H = (S, E, c : E \rightarrow \mathcal{C})$, we will characterize those graph automorphisms of the simple graph (S, E) which can be extended to Lie automorphisms of the associated 2-step nilpotent Lie algebra \mathcal{N}_H .

Let $H = (S, E, c : E \rightarrow \mathcal{C})$ be an edge-colored simple directed graph. Let E_u be the collection of associated undirected edges;

$$E_u = \{\{\alpha, \beta\} : (\alpha, \beta) \in E\}.$$

We color the undirected edges by the same colors, i.e. we use the coloring $c_u : E_u \rightarrow \mathcal{C}$ given by

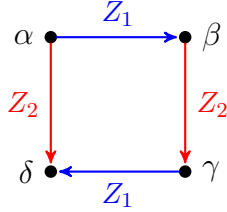
$$c_u(\{\alpha, \beta\}) = c(\alpha, \beta) \text{ if } (\alpha, \beta) \in E.$$

The edge-colored undirected simple graph $H_u = (S, E_u, c_u : E_u \rightarrow \mathcal{C})$ will be called the *underlying undirected graph* of H .

Let E^- denote the set $\{(\beta, \alpha) : (\alpha, \beta) \in E\}$ and \mathcal{C}^- denote the set $\{-Z \in W : Z \in \mathcal{C}\}$. We extend the edge-coloring function c on $E \cup E^-$ as follows: If $c(\alpha, \beta) = Z$, then we define $c(\beta, \alpha)$ by $-Z$.

Definition 4.4. Let $H = (S, E, c : E \rightarrow \mathcal{C})$ be an edge-colored simple directed graph. We say that a color permuting automorphism χ of the underlying undirected graph $H_u = (S, E_u, c_u : E_u \rightarrow \mathcal{C})$ is a *graph Lie automorphism* of H if it induces a permutation on $\mathcal{C} \cup \mathcal{C}^-$, i.e., if there exists a permutation ϕ of $\mathcal{C} \cup \mathcal{C}^-$ such that $c(\chi(\alpha), \chi(\beta)) = \phi(c(\alpha, \beta))$ for all $(\alpha, \beta) \in E \cup E^-$. We denote the group of all such automorphisms by $\text{GLA}(H)$.

Example 4.5. Consider the following directed edge-colored graph H and the associated 2-step nilpotent Lie algebra \mathcal{N}_H as in Example 4.2. The color permuting



automorphism group of the underlying undirected graph H_u , $\text{CPA}(H_u) \cong D_8$ (see Example 2.4). If $\chi = (\alpha\beta)(\gamma\delta)$, then we can see that $\chi \in \text{CPA}(H_u)$. Note that $\mathcal{C} \cup \mathcal{C}^- = \{Z_1, Z_2, -Z_1, -Z_2\}$. We define a permutation ϕ of $\mathcal{C} \cup \mathcal{C}^-$ as follows:

$$\phi(Z_1) = -Z_1, \phi(Z_2) = Z_2, \phi(-Z_1) = Z_1, \phi(-Z_2) = -Z_2.$$

Then $c(\chi(\alpha), \chi(\beta)) = c(\beta, \alpha) = -Z_1 = \phi(Z_1) = \phi(c(\alpha, \beta))$. Similarly one can check that $c(\chi(x), \chi(y)) = \phi(c(x, y))$ for all $(x, y) \in E \cup E^-$. Hence $\chi \in \text{GLA}(H)$.

Now if $\sigma = (\alpha\beta\gamma\delta)$, then $\sigma \in \text{CPA}(H_u)$. Note that $\sigma \notin \text{GLA}(H)$. This is because $c(\sigma(\alpha), \sigma(\beta)) = c(\beta, \gamma) = Z_2$ and $c(\sigma(\gamma), \sigma(\delta)) = c(\delta, \alpha) = -Z_2$. However $c(\alpha, \beta) = c(\gamma, \delta) = Z_1$ and hence there is no permutation ψ of $\mathcal{C} \cup \mathcal{C}^-$ such that $c(\sigma(\alpha), \sigma(\beta)) = \psi(c(\alpha, \beta))$.

We now show that the elements of $\text{GLA}(H)$ give rise to automorphisms of the associated Lie algebra \mathcal{N}_H .

Lemma 4.6. Let $H = (S, E, c : E \rightarrow \mathcal{C})$ be an edge-colored simple directed graph. If $\chi \in \text{GLA}(H)$, then χ can be uniquely extended to a Lie automorphism of \mathcal{N}_H . Therefore, the group $\text{GLA}(H)$ can be realized as a subgroup of $\text{Aut}(\mathcal{N}_H)$.

Proof. Note that $\mathcal{N}_H = V \oplus W$ where V is the \mathbb{R} -vector space with S as a basis and W is the \mathbb{R} -vector space with \mathcal{C} as a basis. In order to extend χ to a Lie automorphism of \mathcal{N}_H , we first extend χ linearly on V and then linearly on W by defining $\chi(Z) = c(\chi(\alpha), \chi(\beta))$ if $Z = c(\alpha, \beta)$. We will denote the extended linear map from \mathcal{N}_H to \mathcal{N}_H by χ as well. Now χ is well defined on \mathcal{N}_H because $\chi \in \text{GLA}(H)$ (see Definition 4.4). It can be seen that χ is onto and hence it is a linear isomorphism.

If $(\alpha, \beta) \in E$ and $c(\alpha, \beta) = Z$, then $\chi([\alpha, \beta]) = \chi(Z) = c(\chi(\alpha), \chi(\beta)) = [\chi(\alpha), \chi(\beta)]$ by definition of the Lie bracket on \mathcal{N}_H . As χ is linear, we have $\chi([X_1, X_2]) = [\chi(X_1), \chi(X_2)]$ for all $X_1, X_2 \in V$. Recall $[Y, U] = 0$ for all $Y \in \mathcal{N}_H$ and $U \in W$. Using the linearity of χ again, we have

$$\chi([X, Y]) = [\chi(X), \chi(Y)]$$

for all $X, Y \in \mathcal{N}_H$ and $\chi \in \text{Aut}(\mathcal{N}_H)$. The uniqueness of the extension is clear from the definition of \mathcal{N}_H . \square

4.7. The directed graph H_n . Throughout we assume that n is an odd integer. We define the directed edge-colored graph H_n whose underlying undirected graph is G_n as introduced in Section 3. We define the vertex set of H_n to be $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and the directed edge set E as follows:

$$E = \left\{ (m+i, m-i) : 0 \leq m \leq n-1, 1 \leq i \leq \frac{n-1}{2} \right\}.$$

The set of colors \mathcal{C} is denoted by $\{Z_i : i \in \mathbb{Z}_n\}$ and the edge-coloring $c : E \rightarrow \mathcal{C}$ is defined by $c(i, j) = Z_{i+j}$ for all $(i, j) \in E$.

Geometrically the orientation of edges of H_n can be visualized as follows. Note that the underlying undirected graph $(H_n)_u$ is nothing but the graph G_n , which can be pictured as a regular n -gon in the plane along with all possible diagonals. To obtain H_n , we orient the n edges of the regular polygon clockwise and then orient the

diagonals in such a way that the edges and diagonals which are parallel receive the same orientation.

For example, the directed edge-colored graph G_5 is as below.

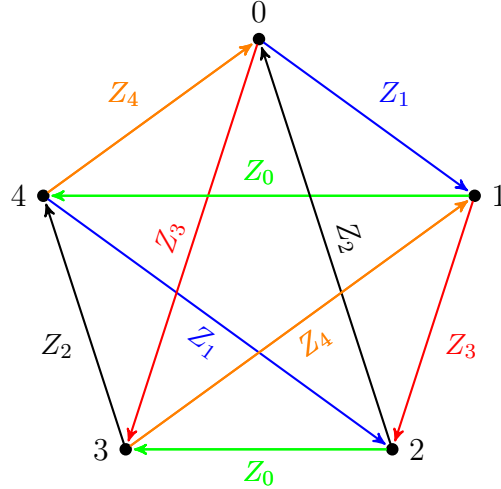


FIGURE 1. H_5

4.8. Graph Lie automorphism group of H_n . First, note that $\text{GLA}(H_n)$ acts transitively on the set of vertices \mathbb{Z}_n . This is because rotations (or translations) are the graph Lie automorphism. To see this, suppose that $\chi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is given by $\chi(i) = i + k$ for all $i \in \mathbb{Z}_n$. We define a permutation ϕ on \mathcal{C} by $\phi(Z_l) = Z_{l+2k}$ for all $l \in \mathbb{Z}_n$. Then if $0 \leq m \leq n - 1$ and $1 \leq i \leq \frac{n-1}{2}$, we have

$$\begin{aligned}
 c \circ \chi((m+i, m-i)) &= c(m+i+2k, m-i+2k) \\
 &= Z_{2m+4k} \\
 &= \phi(Z_{2m+2k}) \\
 &= \phi \circ c((m+i, m-i)).
 \end{aligned}$$

We can extend ϕ on $\mathcal{C} \cup \mathcal{C}^-$ by defining $\phi(-Z_l) = -Z_{l+2k}$. Hence $\chi \in \text{GLA}(H_n)$.

Let $\text{Stab}(0)$ denote the stabilizer of 0 under the action of $\text{GLA}(H_n)$ on \mathbb{Z}_n , i.e. let $\text{Stab}(0) = \{\sigma \in \text{GLA}(H_n) : \sigma(0) = 0\}$. Then by Orbit-stabilizer theorem, we have

$$(1) \quad |\text{GLA}(H_n)| = n|\text{Stab}(0)|.$$

We will prove that $\text{Stab}(0) = \{\pm \text{id}\}$. In other words, we will prove that if $\text{id} \neq \chi \in \text{GLA}(H_n)$ and $\chi(0) = 0$, then χ is a reflection, i.e. $\chi(i) = n - i$ for all $i \in \mathbb{Z}_n$.

Let $R = \{1, \dots, \frac{n-1}{2}\} \subset \mathbb{Z}_n$ and $L = \{\frac{n+1}{2}, \dots, n-1\} \subset \mathbb{Z}_n$. Note that in Figure 1 for H_5 , R (resp. L) consists of the vertices to the right (resp. left) of the vertical line through 0.

Lemma 4.9. If $\tau \in \text{Aut}(\mathbb{Z}_n)$ and $\tau \neq \text{id}$, then $\tau(R) \neq R$.

Proof. Let $\tau(1) = k$. If $k \in L$, we are done as $1 \in R$. Now we assume that $k \in R$, i.e. we assume that $2 \leq k \leq \frac{n-1}{2}$. We claim that there exists q with $1 \leq q \leq \frac{n-1}{2}$ such that $\frac{n+1}{2} \leq qk \leq n-1$. In \mathbb{Z} , we divide $n-1$ by k . Let $q \in \mathbb{N}$ and r with $0 \leq r \leq k-1$ such that $n-1 = qk + r$. Then $q = \frac{n-1}{k} - \frac{r}{k} \leq \frac{n-1}{k} \leq \frac{n-1}{2}$ as $k \geq 2$. Hence $q \leq \frac{n-1}{2}$. Also $n-1 \geq n-1-r > \frac{n-1}{2}$ as $r < \frac{n-1}{2}$. Hence $n-1 \geq qk > \frac{n-1}{2}$. This proves our claim.

As $\tau \in \text{Aut}(\mathbb{Z}_n)$ and $\tau(1) = k$, we have $\tau(q) = qk$. Hence $\tau(R) \neq R$ as $q \in R$ and $qk \in L$. \square

Lemma 4.10. Suppose that $\chi \in \text{GLA}(H_n)$ with $\chi(0) = 0$. If $\chi(k) \in R$ for some $k \in R$, then $\chi(R) = R$.

Proof. Assume that $\chi(k) = i \in R$ where $k \in R$. We note that χ is a color permuting automorphism of the undirected edge-colored complete graph. By Remark 1, $\chi \in \text{Aut}(\mathbb{Z}_n)$. Hence $\chi(n-k) = n-i \in L$. Assume that $\chi(j) \in L$ for some $j \in R$. Then $\chi(n-j) \in R$. Hence $c \circ \chi((k, n-k)) = c(i, n-i) = Z_0$ and $c \circ \chi((j, n-j)) = c(\chi(j), \chi(n-j)) = -Z_0$. We note that $c(k, n-k) = c(j, n-j)$ as both k and j are in R . This is a contradiction to our assumption that $\chi \in \text{GLA}(H_n)$. \square

Proposition 4.11. If $\chi \in \text{GLA}(H_n)$ and $\chi(0) = 0$, then $\chi^2 = \text{id}$.

Proof. If $\chi \in \text{GLA}(H_n)$ and $\chi(0) = 0$, then $\chi \in \text{Aut}(\mathbb{Z}_n)$ by Remark 1. By Lemma 4.9 and Lemma 4.10, we have $\chi(L) = R$ and $\chi(R) = L$. We note that $\chi^2 \in \text{GLA}(H_n)$ and hence $\chi^2(L) = \chi(R) = L$, $\chi^2(R) = \chi(L) = R$. By Lemma 4.9, $\chi^2 = \text{id}$. \square

As noted before, $\text{GLA}(H_n)$ contains all n rotations. Also Proposition 4.11 implies that the only non identity group automorphism which is a graph Lie automorphism must be the reflection about 0. This proves Theorem 1.2.

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