PROPER HOLOMORPHIC SELF-MAPS OF SYMMETRIC POWERS OF BALLS

DEBRAJ CHAKRABARTI AND CHRISTOPHER GROW

Abstract. We show that each proper holomorphic self map of a symmetric power of the unit ball is an automorphism naturally induced by an automorphism of the unit ball, provided the ball is of dimension at least two.

1. Introduction

Let $\mathbb{D}^m$ denote the m-dimensional polydisc in $\mathbb{C}^m$, and for $1 \leqslant k \leqslant m$, denote by σ_k the k-th elementary symmetric polynomial in m variables. The subset of $\mathbb{C}^m$ given by

$$\Sigma^m\mathbb{D} = \{(\sigma_1(z), \sigma_2(z), \ldots, \sigma_m(z)) \in \mathbb{C}^m | z \in \mathbb{D}^m\} \tag{1.1}$$

is known as the *symmetrized polydisc* of m-dimensions. It turns out that $\Sigma^m\mathbb{D}$ is a pseudoconvex domain $\mathbb{C}^n$ with remarkable function theoretic properties, and applications to engineering (see [1] and the work inspired by it). Of particular interest are the symmetries and mapping properties of these domains. In [11], Jarnicki and Pflug determined the biholomorphic automorphisms of $\Sigma^m\mathbb{D}$. More generally, we have the following result of Edigarian and Zwonek on proper self-maps of $\Sigma^m\mathbb{D}$:

Theorem 1 (See [7, 8]). *Let $f : \Sigma^m\mathbb{D} \to \Sigma^m\mathbb{D}$ be a proper holomorphic map, and let $\sigma = (\sigma_1, \ldots, \sigma_m) : \mathbb{C}^m \to \mathbb{C}^m$ be as in* (1.1)*. Then, there exists a proper holomorphic map $B : \mathbb{D} \to \mathbb{D}$ such that*

$$f(\sigma(z^1, \ldots, z^m) = \sigma(B(z^1), \ldots, B(z^m)).$$

Recall that a proper holomorphic map $B : \mathbb{D} \to \mathbb{D}$ is represented by a finite Blaschke product, so the above result gives a complete characterization of proper holomorphic self-maps of $\Sigma^m\mathbb{D}$, so that each such map is induced by a proper holomorphic self-map of the disc.

In this note we prove an analogous result for proper self-maps of complex analytic spaces analogous to the symmetrized polydisc, where the disc is replaced by a higher dimensional ball. To define these spaces, let $\mathbb{B}_s \subset \mathbb{C}^s$ denote the unit ball in $\mathbb{C}^s$, and for some positive integer m, let $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ denote the m-fold *symmetric power* of $\mathbb{B}_s$, i.e., the collection of all *unordered* m-tuples $\langle z^1, z^2, \ldots, z^m\rangle$, where each $z^j \in \mathbb{B}_s$. Note that the construction of the symmetric power is functorial: given any map $g : \mathbb{B}_s \to \mathbb{B}_s$, there is a naturally defined map $g^m_{\mathrm{Sym}} : (\mathbb{B}_s)^m_{\mathrm{Sym}} \to (\mathbb{B}_s)^m_{\mathrm{Sym}}$ given by

$$g^m_{\mathrm{Sym}}(\langle z^1, z^2, \ldots, z^m\rangle) = \langle g(z^1), g(z^2), \ldots, g(z^m)\rangle,$$

Debraj Chakrabarti was partially supported by a grant from the NSF (#1600371), a collaboration grant from the Simons Foundation (# 316632), and an Early Career internal grant from Central Michigan University. Christopher Grow was partially supported by a Research Assistantship from the Central Michigan University mathematics department.

which is easily seen to be well-defined. As discussed in section 2 below, $(\mathbb{B}_s)^m_{\text{Sym}}$ is a complex analytic space, and $\Sigma^m\mathbb{D}$ is biholomorphic to $\mathbb{D}^m_{\text{Sym}}$. Consequently, Theorem 2 is a generalization of Theorem 1. The main result of this paper is:

Theorem 2. *Let $s \geqslant 2$, $m \geqslant 2$, and let $f : (\mathbb{B}_s)^m_{\text{Sym}} \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map. Then, there exists a holomorphic automorphism $g : \mathbb{B}_s \to \mathbb{B}_s$ such that*

$$f = g^m_{\text{Sym}},$$

that is f is the m-th symmetric power of g.

It follows in particular that each proper self-map of $(\mathbb{B}_s)^m_{\text{Sym}}$ is in fact an automorphism. Recall also (see [13, p. 25]) that an automorphism of $\mathbb{B}_s$ is of the form

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \tag{1.2}$$

where $a \in \mathbb{B}_n$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional complex linear subspace spanned by a, $Q_a = \text{id} - P_a$ is the orthogonal projection from $\mathbb{C}^n$ onto the orthogonal complement of the one dimensional complex linear subspace spanned by a, and $s_a = (1 - |a|^2)^{\frac{1}{2}}$.

Note also that the domain $\mathbb{B}_s$ may be replaced in Theorem 2 by any strongly pseudoconvex domain, without any change in the proof. We prefer to state it in this special case for simplicity.

2. Complex Symmetric Powers

Recall that, informally, a *complex analytic space* is made of *local analytic subsets* of $\mathbb{C}^n$ glued together analytically, just as a complex manifold of dimension s is made of open sets of $\mathbb{C}^s$ analytically glued together (see [6, 14, 10] for more information). Recall also that an analytic subset of $\mathbb{C}^n$ is given near each of point of $\mathbb{C}^n$ by the vanishing of a family of analytic functions, and a local analytic subset of $\mathbb{C}^n$ is an analytic subset of an open subset of $\mathbb{C}^n$.

Let X be a complex manifold, or more generally, a complex analytic space. let X^m denote the m-th Cartesian power of X, which is by definition the collection of ordered tuples

$$\{(x^1, \ldots, x^m), x^j \in X, j = 1, \ldots, m\}.$$

X^m is then a complex manifold in an obvious way. The symmetric group S_m of bijective mappings of the set $\{1, \ldots, m\}$ acts on X^m as biholomorphic automorphisms: for $\sigma \in S_m$, and $x = (x^1, \ldots, x^m) \in X^m$ we set

$$\sigma \cdot x = (x^{\sigma(1)}, \ldots, x^{\sigma(m)}).$$

The m-th *symmetric Power* of X, denoted by X^m_{Sym} is the quotient of X^m under the action of S_m defined above, i.e, points of X^m_{Sym} are orbits of the action of S_m on X^m. We denote by

$$\pi : X^m \to X^m_{\text{Sym}}$$

the natural quotient map. It follows from the general theory of complex analytic spaces that X^m_{Sym} has a canonical structure of an analytic space, i.e., the quotient analytic space of X^m under the action of S_m as biholomorphic automorphisms (see [3, 4] and [10]). When X^m_{Sym} is given this complex structure, the map π becomes a proper holomorphic map.

In our application, we are interested in the case when $X = \mathbb{B}_s$, the unit ball in $\mathbb{C}^s$. The symmetric power $(\mathbb{B}_s)^m_{\text{Sym}}$ is then biholomorphic to a local analytic set, in fact to an open subset of a certain affine algebraic variety in $\mathbb{C}^{N(m,s)}$ where

$$N(m, s) = \binom{m+s}{m} - 1.$$

The subvariety of $\mathbb{C}^{N(m,s)}$ in which the image of $(\mathbb{B}_s)^m_{\text{Sym}}$ lies is in fact a realization of the symmetric power $(\mathbb{C}^s)^m_{\text{Sym}}$ and may be constructed using a symmetric analog of the classical Segre embeddding of the product of projective spaces as a subvariety of a higher dimensional projective space. When $s = 1$, we have $N(m, s) = m$, and the embedding $\mathbb{C}^m_{\text{Sym}} \to \mathbb{C}^m$ is actually a biholomorphism, given by

$$z \mapsto (\sigma_1(z), \dots, \sigma_m(z)),$$

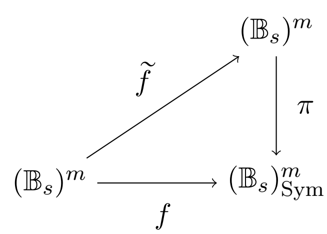
where $\sigma_k(\langle z_1, \dots, z_m\rangle)$ is the k-th elementary symmetric polynomial in the variables $(z_1, \dots, z_m)$. Then the image of $\mathbb{D}^m_{\text{Sym}}$ is a pseudoconvex domain $\Sigma^m\mathbb{D}$ in $\mathbb{C}^m$, called the *symmetrized polydisc.* See [5] for more details. Consequently, the symmetric power $\mathbb{D}^m_{\text{Sym}}$ is biholomorphically identified with the domain $\Sigma^m\mathbb{D}$ in $\mathbb{C}^m$, which shows that Theorem 2 is indeed an extension of Theorem 1.

3. Proper mappings of Cartesian to Symmetric powers

The first step in the proof of Theorem 2 is the following result, which is interesting in its own right:

Theorem 3. *Let $s \geqslant 2$, $m \geqslant 2$, and let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map. Then, there exists a proper holomorphic map $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ such that $f = \pi \circ \widetilde{f}$.*

In other words, the map f can be *lifted* to a proper holomorphic map $\widetilde{f}$ such that the following diagram commutes:



Note however, that the map π is *not* a covering map, so that the classical theory of lifting maps into a covering space is not directly applicable. However, as we will see, we can reduce this problem to a problem involving covering maps by removing the ramification locus and the branching locus of the map π from $(\mathbb{B}_s)^m$ and $(\mathbb{B}_s)^m_{\text{Sym}}$ respectively. We begin by proving two simple lemmas:

Lemma 3.1. *Let X and Y be analytic subsets of $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$, respectively, and let $f : X \to \Omega_2$ be a proper holomorphic map such that $f(X) \subset Y$. Then,*

(1) If $f(X) = Y$, then $\dim X = \dim Y$.
(2) If Y is irreducible and $\dim X = \dim Y$, then $f(X) = Y$.

Proof. By Remmert's Theorem, $f(X)$ is an analytic subset of Ω_2, with $\dim f(X) = \dim f$. Since f is proper, we cannot have $\dim f < \dim X$. Otherwise, the Rank Theorem would imply that for some $y \in f(X)$, $f^{-1}(y)$ is a compact analytic subset of X with positive dimension, which is impossible. Hence, we conclude that $\dim f(X) = \dim X$.

Suppose first $f(X) = Y$. Then, evidently, $\dim X = \dim f(X) = \dim Y$, establishing (1).

Now suppose $\dim X = \dim Y$ and suppose that $f(X) \neq Y$. Then, by well-known properties of analytic sets, $\overline{Y \backslash f(X)}$ is an analytic set, and since Y is closed in Ω_2, $\overline{Y \backslash f(X)}$ is contained in Y. Additionally, since $\dim f(X) = \dim Y$, $\overline{Y \backslash f(X)}$ cannot be all of Y. Now, since $Y = f(X) \cup \overline{Y \backslash f(X)}$, Y is reducible. Hence, if Y is irreducible and $\dim X = \dim Y$, then $f(X) = Y$, completing the proof of (2). □

Lemma 3.2. *Let $A = \{(z^1, \ldots, z^m) \in (\mathbb{B}_s)^m : z^i = z^j$ for some $i \neq j\}$. Then A is an analytic subset of $(\mathbb{B}_s)^m$ of codimension s and $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ of codimension s. The restricted map*

$$\pi : (\mathbb{B}_s)^m \backslash A \to (\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A)$$

is a holomorphic covering map of complex manifolds.

Proof. Let A_{ij} be the linear subspace of $(\mathbb{C}^s)^m \cong \mathbb{C}^{sm}$ given by

$$A_{ij} = \left\{(z^1, \ldots, z^m) | z^i = z^j\right\}.$$

Then A_{ij} is defined by the vanishing of s linearly independent linear functionals $z \mapsto z^i_k - z^j_k$ where $1 \leqslant k \leqslant s$, and consequently A_{ij} is of codimension s in $(\mathbb{C}^s)^m$. Since

$$A = \bigcup_{i<j} \left(A_{ij} \cap (\mathbb{B}_s)^m\right),$$

it now follows that A is an analytic subset of codimension s in $(\mathbb{B}^s)^m$. Since the finite-to-one quotient map $\pi : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\mathrm{Sym}}$ is proper and holomorphic, by Remmert's Theorem, $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$. Since $\pi^{-1}(\pi(A)) = A$, it follows that $\pi|_A$ is a proper map $A \to \pi(A)$, and since π is also surjective, it is easy to see that we must have $\dim A = \dim \pi(A)$. Since $\dim((\mathbb{B}_s)^m) = \dim((\mathbb{B}_s)^m_{\mathrm{Sym}})$, it follows that $\pi(A)$ has the same codimension in $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ as A has in $(\mathbb{B}_s)^m$, which is s.

We note that the restricted map

$$\pi|_{(\mathbb{B}_s)^m \backslash A} : (\mathbb{B}_s)^m \backslash A \to (\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A) \tag{3.1}$$

is still a proper holomorphic map, which is unbranched. Therefore $\pi|_{(\mathbb{B}_s)^m \backslash A}$ is a holomorphic covering. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\mathrm{Sym}}$ be a proper holomorphic map and let A be as defined in Lemma 3.2. Since $\pi(A)$ is an analytic subset in $(\mathbb{B}_s)^m_{\mathrm{Sym}}$, $(\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A)$ is an open, connected set.

Since a holomorphic covering is a local biholomorphism, we see from the covering map (3.1) that $(\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A)$ is an sm-dimensional complex manifold. Moreover, since $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$, we have that $(\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A)$ is connected and dense in $(\mathbb{B}_s)^m_{\mathrm{Sym}}$. Since $\mathrm{reg}((\mathbb{B}_s)^m_{\mathrm{Sym}})$ is an open subset of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ containing $(\mathbb{B}_s)^m_{\mathrm{Sym}} \backslash \pi(A)$,

$\operatorname{reg}((\mathbb{B}_s)^m_{\mathrm{Sym}})$ must be connected, and hence $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ is an irreducible analytic set, with $\dim((\mathbb{B}_s)^m_{\mathrm{Sym}}) = \dim((\mathbb{B}_s)^m_{\mathrm{Sym}}\backslash\pi(A))$. Since f is a proper holomorphic map from $(\mathbb{B}_s)^m$, which is a manifold of dimension sm, to $(\mathbb{B}_s)^m_{\mathrm{Sym}}$, which is an irreducible analytic set of dimension sm, by Lemma 3.1, f is surjective.

Clearly $f|_{f^{-1}(\pi(A))}$ is a proper holomorphic map from $f^{-1}(\pi(A))$, which is an analytic subset of $(\mathbb{B}_s)^m$, onto $\pi(A)$, and so by Lemma 3.1, $\dim(f^{-1}(\pi(A)) = \dim(\pi(A))$. Since we also have $\dim((\mathbb{B}_s)^m) = \dim((\mathbb{B}_s)^m_{\mathrm{Sym}}) = sm$, and we know from Lemma 3.2 that $\pi(A)$ has codimension at least s, $f^{-1}(\pi(A))$ must have codimension at least s in $(\mathbb{B}_s)^m$.

We now recall the following well-known fact: *Let Z be a connected complex manifold without boundary, $Y \subset Z$ be an analytic subset, x a point in $Z\backslash Y$, and $i : Z\backslash Y \to Z$ the inclusion map. Then if the complex codimension of Y is at least 2, then the homomorphism of the fundamental groups*

$$i_* : \pi_1(Z\backslash Y, x) \to \pi_1(Z, x)$$

is an isomorphism. Essentially, this is a reflection of the fact that thanks to the sufficiently large codimension of Y, by a standard transversality argument, there is no problem in homotopically deforming a loop based at x to a loop based at x and not intersecting Y, and further, given two loops based at x homotopic in Z, there is no problem in homotopically deforming them to each other in $Z\backslash Y$. See [9, Théorème 2.3, page 146].

Let $U = (\mathbb{B}_s)^m\backslash f^{-1}(\pi(A))$. Since A and $f^{-1}(\pi(A))$ have complex codimension at least $s \geqslant 2$, both $(\mathbb{B}_s)^m\backslash A$ and U are simply-connected. Hence, $f|_U$ has a holomorphic lift $\widetilde{f}|_U : U \to (\mathbb{B}_s)^m\backslash A$, where $f|_U = \pi \circ \widetilde{f}|_U$. Since $\widetilde{f}|_U$ is bounded on U and $f^{-1}(\pi(A))$ is an analytic set, by Riemann's Continuation Theorem $\widetilde{f}|_U$ extends to a holomorphic function $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ with $f = \pi \circ \widetilde{f}$.

$$\begin{array}{ccc} & & (\mathbb{B}_s)^m\backslash A \\ & \overset{\widetilde{f}|_U}{\nearrow} & \downarrow \pi \\ U & \xrightarrow[f|_U]{} & (\mathbb{B}_s)^m_{\mathrm{Sym}}\backslash\pi(A) \end{array} \qquad\qquad \begin{array}{ccc} & & (\mathbb{B}_s)^m \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \pi \\ (\mathbb{B}_s)^m & \xrightarrow[f]{} & (\mathbb{B}_s)^m_{\mathrm{Sym}} \end{array}$$

It remains to show that $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ is a proper map. Note that $f = \pi \circ \widetilde{f}$ is a proper map, and f is proper. If $\widetilde{f}$ were not proper, one could find a sequence $\{z_n\}_{n=1}^{\infty}$ with no limit points in $(\mathbb{B}_s)^m$ such that $\widetilde{f}(z_n) \to w \in (\mathbb{B}_s)^m$. But this composing with π, we see that f is not proper, which is a contradiction. Therefore $\widetilde{f}$ is proper. □

4. Proof of Theorem 2

Let $f : (\mathbb{B}_s)^m_{\mathrm{Sym}} \to (\mathbb{B}_s)^m_{\mathrm{Sym}}$ be a proper holomorphic map. Then, since π is proper and holomorphic, $h = f \circ \pi$ is a proper holomorphic map $(\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\mathrm{Sym}}$. By Theorem 3, h lifts to a proper holomorphic map $\widetilde{h} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ with $h = \pi \circ \widetilde{h}$. By a classical application of the methods of Remmert and Stein (see [12, page 76]), we conclude that there exist proper holomorphic self-maps of the ball $\mathbb{B}_s$, $\widetilde{h}_i$ for $i = 1, \ldots, m$ and a permutation σ of $\{1, 2, \ldots, m\}$ such that $\widetilde{h}$ has the structure

$$\widetilde{h}(\tau^1, \tau^2, \ldots, \tau^m) = \left(\widetilde{h}_1(\tau^{\sigma(1)}), \widetilde{h}_2(\tau^{\sigma(2)}), \ldots, \widetilde{h}_m(\tau^{\sigma(m)})\right).$$

We get the following commutative diagram:

$$\begin{CD} (\mathbb{B}_s)^m @>{\widetilde{h}}>> (\mathbb{B}_s)^m \\ @V{\pi}VV @VV{\pi}V \\ (\mathbb{B}_s)^m_{\text{Sym}} @>>{f}> (\mathbb{B}_s)^m_{\text{Sym}} \end{CD} \qquad h : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$$

Since $f \circ \pi = \pi \circ \widetilde{h}$, and the left-hand side is invariant under the action of S_m on $(\tau^1, \tau^2, \dots, \tau^m)$, we must have

$$\langle \widetilde{h}_1(\tau^{\sigma(1)}), \widetilde{h}_2(\tau^{\sigma(2)}), \dots, \widetilde{h}_m(\tau^{\sigma(m)}) \rangle = \langle \widetilde{h}_1(\tau^1), \widetilde{h}_2(\tau^2), \dots, \widetilde{h}_m(\tau^m) \rangle \tag{4.1}$$

for every $(\tau^1, \tau^2, \dots, \tau^m) \in (\mathbb{B}_s)^m$ and every $\sigma \in S_m$. Such a relation cannot hold unless there is a self map g of $\mathbb{B}_s$ such that for each $j = 1, \dots, m$, we have

$$\widetilde{h}_j = g,$$

since otherwise we could choose a σ for which the two sides would be distinct. Now, since $f \circ \pi = h$, we have

$$\begin{aligned} f(\langle \tau^1, \tau^2, \dots, \tau^m \rangle) &= \langle g(\tau^1), g(\tau^2), \dots, g(\tau^m) \rangle \\ &= g^m_{\text{Sym}}(\langle \tau^1, \tau^2, \dots, \tau^m \rangle) \end{aligned}$$

Since f is proper and $f = g^m_{\text{Sym}}$, it follows that g is a proper holomorphic self-map of the ball $\mathbb{B}_s$. Thanks to a classical result of Alexander (see [2], and also [13, page 316]), for $s \geqslant 2$, the only proper holomorphic self-mappings of $\mathbb{B}_s$ are the automorphisms of $\mathbb{B}_s$, and it is known that the automorphisms are given by certain multi-dimensional fractional linear maps (see [13]). Hence, g is an automorphism of $\mathbb{B}_s$ (of the form (1.2) and the proof is complete.

References

[1] J. Agler and N.J. Young, *A commutant lifting theorem for a domain in $\mathbb{C}^2$ and spectral interpolation.* J. Funct. Anal. 161 (2), 452–477 (1999)

[2] H. Alexander, *Proper holomorphic mappings in $\mathbb{C}^n$*, Indiana Univ. Math. J. 26 (1977), 137–146.

[3] H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes.* In Algebraic geometry and topology. pp. 90–102. Princeton University Press, Princeton, N. J. (1957)

[4] H. Cartan, *Quotients of complex analytic spaces.* Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960) pp. 1–15 Tata Institute of Fundamental Research, Bombay

[5] D. Chakrabarti and S. Gorai, *Function theory and holomorphic maps on symmetric products of planar domains.* J. Geom. Anal. 25 (2015), no. 4, 2196–2225.

[6] E. M. Chirka, *Complex Analytic Sets*, Kluwer Academic Publishers, 1989.

[7] A. Edigarian, *Proper Holomorphic Self-Mappings of the Symmetrized Bidisc*, Annales Polonici mathematici 84 (2004), 181–184.

[8] A. Edigarian and W. Zwonek, *Geometry of the Symmetrized Polydisc*, Archiv der Mathematik 84 (2005), 364–374.

[9] C. Godbillon, *Éléments du topologie algébrique*, Hermann 1971.

[10] Robert C. Gunning and Hugo Rossi; *Analytic functions of several complex variables.* Reprint of the 1965 original. AMS Chelsea Publishing, Providence, RI, 2009.

[11] M. Jarnicki and P. Pflug, *On automorphisms of the symmetrized bidisc.* Arch. Math. (Basel) 83 (2004), no. 3, 264–266.

[12] R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Mathematics, University of Chicago Press, 1971.
[13] W. Rudin, *Function Theory in the unit ball of* $\mathbb{C}^n$, Grundlehren der mathematischen Wissenschaften 241, Springer-Verlag, 1980.
[14] H. Whitney, *Complex Analytic Varieties*, Addison-Wesley Series in Mathematics, Addison-Wesley, 1972.

Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, USA
E-mail address: chakr2d@cmich.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48823, USA
E-mail address: growchri@msu.edu