FOURIER REPRESENTATIONS IN BERGMAN SPACES

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ABSTRACT

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by Pranav Upadrashta

It is a well known classical theorem of Paley and Wiener that every function in the Hardy space of the upper half plane is the holomorphic Fourier transform of a function square integrable on the positive real line. This correspondence establishes an isometric isomorphism between the Hardy space of the upper half plane and the Hilbert space of square integrable functions on the positive real line. Under this isometric isomorphism, the action of the group of translation symmetries of the upper half plane on the Hardy space corresponds to multiplication by a function in the Hilbert space of square integrable functions.

In this thesis we obtain similar results for Bergman spaces of certain domains which we call *polynomial half spaces*. The upper half plane and Siegel upper half space in higher dimensions are familiar examples of polynomial half spaces. Like the upper half plane, these domains have three one-parameter subgroups of symmetries, namely rotations, translations and scalings. Corresponding to each of the one-parameter subgroup of symmetries of a polynomial half space above, we obtain an isometric isomorphism between the Bergman space of the polynomial half space and a Hilbert space of square integrable functions. This isometric isomorphism respects the action of the corresponding one-parameter subgroup of symmetries on the Bergman space of polynomial half space analogous to the classical Paley-Wiener theorem. We give complete and detailed proofs of the three Fourier representations. Some similar results can be found in literature for the translation group, though perhaps not for the scaling group.

Finally we use the Fourier representations of Bergman space of polynomial half spaces to obtain integral representations of the Bergman kernel of these domains. For the Siegel upper half space, the classical formula for its Bergman kernel is recaptured using our method.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

CHAPTER I

INTRODUCTION

I.1. Motivation

I.1.1. Motivating Example

Let $U = \{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0\}$ denote the upper half plane in $\mathbb{C}$. Recall that the Hardy space $H^2(U)$ of the upper half plane consists of functions which are holomorphic in U and uniformly square integrable on horizontal lines in U. More precisely, $H^2(U)$ is the Banach space of functions $F \in \mathcal{O}(U)$ such that

$$\|F\|^2_{H^2(U)} := \sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^2 \, \mathrm{d}x < \infty.$$

A classical theorem of Paley and Wiener (see [21, Theorem 19.2]) states that every function in $H^2(U)$ arises as the holomorphic Fourier transform of a square integrable function on the positive real line $(0,\infty)$. More precisely, the map $T : L^2(0,\infty) \to H^2(U)$ given by

$$Tf(z) = \int_0^\infty f(t) e^{i2\pi zt} \, \mathrm{d}t \tag{I.1}$$

is an isometric isomorphism of Banach spaces. Consequently, $H^2(U)$ is a Hilbert space.

I.1.2. Relation with the Translation Group

For $\theta \in \mathbb{R}$ and $F \in H^2(U)$, let $\gamma(\theta)F \in H^2(U)$ be given by $(\gamma(\theta)F)(z) = F(z+\theta)$. Since the *translation* $z \mapsto z+\theta$ is a biholomorphic automorphism of U, $\theta \mapsto \gamma(\theta)$ is a unitary action of $\mathbb{R}$ on $H^2(U)$. i.e.,

$$\gamma : \mathbb{R} \to \mathscr{U}\left((H^2(U)\right)$$

is a unitary representation of $\mathbb{R}$ in the Hilbert Space $H^2(U)$. Thus, we have in (I.1)

$$(\gamma(\theta)F)(z) = F(z+\theta) = \int_0^\infty e^{i2\pi\theta t} f(t) e^{i2\pi zt} \, \mathrm{d}t = T(e^{i2\pi\theta(\cdot)} f)(z),$$

for all $t \in (0,\infty)$. That is, for each $\theta \in \mathbb{R}$, the following diagram commutes

$$\begin{array}{ccc} L^2(0,\infty) & \xrightarrow{T} & H^2(U) \\ \Big\downarrow \chi_\theta & & \Big\downarrow \gamma(\theta) \\ L^2(0,\infty) & \xrightarrow{T} & H^2(U) \end{array}$$

where χ_θ denotes the operator which multiplies a function in $L^2(0,\infty)$ by $e^{i2\pi\theta(\cdot)}$. We think of T as a linear change of co-ordinates in the vector space $H^2(U)$ which diagonalizes all the linear operators $\{\gamma(\theta) \,|\, \theta \in \mathbb{R}\}$. When such simultaneous diagonalization holds, we say that the isometric isomorphism $T : L^2(0,\infty) \to H^2(U)$ in (I.1) a *Fourier representation* of $H^2(U)$ corresponding to the translation group, the subgroup of $\mathrm{Aut}(U)$ consisting of maps of the form $z \mapsto z + \theta$ for real θ.

I.2. A Class of Domains with Large Automorphism Group

The goal of this thesis is to study some analogs of (I.1) for Bergman spaces of certain domains in $\mathbb{C}^n$ admitting large groups of automorphisms. We first recall the notion of Bergman space of a domain, and then introduce the class of domains we want to study.

I.2.1. Bergman Space of a Domain

Let Ω be a domain in $\mathbb{C}^n$, and let λ be a positive continuous function on Ω. The space $L^2(\Omega,\lambda)$ consists of all measurable complex valued functions f on Ω such that

$$\|f\|^2_{L^2(\Omega,\lambda)} := \int_\Omega |f(z)|^2 \lambda(z)\, \mathrm{d}V(z) < \infty,$$

where we identify functions that are almost everywhere equal. This is a Hilbert space of L^2 functions on Ω with the natural inner product. The closed subspace $A^2(\Omega,\lambda)$ of $L^2(\Omega,\lambda)$ (see Proposition II.2) consisting of functions holomorphic in Ω is called the *weighted Bergman space* of Ω with *weight* λ. When $\lambda \equiv 1$, we denote $A^2(\Omega,1) = A^2(\Omega)$. The Hilbert space $A^2(\Omega)$ is called the *Bergman space* of Ω.

I.2.2. Polynomial Half Space

We consider a class of domains in $\mathbb{C}^n$ which generalize the upper half plane $U \subset \mathbb{C}$. Let $m = (m_1, \cdots, m_n)$ be a given tuple of positive integers. Given a tuple $\alpha \in \mathbb{N}^n$ of nonnegative integers define the *weight of α with respect to m* to be

$$\mathrm{wt}_m(\alpha) := \sum_{i=1}^{n} \frac{\alpha_i}{2m_i}.$$

A real polynomial $p : \mathbb{C}^n \to \mathbb{R}$ is then called a *weighted homogeneous balanced polynomial* (with respect to the tuple $m \in \mathbb{N}^n$) if p is of the form

$$p(w_1, \cdots, w_n) = \sum_{\mathrm{wt}_m(\alpha)=\mathrm{wt}_m(\beta)=1/2} C_{\alpha,\beta} w^{\alpha} \overline{w}^{\beta}. \tag{I.2}$$

Let $p : \mathbb{C}^n \to \mathbb{R}$ be a weighted homogeneous balanced polynomial such that $p \geq 0$ on $\mathbb{C}^n$. The *polynomial half space $\mathcal{U}_p$* defined by p is the unbounded domain in $\mathbb{C}^{n+1}$

$$\mathcal{U}_p = \{(z, w) \in \mathbb{C} \times \mathbb{C}^n \mid \operatorname{Im} z > p(w)\}. \tag{I.3}$$

We show in Lemma II.6 that a polynomial half space $\mathcal{U}_p$ is biholomorphically equivalent to the bounded domain $\mathcal{E}_p \subset \mathbb{C}^{n+1}$, where

$$\mathcal{E}_p = \{(z, w) \in \mathbb{C} \times \mathbb{C}^n \mid |z|^2 + p(w) < 1\}. \tag{I.4}$$

We call the domain $\mathcal{E}_p$ a *polynomial ellipsoid*. In the trivial case $n = 0$, the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the only example of a polynomial ellipsoid corresponding to the only weighted homogeneous polynomial $p \equiv 0$ on the 0-dimensional space $\mathbb{C}^0 = \{0\}$.The domain $\mathbb{D}$ is biholomorphically equivalent to the upper half plane $U \subset \mathbb{C}$, which is a polynomial half space corresponding to the polynomial $p = 0$. The unit ball in $\mathbb{C}^n$ is another familiar example of polynomial ellipsoid (see Section II.2), which is biholomorphically equivalent to a polynomial half space called the *Siegel upper half space*.

Henceforth, $m = (m_1, \ldots, m_n)$ will denote a tuple of positive integers and p will be a nonnegative weighted homogeneous balanced polynomial with respect to m, as in (I.2).

It is very natural to consider polynomial half spaces in the problem of finding Fourier representations of Bergman spaces, thanks to the following result.

Theorem 1 ([2])**.** *Let Ω be a bounded convex domain with smooth boundary, and of finite type in sense of D'Angelo. Then* $\mathrm{Aut}(\Omega)$ *is noncompact if and only if Ω is biholomorphic to a polynomial ellipsoid.*

The *type* of a bounded domain Ω in sense of D'Angelo (see [6, Pg 118]) at a point $p \in \partial\Omega$ is the maximum order of contact of ambient one-dimensional complex varieties with $\partial\Omega$ at p.

The automorphism group of polynomial half spaces (and therefore polynomial ellipsoids) are noncompact, and contain one-parameter subgroups (see Section I.3 below) with respect to which we can obtain Fourier representations of Bergman spaces of these domains. This explains our interest in these domains.

I.3. Automorphisms of Polynomial Half Spaces.

In this section, we describe the subgroups of $\mathrm{Aut}(\mathcal{U}_p)$, the group of biholomorphic automorphisms of $\mathcal{U}_p$ that are of interest to us. These subgroups described below do not necessarily generate the group $\mathrm{Aut}(\mathcal{U}_p)$.

I.3.1. Rotation Subgroup

In case of the upper half plane $U \subset \mathbb{C}$, the action of the rotation subgroup of $\mathrm{Aut}(U)$ is easier to understand in the bounded model $\mathbb{D}$, the unit disk in $\mathbb{C}$. Similarly, in the case of polynomial half space $\mathcal{U}_p$, the action of the compact one-parameter subgroup of automorphisms is easier to understand in the bounded model $\mathcal{E}_p$, which we now describe.

It is apparent that for $\theta \in \mathbb{R}$, the map $\sigma_\theta : \mathcal{E}_p \to \mathcal{E}_p$ given by

$$\sigma_\theta(z, w) = (e^{i\theta} z, w), \text{ for } z \in \mathbb{C}, w \in \mathbb{C}^n \text{ such that } (z, w) \in \mathcal{E}_p \tag{I.5}$$

is an automorphism of $\mathcal{E}_p = \{(z,w) \in \mathbb{C} \times \mathbb{C}^n | |z|^2 + p(w) < 1\}$. We call such an automorphism a *rotation* of $\mathcal{E}_p$. The rotations of $\mathcal{E}_p$ form a compact one-parameter subgroup of the group $\mathrm{Aut}(\mathcal{E}_p)$ of biholomorphic automorphisms of $\mathcal{E}_p$, isomorphic to the circle group. Since there is a biholomorphic equivalence $\Lambda : \mathcal{U}_p \to \mathcal{E}_p$ (see Lemma II.6), therefore, for each $\theta \in \mathbb{R}$, the map $\Lambda^{-1} \circ \sigma_\theta \circ \Lambda : \mathcal{U}_p \to \mathcal{U}_p$ is an automorphism of the polynomial half space $\mathcal{U}_p$.

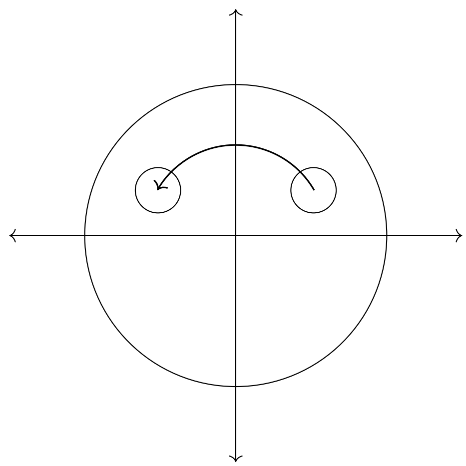

Figure 1. Action of the Rotation Subgroup on $\mathbb{D}$

I.3.2. Translation Subgroup

We next describe a noncompact one-parameter subgroup of automorphism group $\mathrm{Aut}(\mathcal{U}_p)$ of the polynomial half space $\mathcal{U}_p$.

For $\theta \in \mathbb{R}$, the map $\tau_\theta : \mathcal{U}_p \to \mathcal{U}_p$ given by

$$\tau_\theta(z,w) = (z+\theta, w), \text{ for } z \in \mathbb{C}, w \in \mathbb{C}^n \text{ such that } (z,w) \in \mathcal{U}_p \qquad \text{(I.6)}$$

is an automorphism of $\mathcal{U}_p = \{(z,w) \in \mathbb{C} \times \mathbb{C}^n \mid \mathrm{Im}\, z > p(w)\}$, since $\mathrm{Im}\,(z+\theta) = \mathrm{Im}\, z > p(w)$. We call τ_θ a *translation* of $\mathcal{U}_p$. The translations form a one-parameter subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ isomorphic to $\mathbb{R}$.

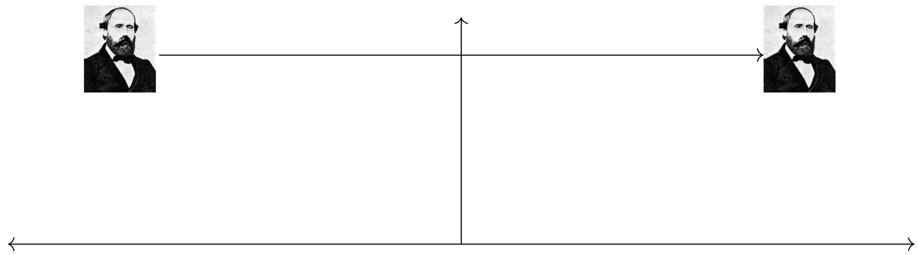

Figure 2. Action of the Translation Subgroup on U

I.3.3. Scaling Subgroup

We now describe another non-compact one-parameter subgroup of automorphism group $\mathrm{Aut}(\mathcal{U}_p)$ of a polynomial half space $\mathcal{U}_p$.

Recall that $m = (m_1, \ldots, m_n)$ is the tuple of positive integers with respect to which p is a weighted homogeneous balanced polynomial. For $\theta > 0$, let $\widehat{\rho}_\theta : \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$\widehat{\rho}_\theta(w) = \left(\theta^{1/2m_1} w_1, \ldots, \theta^{1/2m_n} w_n\right), \quad \text{for } w \in \mathbb{C}^n. \tag{I.7}$$

Then the map $\rho_\theta : \mathcal{U}_p \to \mathcal{U}_p$ given by

$$\rho_\theta(z,w) = (\theta z, \widehat{\rho}_\theta(w)) = \left(\theta z, \theta^{1/2m_1} w_1, \cdots, \theta^{1/2m_n} w_n\right) \text{ for } (z,w) \in \mathcal{U}_p \tag{I.8}$$

can be shown to be an automorphism of $\mathcal{U}_p$ (see Lemma II.5). We call such an automorphism a *scaling* of $\mathcal{U}_p$. The scalings of $\mathcal{U}_p$ form a one-parameter subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ isomorphic to the multiplicative group of positive real numbers.

Note that, in case of the upper half plane $U \subset \mathbb{C}$, these consist of dilations $z \mapsto \theta z$, where $\theta > 0$.

I.4. Results

In this thesis, we obtain Fourier representations analogous to (I.1) corresponding to the action of the rotation, translation and scaling groups on Bergman spaces of polynomial half spaces.

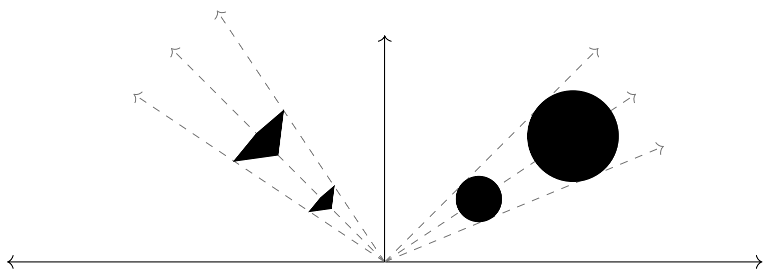

Figure 3. Action of the Scaling Subgroup on U

Recall that for a domain $\Omega \subset \mathbb{C}^n$, if $\phi : \Omega \to \Omega$ is an automorphism then the map ϕ^* from $A^2(\Omega)$ to itself given by

$$\phi^* f(z) = f(\phi(z)) \cdot \det \phi'(z), \quad \text{for all } z \in \Omega, f \in A^2(\Omega) \tag{I.9}$$

is an isometric isomorphism of $A^2(\Omega)$ with itself (see Lemma II.3). In other words, the map $\phi \mapsto \phi^*$ is a unitary representation of $\mathrm{Aut}(\Omega)$ in $A^2(\Omega)$.

The rotation group being compact, in this case (Theorem 2) we obtain a Fourier representation of $A^2(\mathcal{E}_p)$ with respect to this group as an infinite series instead of as an integral as in Theorems 3 and 4. We include this theorem here for completeness. In Theorems 3 and 4 we obtain Fourier representations of $A^2(\mathcal{U}_p)$ as a Hilbert space of L^2 functions on a subset of $\mathbb{R} \times \mathbb{C}^n$, such that the functions in the Hilbert space are holomorphic in the complex parameter.

I.4.1. Rotation Subgroup

Let $\mathbb{B}_p \subset \mathbb{C}^n$ be the domain given by

$$\mathbb{B}_p = \{w \in \mathbb{C}^n \mid p(w) < 1\}. \tag{I.10}$$

For $k \in \mathbb{N}$, let $\mathcal{W}_p(k)$ be the weighted Bergman space on $\mathbb{B}_p$ with respect to the weight $w \mapsto (1 - p(w))^{k+1}$, i.e.,

$$\mathcal{W}_p(k) = A^2\left(\mathbb{B}_p, (1-p)^{k+1}\right)$$

When $n=0$, the constant polynomial $p=0$ is the only weighted homogeneous balanced polynomial, so $\mathbb{B}_p = \mathbb{C}^0 = \{0\}$. Then the weight $w \mapsto (1-p(w))^{k+1}$ is identically 1 for all $k \in \mathbb{N}$, and we endow $\mathbb{C}^0$ with the counting measure. Then we interpret $\mathcal{W}_p(k)$ as the space of all maps from $\mathbb{C}^0$ to $\mathbb{C}$ that are square integrable on $\mathbb{C}^0$, i.e., $\mathcal{W}_p(k) = \mathbb{C}$ for all $k \in \mathbb{N}$.

Let $\mathcal{Y}_p$ be the Hilbert space of sequences $a = (a_k)_{k=0}^\infty$ where for each $k \in \mathbb{N}$, $a_k \in \mathcal{W}_p(k)$ and

$$\|a\|_{\mathcal{Y}_p}^2 := \pi \sum_{k=0}^\infty \frac{1}{k+1} \|a_k\|_{\mathcal{W}_p(k)}^2 < \infty. \tag{I.11}$$

A convenient language to express constructions like (I.11) is that of *direct integrals* (see [9, Sec 7.4]). Let μ be the measure on $\mathbb{N}$ which assigns to each nonnegative integer k, the mass $\mu(k) = \frac{\pi}{k+1}$. Notice that the function $a \in \mathcal{Y}_p$ is such that for each $k \in \mathbb{N}$, $a_k \in \mathcal{W}_p(k)$. Then the space $\mathcal{Y}_p$ is the direct integral

$$\mathcal{Y}_p = \int^{\oplus} \mathcal{W}_p(k)\, \mathrm{d}\mu(k). \tag{I.12}$$

Notice that in this case the direct integral reduces to a weighted direct sum of Hilbert spaces.

Theorem 2. *The map $T : \mathcal{Y}_p \to A^2(\mathcal{E}_p)$ given by*

$$Ta(z,w) = \sum_{k=0}^\infty a_k(w) z^k, \quad \textit{for all } (z,w) \in \mathcal{E}_p \tag{I.13}$$

is an isometric isomorphism of Hilbert spaces.

Note that when $n=0$, this reduces to a familiar result about Taylor expansions of a function in Bergman space $A^2(\mathbb{D})$ of the unit disc $\mathbb{D}$ (see Proposition III.1).

Let σ_θ be a rotation of $\mathcal{E}_p$ as in (I.5), when $\theta \in \mathbb{R}$. Then for each $F \in A^2(\mathcal{E}_p)$, the map $F \mapsto \sigma_\theta^* F$ determines an action of the rotation subgroup of $\mathrm{Aut}(\mathcal{E}_p)$ on $A^2(\mathcal{E}_p)$, where $\sigma_\theta^* F$ is defined as in (I.9). By Theorem 2, there is a sequence $a = (a_0, a_1, \dots) \in \mathcal{Y}_p$ such that $F = Ta$, where T is as in (I.13). Thus, for $(z,w) \in \mathcal{E}_p$, we have

$$\sigma_\theta^* F(z,w) = F(\sigma_\theta(z,w)) \det \sigma_\theta'(z,w) = \left(\sum_{k=0}^\infty a_k(w) (e^{i\theta} z)^k \right) \cdot e^{i\theta}$$

$$= T(a_0 e^{i\theta}, e^{i2\theta} a_1, e^{i3\theta} a_2, \ldots). \tag{I.14}$$

Therefore T is a Fourier representation of $A^2(\mathcal{E}_p)$ corresponding to the rotation subgroup of $\mathrm{Aut}(\mathcal{E}_p)$. It follows from equation (I.14) that the isometric isomorphism T in Theorem 2 diagonalizes the action of rotation subgroup of $\mathrm{Aut}(\mathcal{E}_p)$ on $A^2(\mathcal{E}_p)$.

I.4.2. Translation Subgroup

We now introduce $\mathcal{H}_p$, which is a space of L^2 functions, such that functions in $A^2(\mathcal{U}_p)$ are holomorphic Fourier transforms of functions in $\mathcal{H}_p$ (see Theorem 3).

Let $\mathcal{H}_p$ be the Hilbert space of measurable functions g on $(0,\infty) \times \mathbb{C}^n$ such that

$$\|g\|^2_{\mathcal{H}_p} := \int_{\mathbb{C}^n} \int_0^\infty |g(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \,\mathrm{d}V(w)\,\mathrm{d}t < \infty, \tag{I.15}$$

and

$$\frac{\partial g}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions, } 1 \le j \le n,$$

where $w_1, \ldots, w_n$ are co-ordinates of $\mathbb{C}^n$. In other words, functions in $\mathcal{H}_p$ are square integrable on $(0,\infty) \times \mathbb{C}^n$ with respect to the weight $(t,w) \mapsto e^{-4\pi p(w)t}/4\pi t$, and are holomorphic in the variable $w \in \mathbb{C}^n$.

Theorem 3. *The map $T_S : \mathcal{H}_p \to A^2(\mathcal{U}_p)$ given by*

$$T_S f(z,w) = \int_0^\infty f(t,w) e^{i2\pi zt}\,\mathrm{d}t, \quad \textit{for all } (z,w) \in \mathcal{U}_p \tag{I.16}$$

is an isometric isomorphism of Hilbert spaces.

For $\theta \in \mathbb{R}$, let τ_θ be a translation of $\mathcal{U}_p$ as in (I.6). Then for each $F \in A^2(\mathcal{U}_p)$ the map $F \mapsto \tau_\theta^* F$ determines an action of the translation subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ on $A^2(\mathcal{U}_p)$, where $\tau_\theta^* F$ is defined analogous to (I.9). By Theorem 3, there is an $f \in \mathcal{H}_p$ such that $F = T_S f$, where T_S is as in (I.16). Thus, for $(z,w) \in \mathcal{U}_p$ we have

$$\tau_\theta^* F(z,w) = F(\tau_\theta(z,w)) \cdot \det \tau_\theta'(z,w) = \left(\int_0^\infty f(t,w) e^{i2\pi(z+\theta)t}\,\mathrm{d}t \right) \cdot 1 = T_S(e^{i2\pi\theta(\cdot)} f).$$

Therefore (I.16) is a Fourier representation of $A^2(\mathcal{U}_p)$ corresponding to the translation subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ as the isometric isomorphism T_S in Theorem 3 diagonalizes the action of the translation subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ on $A^2(\mathcal{U}_p)$.

I.4.3. Scaling Subgroup

To study Fourier representations with respect to the scaling group we introduce $\mathcal{X}_p$, another Hilbert space isometrically isomorphic to $A^2(\mathcal{U}_p)$ (see Theorem 4). For $s \in (-1,1)$ and $t \in \mathbb{R}$ let λ be given by

$$\lambda(s,t) = \begin{cases} \dfrac{1}{4\pi t}\left(e^{-4\pi \sin^{-1}(s)t} - e^{-4\pi(\pi - \sin^{-1}(s))t}\right), \text{ if } t \neq 0 \\ \pi - 2\sin^{-1} s, \quad \text{if } t = 0. \end{cases} \tag{I.17}$$

Since $\lim_{t\to 0}\lambda(s,t) = \pi - 2\sin^{-1}(s)$, we see that for all $s \in (-1,1)$, the function $\lambda(s,\cdot)$ is continuous on $\mathbb{R}$.

Recall from (I.10) that $\mathbb{B}_p = \{w \in \mathbb{C}^n \mid p(w) < 1\}$. Let $\mathcal{X}_p$ be the Hilbert space consisting of measurable functions f on $\mathbb{R}\times\mathbb{B}_p$ such that

$$\|f\|^2_{\mathcal{X}_p} := \int_{\mathbb{R}}\int_{\mathbb{B}_p} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\, \mathrm{d}V(\zeta)\, \mathrm{d}t < \infty,$$

and such that

$$\frac{\partial f}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions, } 1 \le j \le n,$$

where $w_1,\dots,w_n$ are co-ordinates of $\mathbb{B}_p$. That is, $\mathcal{X}_p$ is the space of measurable functions on $\mathbb{R}\times\mathbb{B}_p$ which are holomorphic in the variable $\zeta \in \mathbb{B}_p$ and square integrable with respect to the weight $(t,\zeta) \mapsto \lambda(p(\zeta),t)$, where for $\zeta \in \mathbb{B}_p$,

$$\lambda\left(p(\zeta),t\right) = \begin{cases} \dfrac{1}{4\pi t}\left(\exp\left(-4\pi t \sin^{-1} p(\zeta)\right) - \exp\Big(-4\pi t\left(\pi - \sin^{-1} p(\zeta)\right)\Big)\right), \text{ if } t \neq 0 \\ \pi - 2\sin^{-1}\left(p(\zeta)\right), \quad \text{if } t = 0. \end{cases} \tag{I.18}$$

Let $\gamma \in \mathbb{C}$, and suppose that γ does not lie on the negative real axis. Let the map $\widehat{\rho}_\gamma : \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$\widehat{\rho}_\gamma(w_1, \ldots, w_n) = \left(\gamma^{1/2m_1} w_1, \ldots, \gamma^{1/2m_n} w_n\right) \quad \text{for all } w \in \mathbb{C}^n. \tag{I.19}$$

where the powers of γ are defined using a branch of the logarithm which coincides with the natural logarithm on the positive real line. Note that this extends the map (I.7) on page 8 to the complex plane.

Theorem 4. *Let $1/\mu = 1/m_1 + \cdots + 1/m_n$. For $(z,w) \in \mathcal{U}_p$, the map $T_V : \mathcal{X}_p \to A^2(\mathcal{U}_p)$ given by*

$$\begin{aligned} T_V g(z,w) &= \int_{\mathbb{R}} g\left(t, \widehat{\rho}_{1/z}(w)\right) \frac{z^{i2\pi t}}{z^{1+1/2\mu}} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} g\left(t, \frac{w_1}{z^{1/2m_1}}, \cdots, \frac{w_n}{z^{1/2m_n}}\right) \frac{z^{i2\pi t}}{z^{1+1/2\mu}} \, \mathrm{d}t \end{aligned} \tag{I.20}$$

is an isometric isomorphism of Hilbert spaces.

Let $\rho_\theta \in \mathrm{Aut}(\mathcal{U}_p)$ be as in (I.8), for $\theta > 0$. Then for each $F \in A^2(\mathcal{U}_p)$, the map $F \mapsto \rho_\theta^* F$ determines the action of the scaling subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ on $A^2(\mathcal{U}_p)$, where $\rho_\theta^* F$ is defined analogous to (I.9). By Theorem 4, there is an $f \in \mathcal{X}_p$ such that $F = T_V f$, where T_V is as in (I.20). Thus, for $(z,w) \in \mathcal{U}_p$ we have

$$\begin{aligned} \rho_\theta^* F(z,w) &= F(\rho_\theta(z,w)) \cdot \det \rho_\theta'(z,w) \\ &= \left(\int_{\mathbb{R}} f\left(t, \widehat{\rho}_{1/\theta z}(\widehat{\rho}_\theta(w))\right) \frac{(\theta z)^{i2\pi t}}{(\theta z)^{1+1/2\mu}} \, \mathrm{d}t \right) \cdot \theta^{1+1/2\mu} \\ &= T_V\left(\theta^{i2\pi(\cdot)} f\right). \end{aligned}$$

Thus, T_V is a Fourier representation of $A^2(\mathcal{U}_p)$ corresponding to the scaling subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ as the isometric isomorphism T_V in Theorem 4 diagonalizes the action of the scaling subgroup of $\mathrm{Aut}(\mathcal{U}_p)$ on $A^2(\mathcal{U}_p)$.

I.4.4. Bergman Kernel of Polynomial Half Spaces

Recall that associated to a weighted Bergman space $A^2(\Omega,\lambda)$ is a function $K:\Omega\times\Omega\to\mathbb{C}$ such that for all $f\in A^2(\Omega,\lambda)$, we have

$$f(z)=\int_{\Omega} f(w)K(z,w)\lambda(w)\,\mathrm{d}V(w)\ \text{ for all } z\in\Omega.$$

The *reproducing kernel* K for the weighted Bergman space $A^2(\Omega,\lambda)$ (see Proposition IV.1) is the unique function with this property that is holomorphic in the variable z and anti-holomorphic in the variable w. Also, recall that the reproducing kernel for $A^2(\Omega)$ is called the *Bergman kernel* of Ω (see [15, Chapter 1]).

The Bergman kernel of a domain Ω can sometimes be determined explicitly when Ω has a large group of symmetries, as in the case of the ball. For the polynomial half space $\mathcal{U}_p$, the asymptotic behavior of the Bergman kernel on the diagonal is determined by the symmetries of $\mathcal{U}_p$, as we show in Theorem 5 below. This result is a special case of a more general theorem in [4].

Theorem 5. *Let $\mathcal{U}_p$ be a polynomial half space and for $c>0$, let Γ_c be the subset of $\mathcal{U}_p$,*

$$\Gamma_c=\{(z,w)\in\mathcal{U}_p \mid |z|>c\,|w|\}. \tag{I.21}$$

Let K be the Bergman kernel for $\mathcal{U}_p$ and for $(z,w)\in\mathcal{U}_p$, let

$$K_{\text{diag}}(z,w)=K(z,w;z,w)$$

be the Bergman kernel of $\mathcal{U}_p$ on the diagonal of $\mathcal{U}_p\times\mathcal{U}_p$. Then we have for each $c>0$,

$$\lim_{\substack{(z,w)\to(0,0)\\(z,w)\in\Gamma_c}} K_{\text{diag}}(z,w)d(z,w)^{2+1/\mu}=K_{\text{diag}}(i,0), \tag{I.22}$$

where $1/\mu=\sum_{j=1}^{n}1/m_j$ and $d(z,w)$ is the distance of a point $(z,w)\in\mathcal{U}_p$ to $\partial\mathcal{U}_p$.

Remark. The limit in the left hand side of equation (I.22) corresponds to the nontangential convergence of $(z,w)\in\mathcal{U}_p$ to the point $(0,0)\in\partial\mathcal{U}_p$.

Recall from (I.12) that $\mathcal{Y}_p$ admits the following direct integral representation

$$\mathcal{Y}_p = \int^{\oplus} \mathcal{W}_p(k)\, \mathrm{d}\mu(k),$$

where for $k \in \mathbb{N}$, $\mathcal{W}_p(k)$ is the weighted Bergman space on $\mathbb{B}_p$ with weight $w \mapsto (1-p(w))^{k+1}$. Using this together with the fact that $A^2(\mathcal{E}_p)$ is isometrically isomorphic to $\mathcal{Y}_p$ allows us to express the Bergman kernel of $\mathcal{E}_p$ in terms of the reproducing kernels for $\mathcal{W}_p(k)$ below.

Theorem 6. *For $k \in \mathbb{N}$, let $Y_p(k;\cdot,\cdot)$ be the reproducing kernel for the weighted Bergman space $\mathcal{W}_p(k)$ on the domain $\mathbb{B}_p \subset \mathbb{C}^n$. Then for $z, Z \in \mathbb{C}$ and $w, W \in \mathbb{C}^n$ such that $(z,w), (Z,W) \in \mathcal{E}_p$, the Bergman kernel B of $\mathcal{E}_p$ is given by*

$$B(z,w;Z,W) = \sum_{k=0}^{\infty} \frac{k+1}{\pi} Y_p(k;w,W) z^k \overline{Z}^k.$$

For $t > 0$, let $\mathcal{S}_p(t)$ be the weighted Bergman space on $\mathbb{C}^n$ with weight $w \mapsto e^{-4\pi p(w)t}$, i.e.,

$$\mathcal{S}_p(t) = A^2\left(\mathbb{C}^n, e^{-4\pi pt}\right). \tag{I.23}$$

Then we have the following direct integral representation (see Proposition II.7, Section II.3) of $\mathcal{H}_p$

$$\mathcal{H}_p = \int^{\oplus} \mathcal{S}_p(t) \frac{\mathrm{d}t}{4\pi t}, \tag{I.24}$$

where the integral is taken with respect to the Haar measure $\mathrm{d}t/4\pi t$ on $(0,\infty)$. Then we may represent the Bergman kernel for $A^2(\mathcal{U}_p)$ in terms of reproducing kernel for $\mathcal{S}_p(t)$ as below.

Theorem 7. *Let the Bergman kernel for $\mathcal{S}_p(t)$ be denoted by $H_p(t;\cdot,\cdot)$. Then for $(z,w), (Z,W) \in \mathcal{U}_p$, the Bergman kernel K of $\mathcal{U}_p$ is given by*

$$K(z,w;Z,W) = 4\pi \int_0^{\infty} t H_p(t;w,W) e^{i2\pi(z-\overline{Z})t}\, \mathrm{d}t \tag{I.25}$$

For $t \in \mathbb{R}$ let $\mathcal{Q}_p(t)$ be the weighted Bergman space on $\mathbb{B}_p$ with weight $w \mapsto \lambda(p(w),t)$, where λ is as in equation (I.18) i.e.,

$$\mathcal{Q}_p(t) = A^2\left(\mathbb{B}_p, \lambda(p,t)\right). \tag{I.26}$$

Then the Hilbert space $\mathcal{X}_p$ admits the following direct integral representation (see Proposition II.9, Section II.3)

$$\mathcal{X}_p(t) = \int^{\oplus} \mathcal{Q}_p(t)\,\mathrm{d}t, \tag{I.27}$$

where the integral is taken over $\mathbb{R}$. Then we may represent the Bergman kernel for $A^2(\mathcal{U}_p)$ in terms of reproducing kernel for $\mathcal{Q}_p(t)$ as below.

Theorem 8. *Let the reproducing kernel for $\mathcal{Q}_p(t)$ be denoted by $X_p(t;\cdot,\cdot)$. Then for $(z,w),(Z,W) \in \mathcal{U}_p$, the Bergman kernel K of $\mathcal{U}_p$ is given by*

$$K(z,w;Z,W) = \int_{\mathbb{R}} X_p(t;w,W)\frac{z^{2\pi it}\overline{Z}^{-2\pi it}}{(z\overline{Z})^{1+1/2\mu}}\,\mathrm{d}t. \tag{I.28}$$

I.5. Outline of this Thesis

In Chapter 2, we recall basic facts about Bergman spaces and present proofs of propositions that we will need to prove our main results in Chapter 3 and Chapter 4. We will also discuss geometric properties of polynomial ellipsoids and give a few examples. Finally, we discuss the direct integral representation of Hilbert spaces $\mathcal{H}_p$ and $\mathcal{X}_p$. In Chapter 3, we present proofs of Theorems 2, 3, and 4. We first prove analogous results in one variable, and use these in the proofs of Theorems 2, 3, and 4. Finally in Chapter 4, we recall notions about Hilbert spaces that are necessary to obtain integral representations of Bergman kernel of $\mathcal{U}_p$ as in Theorems 7 and 8, and complete the proofs of Theorems 6, 7 and 8.

I.6. Historical Note

The classical Paley-Wiener theorem was obtained in [17] for the Hardy space of the upper half plane. A Fourier representation for Hardy space of tube domains in $\mathbb{C}^n$ was obtained by S. Bochner ([5]); tube domains are a different set of generalizations of the upper half plane in $\mathbb{C}$ to higher dimensions. A Fourier representation of Bergman space of tube domains over self dual cones was obtained by O. Rothaus ([20]) and this was generalized to Bergman space of tubes over more general cones by A. Koranyi ([13]). The Fourier type representation of Bergman space of the upper half plane with respect to the translation group has been rediscovered many times (see [10, 22]), most recently by P. Duren, et. al ([7]). A. Koranyi along with E. Stein also obtained a Fourier representation for Hardy spaces of generalized half spaces ([14]). R. Ogden and S. Vági ([16]) obtained Fourier representations of functions in Hardy space of Siegel domains of type II, with respect to the translation group.

For Bergman space of the Siegel upper half space $U_n \subset \mathbb{C}^n$, M. Peloso et al., ([1]) obtained Fourier representations of Bergman space of U_n with respect to the Heisenberg group. N. Vasilevski et al., ([18]) obtained representations of Bergman space of Siegel upper half space $U_n \subset \mathbb{C}^n$ as L^2 spaces, with respect to maximal abelian subgroups of $\mathrm{Aut}(U_n)$. Their representation however is not holomorphic in the complex parameter.

Integral representation of the Bergman kernel of polynomial half space as in (I.25) was already obtained by F. Haslinger ([12]), by differentiating a similar formula for the Szegö kernel.

It seems that Fourier representations of Bergman spaces of polynomial half spaces (even for the upper half plane) with respect to the scaling group have not been noted before.

CHAPTER II

PRELIMINARIES

In this chapter we state and prove familiar properties of the weighted Bergman space $A^2(\Omega,\lambda)$ of a domain $\Omega\subset\mathbb{C}^n$. We first show that the weighted Bergman space $A^2(\Omega,\lambda)$ of Ω is a closed subspace of $L^2(\Omega,\lambda)$. Then we show that Bergman spaces $A^2(\Omega_1)$ and $A^2(\Omega_2)$ of biholomorphically equivalent domains Ω_1 and Ω_2 in $\mathbb{C}^n$ are isometrically isomorphic. We also show that a polynomial half space $\mathcal{U}_p$ is biholomorphically equivalent to a polynomial ellipsoid $\mathcal{E}_p$. Then we give a few examples of polynomial half spaces. Finally we obtain the direct integral representations (II.5) and (II.19) of the Hilbert spaces $\mathcal{H}_p$ and $\mathcal{X}_p$ introduced in Theorems 3 and 4 respectively.

II.1. Bergman Spaces

Proposition II.1 (Bergman Inequality)**.** *Let $A^2(\Omega,\lambda)$ be the weighted Bergman space of Ω with weight λ. Then, for each compact set $K\subset\Omega$, there is a constant $C_K>0$ such that*

$$\sup_{z\in K}|f(z)|\le C_K\,\|f\|_{A^2(\Omega,\lambda)}\,. \tag{II.1}$$

Proof. Define the number $r>0$ in the following way. If $\Omega=\mathbb{C}^n$, let $r=1$, and if $\Omega\neq\mathbb{C}^n$, $r=\frac{1}{2}\left(\operatorname{dist}(K,\partial\Omega)\right)$. Let

$$K_r=\{z\in\mathbb{C}^n\mid \operatorname{dist}(z,K)\le r\}.$$

Then $K_r\subset\Omega$ and for all $z\in K$, $\overline{B(z,r)}\subset K_r$. Here $\overline{B(z,r)}$ is the closed ball of radius r and center z. Then, by mean value property of harmonic functions applied to real and imaginary parts of $f\in A^2(\Omega,\lambda)$, we have

$$f(z)=\frac{1}{\mathrm{Vol}\,(B(z,r))}\int_{B(z,r)}f(\zeta)\,\mathrm{d}V(\zeta).$$

Consequently, we have

$$|f(z)|=\frac{1}{\mathrm{Vol}\,(B(z,r))}\left|\int_{B(z,r)}f(\zeta)\,\mathrm{d}V(\zeta)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\text{Vol }(B(z,r))}\int_{B(z,r)}|f(z)|\,\mathrm{d}V(\zeta)\\
&\leq \frac{1}{\text{Vol }(B(z,r))}\left(\int_{B(z,r)}\frac{\mathrm{d}V(\zeta)}{\lambda(\zeta)}\right)^{1/2}\left(\int_{B(z,r)}|f(z)|^2\lambda(\zeta)\,\mathrm{d}V(\zeta)\right)^{1/2}\\
&\leq \frac{1}{(\text{Vol }B(z,r))^{1/2}}\left\|\frac{1}{\lambda}\right\|_{\overline{B(z,r)}}^{1/2}\|f\|_{L^2(B(z,r),\lambda)}\\
&\leq \frac{1}{(\text{Vol }B(z,r))^{1/2}}\left\|\frac{1}{\lambda}\right\|_{K_r}^{1/2}\|f\|_{A^2(\Omega,\lambda)}\\
&= \frac{1}{(\text{Vol }B(0,1))^{1/2}\,r^n}\left\|\frac{1}{\lambda}\right\|_{K_r}^{1/2}\|f\|_{A^2(\Omega,\lambda)}\\
&:= C_K\,\|f\|_{A^2(\Omega,\lambda)}\,.
\end{aligned}
$$

Here $C_K > 0$ is finite, because K_r is compact and λ is strictly positive on Ω so $\|1/\lambda\|_{K_r}$, the minimum value of λ on K_r is a positive quantity. □

Corollary II.2. *Let Ω be a domain in $\mathbb{C}^n$. The weighted Bergman space $A^2(\Omega,\lambda)$ of Ω with weight λ is a closed subspace of $L^2(\Omega,\lambda)$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $A^2(\Omega,\lambda)$, which converges in $L^2(\Omega,\lambda)$. Then by Bergman inequality (II.1) it follows that the sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on compact sets of Ω, and consequently the limiting function is holomorphic in Ω. Thus, the limiting function is in $A^2(\Omega,\lambda)$ which shows that $A^2(\Omega,\lambda)$ is closed in $L^2(\Omega,\lambda)$. □

Lemma II.3 (Biholomorphic Invariance of Bergman Space)**.** *Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic map between two domains Ω_1 and Ω_2 in $\mathbb{C}^n$. The map Φ induces an isometric isomorphism $\Phi^* : A^2(\Omega_2) \to A^2(\Omega_1)$ between the Bergman spaces of Ω_2 and Ω_1, where for all $F \in A^2(\Omega_2)$ and all $z \in \Omega_1$ we have*

$$
(\Phi^* F)(z) = F(\Phi(z)) \cdot \det \Phi'(z).
$$

Proof. Let $F \in A^2(\Omega_2)$. Then we have

$$
\|F\|^2_{A^2(\Omega_2)} = \int_{\Omega_2} |F(\zeta)|^2\,\mathrm{d}V(\zeta).
$$

Applying the change of variables formula to the map Φ,

$$\begin{aligned}\|F\|^2_{A^2(\Omega_2)} &= \int_{\Omega_1} |F(\Phi(z))|^2 \left|\det \Phi'(z)\right|^2 \,\mathrm{d}V(z) \\ &= \int_{\Omega_1} \left|F(\Phi(z)) \det \Phi'(z)\right|^2 \,\mathrm{d}V(z) \\ &= \left\|(F\circ\Phi)\cdot \det\Phi'\right\|^2_{A^2(\Omega_1)}.\end{aligned}$$

This shows that $F \in A^2(\Omega_2)$ if and only if $(F\circ\Phi)\cdot\det\Phi' \in A^2(\Omega_1)$. This also shows that Φ^* is an isometry and injective. Let $\Psi : \Omega_2 \to \Omega_1$ be the inverse of Φ. Given $F \in A^2(\Omega_1)$, let $G : \Omega_2 \to \mathbb{C}$ be given by

$$G(z) = F(\Psi(z)) \cdot \det\Psi'(z).$$

Then we have

$$G\circ\Phi(z)\cdot\det\Phi'(z) = F(\Psi(\Phi(z)))\cdot\det\Psi'(\Phi(z))\cdot\det\Phi'(z) = F(z).$$

This shows that $(G\circ\Phi)\cdot\det\Phi' \in A^2(\Omega_1)$. Thus, it follows that $G \in A^2(\Omega_2)$ and $\Phi^*G = F$. Consequently, Φ^* is surjective and an isometric isomorphism as claimed. □

II.2. Geometry of Polynomial Half Spaces

Lemma II.4. *Let $\gamma \in \mathbb{C}\setminus\{z\in\mathbb{C} \,|\, \operatorname{Im} z = 0, \operatorname{Re} z \le 0\}$, and let $p : \mathbb{C}^n \to \mathbb{R}$ be a weighted homogeneous balanced polynomial. Recall from* (I.19) *that $\widehat{\rho}_\gamma : \mathbb{C}^n \to \mathbb{C}^n$ is given by*

$$\widehat{\rho}_\gamma(\zeta_1,\ldots,\zeta_n) = \left(\gamma^{1/2m_1}\zeta_1,\ldots,\gamma^{1/2m_n}\zeta_n\right),\ \textit{for } \zeta\in\mathbb{C}^n.$$

Then we have $p(\widehat{\rho}_\gamma(\zeta)) = |\gamma|\, p(\zeta)$.

Proof. The function $L : \mathbb{C}\setminus\{z\in\mathbb{C}\,|\,\operatorname{Im} z = 0, \operatorname{Re} z\le 0\} \to \mathbb{C}$ given by

$$L(\gamma) = \log|\gamma| + i\arg\gamma,\ -\pi < \arg\gamma < \pi$$

is a well defined branch of logarithm, using which we define the powers $\gamma^{1/2m_1},\dots,\gamma^{1/2m_n}$. Consequently, for a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ we have

$$
\begin{aligned}
\left(\widehat{\rho}_\gamma(\zeta)\right)^\alpha &= \left(\gamma^{1/2m_1}\zeta_1\right)^{\alpha_1}\dots\left(\gamma^{1/2m_n}\zeta_n\right)^{\alpha_n} \\
&= \gamma^{\sum_{j=1}^n \alpha_j/2m_j}\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n} \\
&= \gamma^{\mathrm{wt}_m(\alpha)}\zeta^\alpha.
\end{aligned}
$$

Thus, for a weighted homogeneous balanced polynomial p we have

$$
\begin{aligned}
p\left(\widehat{\rho}_\gamma(\zeta)\right) &= \sum_{\mathrm{wt}_m(\alpha)=\mathrm{wt}_m(\beta)=1/2} C_{\alpha,\beta}\left(\widehat{\rho}_\gamma(\zeta)\right)^\alpha\left(\overline{\widehat{\rho}_\gamma(\zeta)}\right)^\beta \\
&= \sum_{\mathrm{wt}_m(\alpha)=\mathrm{wt}_m(\beta)=1/2} C_{\alpha,\beta}\gamma^{\mathrm{wt}_m(\alpha)}\zeta^\alpha\overline{\gamma}^{\mathrm{wt}_m(\beta)}\overline{\zeta}^\beta \\
&= \sum_{\mathrm{wt}_m(\alpha)=\mathrm{wt}_m(\beta)=1/2} C_{\alpha,\beta}\gamma^{\mathrm{wt}_m(\alpha)}\zeta^\alpha\overline{\gamma^{\mathrm{wt}_(\beta)}}\overline{\zeta}^\beta \\
&= \sum_{\mathrm{wt}_m(\alpha)=\mathrm{wt}_m(\beta)=1/2} C_{\alpha,\beta}\gamma^{1/2}\overline{\gamma^{1/2}}\zeta^\alpha\overline{\zeta}^\beta \\
&= |\gamma|\,p(\zeta).
\end{aligned}
$$

□

For a weighted homogeneous balanced polynomial p, recall that the polynomial half space $\mathcal{U}_p$ is a domain in $\mathbb{C}^{n+1}$ given by

$$\mathcal{U}_p=\{(z,w)\in\mathbb{C}\times\mathbb{C}^n\mid \operatorname{Im} z>p(w)\},$$

and the polynomial ellipsoid defined by p is given by

$$\mathcal{E}_p=\{(z,w)\in\mathbb{C}\times\mathbb{C}^n\mid |z|^2+p(w)<1\}.$$

Corollary II.5. *Recall from* (I.7) *that for* $\theta>0$*, and* $w=(w_1,\dots,w_n)\in\mathbb{C}^n$*,* $\widehat{\rho}_\theta$ *is given by*

$$\widehat{\rho}_\theta(w_1,\cdots,w_n)=\left(\theta^{1/2m_1}w_1,\cdots,\theta^{1/2m_n}w_n\right).$$

Then the map $\widehat{\rho} : \mathcal{U}_p \to \mathcal{U}_p$ *given by (as in* (I.8)*)*

$$\rho_\theta(z,w) = (\theta z, \widehat{\rho}_\theta(w)) \text{ for } (z,w) \in \mathcal{U}_p$$

is an automorphism of $\mathcal{U}_p$*.*

Proof. By Lemma II.4 we have $p(\rho_\theta(w)) = p(\widehat{\rho}_\theta(w)) = \theta p(w)$, since θ is a positive number. Thus, we have

$$\operatorname{Im} \rho_\theta(z) = \operatorname{Im} \theta z = \theta \operatorname{Im} z > \theta p(w) = p(\rho_\theta(w)).$$

This shows that $\rho_\theta(z,w)$ is in $\mathcal{U}_p$. Since

$$\rho_\theta \circ \rho_{1/\theta}(z,w) = \rho_{1/\theta} \circ \rho_\theta(z,w) = (z,w),$$

it follows that ρ_θ is a bijection. Since ρ_θ is a linear map, it is also holomorphic in $\mathcal{U}_p$. Thus, ρ_θ is an automorphism of $\mathcal{U}_p$, as it is a holomorphic bijection (see [19, Theorem 2.14]). □

Lemma II.6. *The map* $\Lambda : \mathcal{U}_p \to \mathcal{E}_p$ *given by*

$$\begin{aligned} \Lambda(z, w_1, \ldots, w_n) &= \left(\frac{1+iz/4}{1-iz/4}, \frac{w_1}{(1-iz/4)^{1/m_1}}, \cdots, \frac{w_n}{(1-iz/4)^{1/m_n}} \right) \\ &= \left(\frac{1+iz/4}{1-iz/4}, \widehat{\rho}_{(1-iz/4)^{-2}}(w) \right), \end{aligned} \tag{II.2}$$

where $\widehat{\rho}$ *is as in* (I.19)*, is a biholomorphic equivalence.*

Proof. Since $\operatorname{Im} z > p(w) > 0$, $(1-iz/4) \in \mathbb{C} \setminus \{z \in \mathbb{C} \,|\, \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. Thus, by Lemma II.4 we have

$$p\left(\frac{w_1}{(1-iz/4)^{1/m_1}}, \cdots, \frac{w_n}{(1-iz/4)^{1/m_n}} \right) = p\left(\widehat{\rho}_{(1-iz/4)^{-2}}(w) \right) = \frac{p(w)}{|1-iz/4|^2}.$$

Consequently we have

$$\begin{aligned} \left| \frac{1+iz/4}{1-iz/4} \right|^2 + p\left(\frac{w_1}{(1-iz/4)^{1/m_1}}, \cdots, \frac{w_n}{(1-iz/4)^{1/m_n}} \right) &= \frac{|1+iz/4|^2 + p(w)}{|1-iz/4|^2} \\ &= \frac{1+|z/4|^2 - \operatorname{Im} z/2 + p(w)}{1+|z/4|^2 + \operatorname{Im} z/2} \end{aligned}$$

$$\leq \frac{1+|z/4|^2 - \operatorname{Im} z/2 + \operatorname{Im} z}{1+|z/4|^2 + \operatorname{Im} z/2}$$

$$= 1.$$

This shows that $\Lambda(z,w) \in \mathcal{E}_p$. For $(z,w) \in \mathcal{E}_p$, the inverse of Λ is the map $\Lambda^{-1} : \mathcal{E}_p \to \mathcal{U}_p$ given by

$$\Lambda^{-1}(z, w_1, \cdots, w_n) = \left(4i\frac{1-z}{1+z}, \frac{2^{1/m_1} w_1}{(1+z)^{1/m_1}}, \cdots, \frac{2^{1/m_n} w_n}{(1+z)^{1/m_n}}\right).$$

Since Λ is a holomorphic map and a set theoretic bijection, it follows that Λ is a biholomorphism (see [19, Theorem 2.14] for instance). □

We now give some examples of polynomial half spaces.

II.2.1. Siegel Upper Half Space

Let $p(w) = \sum_{j=1}^{n} |w_j|^2$. Then p is a weighted homogeneous balanced polynomial with respect to the tuple $(1, \ldots, 1) \in \mathbb{N}^n$. We see that the polynomial half space in this case is

$$\mathcal{U}_{n+1} = \{(z,w) \in \mathbb{C} \times \mathbb{C}^n \mid \operatorname{Im} z > |w|^2\}.$$

$\mathcal{U}_{n+1}$ is biholomorphically equivalent to the unit ball $\mathbb{B}^{n+1} = \{(z,w) \in \mathbb{C}^{n+1} \mid |z|^2 + |w|^2 < 1\}$, and the bihilomorphism $\Lambda : \mathcal{U}_{n+1} \to \mathbb{B}^{n+1}$ in Lemma II.6 is the Cayley map given by

$$\Lambda(z, w_1, \ldots, w_n) = \left(\frac{1-iz/4}{1+iz/4}, \frac{w_1}{1-iz/4}, \cdots, \frac{w_n}{1-iz/4}\right).$$

Note that when $n = 0$, the domain $\mathcal{U}_1$ is the upper half plane U in $\mathbb{C}$ and the map above reduces to the familiar one variable Cayley map between $\mathcal{U}_1 = U$ and the disc $\mathbb{D}$.

II.2.2. Bedford-Pinchuk Domains.

Let m be a positive integer. Then the solution $\alpha \in \mathbb{N}$ to the equation $\mathrm{wt}_m(\alpha) = 1/2$ is $\alpha = m$. Then a weighted homogeneous balanced polynomial p with respect to this integer is of the form $p(w) = C|w|^{2m}$. Taking $C = 1$, we get

$$\mathcal{E}_m = \{(z,w) \in \mathbb{C}^2 \mid \operatorname{Im} z > |w|^{2m}\}.$$

The polynomial half space $\mathcal{E}_m$ is biholomorphically equivalent to the Bedford-Pinchuk domain

$$E_m = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^{2m} < 1\} \tag{II.3}$$

where the bihilomorphism $\Lambda : \mathcal{E}_m \to E_m$ in Lemma II.6 is the given by

$$\Lambda(z,w) = \left(\frac{1 - iz/4}{1 + iz/4}, \frac{w}{(1 - iz/4)^{1/m}} \right).$$

Interestingly, the domains $E_m \subset \mathbb{C}^2$ are the only 'nice' domains in $\mathbb{C}^2$ that have large automorphism groups as the following theorem due to Bedford and Pinchuk shows.

Theorem 9 ([3])**.** *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with real analytic boundary. If* $\operatorname{Aut}(\Omega)$ *is noncompact, then it is biholomorphically equivalent to the domain E_m given by (II.3).*

II.3. Direct Integral Representations

In this section, we express the Hilbert spaces $\mathcal{H}_p$ and $\mathcal{X}_p$ as a direct integral of weighted Bergman spaces on domains in $\mathbb{C}^n$. In view of this, we may interpret Theorems 3 and 4 as representing the Bergman space of the polynomial half space $\mathcal{U}_p$ in $\mathbb{C}^{n+1}$ as a direct integral of weighted Bergman spaces on domains in $\mathbb{C}^n$. We now introduce the notion of a direct integral of Hilbert spaces.

Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, i.e., μ is a σ-finite measure on the σ-algebra $\mathscr{M}$ of sets in X. Let $\{H_t\}_{t \in X}$ be a collection of separable Hilbert spaces of equal dimension. A *section* f of the collection $\{H_t\}_{t \in X}$ is a function on X such that $f(t) \in H_t$ for each $t \in X$. A countable set of sections $\{e_j\}_{j=1}^{\infty}$ is called a *measurability structure* on the collection of Hilbert spaces $\{H_t\}_{t \in X}$ if the functions $t \mapsto \langle e_j(t), e_k(t) \rangle_{H_t}$ are measurable for all $j, k \in \mathbb{N}$ and the linear span of $\{e_j(t)\}_{j=1}^{\infty}$ is dense in H_t for each $t \in X$. A section f is said to be *measurable* if the function $t \mapsto \langle e_j(t), f(t)\rangle_{H_t}$ is measurable for each $j \in \mathbb{N}$.

Suppose the following data is given: (1) a σ-finite measure space $(X, \mathscr{M}, \mu)$, (2) a collection $\{H_t\}_{t\in X}$ of separable Hilbert spaces of equal dimension, (3) a measurability structure on $\{H_t\}_{t\in X}$. Then the *direct integral* of H_t's with respect to the measure μ denoted by

$$\int^{\oplus} H_t \, \mathrm{d}\mu(t)$$

is the space of equivalence classes of almost everywhere equal measurable sections f for which

$$\|f\|^2 = \int_X \|f(t)\|_{H_t}^2 \, \mathrm{d}\mu(t) < \infty.$$

It can be shown that the direct integral of a collection of Hilbert spaces is a Hilbert space with the natural inner product.

Recall that the p is a nonnegative weighted homogeneous balanced polynomial with respect to the tuple $m = (m_1, \ldots, m_n)$ of positive integers. With this notation let

$$M = \text{l.c.m}(2, m_1, \cdots, m_n).$$

Then we call a polynomial of the form

$$g(w) = \sum_{\substack{\mathrm{wt}_m \alpha = k/M \\ \alpha \in \mathbb{N}^n}} C_\alpha w^\alpha, \; w \in \mathbb{C}^n \tag{II.4}$$

a *weighted homogeneous polynomial of weighted degree k/M*.

II.3.1. Direct Integral Representation of $\mathcal{H}_p$

Recall that the Hilbert space $\mathcal{H}_p$ consists of measurable functions $f : (0,\infty) \times \mathbb{C}^n \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{H}_p}^2 = \int_0^\infty \int_{\mathbb{C}^n} |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \, \mathrm{d}V(w) \, \mathrm{d}t < \infty$$

and

$$\frac{\partial f}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions for } 1 \leq j \leq n,$$

where $w_1, \cdots, w_n$ are co-ordinates of $\mathbb{C}^n$.

Proposition II.7. *For $t > 0$, let $\mathcal{S}_p(t)$ be the weighted Bergman space on $\mathbb{C}^n$ with the weight $w \mapsto e^{-4\pi p(w)t}$, i.e.,*

$$\mathcal{S}_p(t) = A^2(\mathbb{C}^n, e^{-4\pi pt}).$$

Then $\{\mathcal{S}_p(t)\}_{t>0}$ is a collection of Hilbert spaces indexed by $(0,\infty)$. For a tuple $\alpha \in \mathbb{N}^n$, let q_α be given by

$$q_\alpha(t)(w) = w^\alpha, \textit{ for all } t > 0, w \in \mathbb{C}^n.$$

Then q_α is a section on the collection $\{\mathcal{S}_p(t)\}_{t>0}$ of Hilbert spaces, i.e., $q_\alpha(t) \in \mathcal{S}_p(t)$ for all $t > 0$, which is a measurability structure on $\{\mathcal{S}_p(t)\}_{t>0}$. Then the Hilbert space $\mathcal{H}_p$ (as above) admits the direct integral representation

$$\mathcal{H}_p = \int^{\oplus} \mathcal{S}_p(t) \frac{\mathrm{d}t}{d\pi t}. \tag{II.5}$$

Proof. It is not difficult to verify that $q_\alpha(t) \in \mathcal{S}_p(t)$ for all $t > 0$, because the weight $w \mapsto e^{-4\pi p(w)t}$ decays faster than any monomial $q_\alpha(t)$ grows. We now show that the collection of sections $\{q_\alpha\}_{\alpha \in \mathbb{N}^n}$ is a measurability structure on $\mathcal{S}_p(t)$.

For multi-indices $\alpha, \beta \in \mathbb{N}^n$, we have

$$\left\langle q_\alpha(t), q_\beta(t) \right\rangle_{\mathcal{S}_p(t)} = \int_{\mathbb{C}^n} w^\alpha \overline{w}^\beta e^{-4\pi p(w)t} \, \mathrm{d}V(w).$$

By the dominated convergence theorem it follows that the map $t \mapsto \langle q_\alpha(t), q_\beta(t) \rangle_{\mathcal{S}_p(t)}$ is continuous and hence measurable on $(0,\infty)$. It remains to be seen that for all $t \in (0,\infty)$ the linear span of $\{q_\alpha(t)\}_{\alpha \in \mathbb{N}^n}$, i.e., the polynomials are dense in $\mathcal{S}_p(t)$.

First, we show that if $\alpha, \beta \in \mathbb{N}^n$ are two multi-indices such that $\mathrm{wt}_m(\alpha) \neq \mathrm{wt}_m(\beta)$, then for all $t \in (0,\infty)$, the monomial $q_\alpha(t)$ is orthogonal to the monomial $q_\beta(t)$ in $\mathcal{S}_p(t)$. Fix a $\theta \in (-\pi, \pi)$.

Then by Lemma II.4 we have

$$p\left(\widehat{\rho}_{e^{i\theta}}(\zeta)\right) = \left|e^{i\theta}\right| p(\zeta) = p(\zeta),$$

where $\widehat{\rho}_{e^{i\theta}}$ is as in Lemma II.4. Making a change of variables $\zeta = \widehat{\rho}_{e^{i\theta}}(w)$ below,we obtain

$$\begin{aligned}\left\langle q_\alpha(t), q_\beta(t)\right\rangle_{\mathcal{S}_p(t)} &= \int_{\mathbb{C}^n} \zeta^\alpha \overline{\zeta}^\beta e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta)\\ &= \int_{\mathbb{C}^n} \left(\widehat{\rho}_{e^{i\theta}}(w)\right)^\alpha \overline{\left(\widehat{\rho}_{e^{i\theta}}(w)\right)}^\beta e^{-4\pi p(\widehat{\rho}_{e^{i\theta}}(w))t}\,\mathrm{d}V(w)\\ &= \int_{\mathbb{C}^n} e^{i\theta(\mathrm{wt}_m(\alpha)-\mathrm{wt}_m(\beta))} w^\alpha \overline{w}^\beta e^{-4\pi p(w)t}\,\mathrm{d}V(w).\end{aligned}$$

Thus we have

$$\left(1 - e^{i\theta(\mathrm{wt}_m(\alpha)-\mathrm{wt}_m(\beta))}\right)\left\langle q_\alpha(t), q_\beta(t)\right\rangle_{\mathcal{S}_p(t)} = 0,$$

from which it follows that $q_\alpha(t)$ is orthogonal to $q_\beta(t)$ in $\mathcal{S}_p(t)$ since $\mathrm{wt}_m(\alpha) \neq \mathrm{wt}_m(\beta)$. It follows from this that a weighted homogeneous polynomial of weighted degree k/M is orthogonal to a weighted homogeneous polynomial of weighted degree j/M in $\mathcal{S}_p(t)$ for all $t > 0$.

Now, we are ready to show that polynomials are dense in $\mathcal{S}_p(t)$. Suppose $f \in \mathcal{S}_p(t)$ is a function which is orthogonal to every polynomial. Then, we wish to show that $f = 0$, almost everywhere in $\mathbb{C}^n$. Since f is entire, it has a power series expansion that converges normally on compact subsets of $\mathbb{C}^n$. We may rearrange such a series to obtain a representation of the form

$$f(\zeta) = \sum_{k=0}^{\infty} f_k(\zeta),$$

that converges to f uniformly on compact subsets of $\mathbb{C}^n$, and where f_k is a weighted homogeneous polynomial of weighted degree k/M of the form (II.4). Since f is orthogonal to all weighted homogeneous polynomials f_j of weighted degree j/M, for all $j \in \mathbb{N}$, we have

$$\begin{aligned}0 &= \left\langle f, f_j\right\rangle_{\mathcal{S}_p(t)} \qquad \text{(II.6)}\\ &= \int_{\mathbb{C}^n} f(\zeta)\overline{f_j(\zeta)} e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{r\to\infty} \int_{\{|\zeta|<r\}} f(\zeta)\overline{f_j(\zeta)} e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta) \\
&= \lim_{r\to\infty} \int_{\{|\zeta|<r\}} \sum_{k=0}^{\infty} f_k(\zeta)\overline{f_j(\zeta)} e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta) \\
&= \lim_{r\to\infty} \sum_{k=0}^{\infty} \int_{\{|\zeta|<r\}} f_k(\zeta)\overline{f_j(\zeta)} e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta) \\
&= \lim_{r\to\infty} \int_{\{|\zeta|<r\}} \left|f_j(\zeta)\right|^2 e^{-4\pi p(\zeta)t}\,\mathrm{d}V(\zeta)
\end{aligned}
$$

which shows that $f_j = 0$ almost everywhere in $\mathbb{C}^n$. Since $j \in \mathbb{N}$ was arbitrary, it follows that $f = 0$ almost everywhere in $\mathbb{C}^n$. This shows that the polynomials are dense in $\mathcal{S}_p(t)$ for all $t \in (0,\infty)$, which shows that the collection of sections $\{q_\alpha\}_{t>0}$ is a measurability structure on $\{\mathcal{S}_p(t)\}_{t>0}$.

Suppose $f \in \mathcal{H}_p$. We then show that $f \in \int^{\oplus} \mathcal{S}_p(t)\frac{\mathrm{d}t}{4\pi t}$. Since $f \in \mathcal{H}_p$, we have

$$
\|f\|^2_{\mathcal{H}_p} = \int_0^\infty \int_{\mathbb{C}^n} |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t}\,\mathrm{d}V(w)\,\mathrm{d}t < \infty.
$$

Thus, by Fubini's theorem, for almost all $t > 0$ we have

$$
\int_{\mathbb{C}^n} |f(t,w)|^2 e^{-4\pi p(w)t}\,\mathrm{d}V(w) < \infty,
$$

i.e., $f(t,\cdot) \in L^2(\mathbb{C}^n, e^{-4\pi pt})$. We now show that $f(t,\cdot)$ is holomorphic in $\mathbb{C}^n$ for almost all $t > 0$. To show this, it suffices to show that $f(t,\cdot)$ is holomorphic in each of the variables $w_1,\dots,w_n$, by Hartog's theorem on separate analyticity. By Weyl's lemma, it suffices to show that for all $1 \le j \le n$,

$$
\frac{\partial f}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions.}
$$

Let $\alpha \in \mathscr{C}^\infty_c((0,\infty)\times\mathbb{C}^n)$ be of the form $\alpha(t,w) = \alpha_1(t)\alpha_2(w)$, where $\alpha_1 \in \mathscr{C}^\infty_c((0,\infty))$, and $\alpha_2 \in \mathscr{C}^\infty_c(\mathbb{C}^n)$. Since $f \in \mathcal{H}_p$, for $1 \le j \le n$, $\frac{\partial f}{\partial \overline{w}_j} = 0$ in the sense of distributions the action of the distributional derivative of f with respect to $\overline{w}_j$ on the test function α above gives

$$
\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{C}^n} -f(t,w)\frac{\partial \alpha}{\partial \overline{w}_j}\,\mathrm{d}V(w)\,\mathrm{d}t \\
&= \int_0^\infty \alpha_1(t)\left(\int_{\mathbb{C}^n} -f(t,w)\frac{\partial \alpha_2}{\partial \overline{w}_j}\,\mathrm{d}V(w)\right)\mathrm{d}t.
\end{aligned}
$$

Since this holds for all $\alpha_1 \in \mathscr{C}_c^\infty((0,\infty))$, we have

$$\int_{\mathbb{C}^n} -f(t,w)\frac{\partial \alpha_2}{\partial \overline{w}_j}\,\mathrm{d}V(w) = 0,$$

for almost all $t \in (0,\infty)$ and all $\alpha_2 \in \mathscr{C}_c^\infty(\mathbb{C}^n)$. This shows that for almost all $t \in (0,\infty)$, we have $\frac{\partial f(t,\cdot)}{\partial \overline{w}_j} = 0$ in the sense of distributions and thus, $f(t,\cdot) \in \mathcal{S}_p(t)$. Moreover,

$$\begin{aligned}\int_0^\infty \|f(t,\cdot)\|^2_{\mathcal{S}_p(t)} \frac{\mathrm{d}t}{4\pi t} &= \int_0^\infty \int_{\mathbb{C}^n} |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t}\,\mathrm{d}V(w)\,\mathrm{d}t \\ &= \|f\|^2_{\mathcal{H}_p} < \infty. \end{aligned} \tag{II.7}$$

This shows that $f \in \int^\oplus \mathcal{S}_p(t)\frac{\mathrm{d}t}{4\pi t}$.

Suppose $f \in \int^\oplus \mathcal{S}_p(t)\frac{\mathrm{d}t}{4\pi t}$. In view of (II.7) above, it follows that $\|f\|_{\mathcal{H}_p} < \infty$. Moreover, it follows from (II.7) that $f \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{C}^n)$, and since $f(t,\cdot) \in \mathcal{O}(\mathbb{C}^n)$ for almost every $t \in (0,\infty)$ it follows that

$$\frac{\partial f}{\partial \overline{w}_j} = 0, \text{ in the sense of distributions for } j = 1,\ldots,n,$$

and consequently, $f \in \mathcal{H}_p$. □

II.3.2. Direct Integral Representation of $\mathcal{X}_p$

Recall that p is a nonnegative weighted homogeneous balanced polynomial with respect to the tuple $m = (m_1,\ldots,m_n)$ and $M = \mathrm{l.c.m}(2,m_1,\ldots,m_n)$. Also recall that $\mathbb{B}_p \subset \mathbb{C}^n$ is the domain

$$\mathbb{B}_p = \{w \in \mathbb{C}^n \mid p(w) < 1\}.$$

We now state and prove a result that we need to obtain the direct integral representation of $\mathcal{X}_p$.

Lemma II.8. *A function $f \in \mathcal{O}(\mathbb{B}_p)$ admits a series expansion in weighted homogeneous polynomials, that is we may write*

$$f(z) = \sum_{k=0}^\infty f_k(z), \quad \textit{for all } z \in \mathbb{B}_p \tag{II.8}$$

where for each $k \geq 0$, f_k is a weighted homogeneous polynomial of weighted degree k/M as in (II.4). *The series* (II.8) *converges uniformly on compact subsets of* $\mathbb{B}_p$.

Proof. For $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, let $\delta_\zeta : \mathbb{C} \to \mathbb{C}^n$ be the map

$$\delta_\zeta(\mu) = \left(\mu^{M/2m_1}\zeta_1, \cdots, \mu^{M/2m_n}\zeta_n\right), \text{ for } \mu \in \mathbb{C}. \tag{II.9}$$

Note that since $M = \text{l.c.m}(2, m_1, \ldots, m_n)$, the powers of μ above are all positive integers, and a calculation shows that

$$p(\delta_\zeta(\mu)) = |\mu|^M p(\zeta). \tag{II.10}$$

Fix a $z \in \mathbb{B}_p \setminus \{0\}$, and let

$$\omega = \delta_z(1/|z|) = \left(\frac{z_1}{|z|^{M/2m_1}}, \cdots, \frac{z_n}{|z|^{M/2m_n}}\right).$$

We now show that the set of all $\mu \in \mathbb{C}$ such that $\delta_\omega(\mu) \in \mathbb{B}_p$ is given by

$$D(0, R(\omega)) = \{\mu \in \mathbb{C} \mid |\mu| < R(\omega)\}, \tag{II.11}$$

where $R(\omega)$ is a positive number depending on ω. Indeed, if $\mu \in \mathbb{C}$ is such that $\zeta = \delta_\omega(\mu)$ lies in $\mathbb{B}_p$, we must have

$$p(\delta_\omega(\mu)) < 1. \tag{II.12}$$

On the other hand, by using (II.10) we get

$$p(\delta_\omega(\mu)) = |\mu|^M p(\omega) = |\mu|^M p(\delta_z(1/|z|)) = \left(\frac{\mu}{|z|}\right)^M p(z),$$

so equation (II.12) becomes

$$|\mu| < \frac{|z|}{(p(z))^{1/M}} = \frac{1}{(p(\omega))^{1/M}} := R(\omega). \tag{II.13}$$

Since f is in $\mathcal{O}(\mathbb{B}_p)$, in a neighborhood of 0, it admits a power series expansion

$$f(\zeta) = \sum_{\alpha \in \mathbb{N}^n} A_\alpha \zeta^\alpha.$$

This series converges normally in an open set containing 0, so we rearrange the terms to get

$$f(\zeta) = \sum_{k=0}^{\infty} f_k(\zeta), \tag{II.14}$$

where f_k is a weighted homogeneous polynomial of weighted degree k/M. To show that the series in (II.14) converges at the point $z \in \mathbb{B}_p$ we restrict f to the complex analytic disc $\{\zeta \in \mathbb{C}^n \mid \zeta = \delta_\omega(\mu), \mu \in D(0,R(\omega))\}$ to obtain

$$\varphi(\mu) := f(\delta_\omega(\mu)) = \sum_{k=0}^{\infty} f_k(\delta_\omega(\mu)) = \sum_{k=0}^{\infty} f_k(\omega)(|\mu|^M)^{k/M} = \sum_{k=0}^{\infty} f_k(\omega)\,|\mu|^k, \tag{II.15}$$

where we used the fact that f_k is a weighted homogeneous polynomial of weighted degree k/M to obtain the penultimate equality above. Since $f \in \mathcal{O}(\mathbb{B}_p)$, φ is holomorphic in the disc $D(0,R(\omega))$ given by (II.11). Now, it follows from equation (II.13) that $|z| < R(\omega)$, and consequently the series (II.15) converges for $\mu = |z|$. Since $\varphi(|z|) = f(\delta_\omega(|z|)) = f(z)$, it follows that the series (II.14) converges at the point z.

Now we wish to show that the series (II.14) converges uniformly on compact subsets of $\mathbb{B}_p$. Suppose a compact subset K of $\mathbb{B}_p$ is given. Then, there are numbers $0 < s,q < 1$ such that $K \subset \{z \in \mathbb{C}^n \mid p(z) < q^M s\}$. Letting $\omega = \delta_z(1/|z|)$ as before, we use (II.10) to get

$$p(z) = p(\delta_\omega(|z|)) = |z|^M\, p(\omega).$$

Then for every $z \in K$, we have $p(z) = |z|^M\, p(\omega) < q^M r$ from which it follows that

$$|z| < q\left(\frac{s}{p(\omega)}\right)^{1/M} := qr(\omega). \tag{II.16}$$

Then we have

$$|f_k(z)| = |f_k(\delta_\omega(|z|))| = |f_k(\omega)|\left(|z|^M\right)^{k/M} \le |f_k(\omega)|\, r^k(\omega) q^k. \tag{II.17}$$

It follows from equations (II.13) and (II.16) that $r(\omega) < R(\omega)$. Thus, to estimate $|f_k(\omega)|$, we note that it is the coefficient of μ^k in series (II.15) and apply Cauchy integral formula to get

$$f_k(\omega) = \frac{1}{2\pi i} \int_{|\mu|=r(\omega)} \frac{f_k(\delta_\omega(\mu))}{\mu^{k+1}} \, \mathrm{d}\mu. \tag{II.18}$$

The set $\{\zeta \in \mathbb{B}_p \mid \zeta = \delta_\omega(\mu), \mu \in \mathbb{C} \text{ and } |\mu| = r(\omega)\}$ is a compact set in $\mathbb{B}_p$, so there exists an $N > 0$ such that $\left|f_k(\delta_\mu(\omega))\right| \leq N$ for all $\mu \in \mathbb{C}$ such that $|\mu| = r(\omega)$. Using this, we obtain Cauchy estimates in (II.18) as

$$|f_k(\delta_\omega(\mu))| \leq \frac{N}{r^k(\omega)},$$

and thus, equation (II.17) reduces to

$$|f_k(z)| \leq Nq^k.$$

This shows that the series (II.14) converges uniformly on K, since $0 < q < 1$ is independent of the choice of z. □

Recall from equation (I.18) that for $\zeta \in \mathbb{B}_p$, $\lambda(p(\zeta),t)$ is given by

$$\lambda(p(\zeta),t) = \begin{cases} \dfrac{1}{4\pi t}\left(e^{-4\pi t \sin^{-1} p(\zeta)} - e^{-4\pi t(\pi - \sin^{-1} p(\zeta))}\right) & \text{if } t \neq 0 \\ \pi - 2\sin^{-1}(p(\zeta)) & \text{if } t = 0. \end{cases}$$

Also recall that the Hilbert space $\mathcal{X}_p$ consists of all measurable functions f such that

$$\|f\|^2_{\mathcal{X}_p} := \int_{\mathbb{R}} \int_{\mathbb{B}_p} |f(t,\zeta)|^2 \lambda(p(\zeta),t) \, \mathrm{d}V(\zeta) \, \mathrm{d}t < \infty$$

and

$$\frac{\partial f}{\partial \overline{\zeta}_j} = 0, \quad \text{in the sense of distributions for } 1 \leq j \leq n,$$

where $\zeta_1, \cdots, \zeta_n$ are co-ordinates of $\mathbb{B}_p$.

Proposition II.9. *For $t \in \mathbb{R}$, let $\mathcal{Q}_p(t)$ be the weighted Bergman space on $\mathbb{B}_p$ with the weight $\zeta \mapsto \lambda(p(\zeta),t)$, i.e.,*

$$\mathcal{Q}_p(t) = A^2(\mathbb{B}_p, \lambda(p,t)).$$

Then $\{\mathcal{Q}_p(t)\}_{t\in\mathbb{R}}$ is a collection of Hilbert spaces indexed by $\mathbb{R}$. For a tuple $\alpha \in \mathbb{N}^n$, let q_α be given by

$$q_\alpha(t)(w) = w^\alpha, \textit{for all } t \in \mathbb{R}, w \in \mathbb{B}_p.$$

Then q_α is a section of the collection $\{\mathcal{Q}_p(t)\}_{t\in\mathbb{R}}$ of Hilbert spaces, i.e., $q_\alpha(t) \in \mathcal{Q}_p(t)$ for each $t \in \mathbb{R}$, which is a measurability structure on $\{\mathcal{Q}_p(t)\}_{t\in\mathbb{R}}$. Then the Hilbert space $\mathcal{X}_p$ (as above) has the direct integral representation

$$\mathcal{X}_p = \int^{\oplus} \mathcal{Q}_p(t)\,\mathrm{d}t. \tag{II.19}$$

Proof. It is not difficult to verify that $q_\alpha(t) \in \mathcal{Q}_p(t)$ for all $t \in \mathbb{R}$, because the weight λ and the function $q_\alpha(t)$ are both continuous upto the boundary of $\mathbb{B}_p$. We now show that the collection of sections $\{q_\alpha\}_{\alpha\in\mathbb{N}^n}$ is a measurability structure on $\{\mathcal{Q}_p(t)\}_{t\in\mathbb{R}}$.

For multi-indices $\alpha, \beta \in \mathbb{N}^n$, we have

$$\langle q_\alpha(t), q_\beta(t)\rangle_{\mathcal{Q}_p(t)} = \int_{\mathbb{B}_p} w^\alpha \overline{w}^\beta \lambda(p(w), t)\,\mathrm{d}V(w).$$

By the dominated convergence theorem it follows that the map $t \mapsto \langle q_\alpha(t), q_\beta(t)\rangle_{\mathcal{Q}_p(t)}$ is continuous and hence measurable on $\mathbb{R}$. It remains to be seen that for all $t \in \mathbb{R}$, the linear span of $\{q_\alpha(t)\}_{\alpha\in\mathbb{N}^n}$, i.e., the polynomials are dense in $\mathcal{Q}_p(t)$.

First we show that if $\alpha, \beta \in \mathbb{N}^n$ are two multi-indices such that $\mathrm{wt}_m(\alpha) \neq \mathrm{wt}_m(\beta)$, then $q_\alpha(t)$ is orthogonal to $q_\beta(t)$ in $\mathcal{Q}_p(t)$ for all $t \in \mathbb{R}$. Fix a $\theta \in (-\pi, \pi)$. Then by Lemma II.4 we have

$$p\left(\widehat{\rho}_{e^{i\theta}}(\zeta)\right) = \left|e^{i\theta}\right| p(\zeta) = p(\zeta),$$

where $\widehat{\rho}_{e^{i\theta}}$ is as in Lemma II.4. Making a change of variables $\zeta = \widehat{\rho}_{e^{i\theta}}(w)$, we obtain

$$\begin{aligned}\langle q_\alpha(t), q_\beta(t)\rangle_{\mathcal{Q}_p(t)} &= \int_{\mathbb{B}_p} \zeta^\alpha \overline{\zeta}^\beta \lambda(p(\zeta), t)\,\mathrm{d}V(\zeta)\\ &= \int_{\mathbb{B}_p} \left(\widehat{\rho}_{e^{i\theta}}(w)\right)^\alpha \overline{\left(\widehat{\rho}_{e^{i\theta}}(w)\right)}^\beta \lambda(p(\widehat{\rho}_{e^{i\theta}}(w)), t)\,\mathrm{d}V(w)\end{aligned}$$

$$= \int_{\mathbb{B}_p} e^{i\theta(\mathrm{wt}_m(\alpha)-\mathrm{wt}_m(\beta))} w^\alpha \overline{w}^\beta \lambda(p(w),t)\, \mathrm{d}V(w).$$

Thus we have

$$\left(1 - e^{i\theta(\mathrm{wt}_m(\alpha)-\mathrm{wt}_m(\beta))}\right) \langle q_\alpha(t), q_\beta(t)\rangle_{\mathcal{Q}_p(t)} = 0,$$

from which it follows that $q_\alpha(t)$ is orthogonal to $q_\beta(t)$ in $\mathcal{Q}_p(t)$, since $\mathrm{wt}_m(\alpha) \neq \mathrm{wt}_m(\beta)$. It follows from this that a weighted homogeneous polynomial of weighted degree k/M is orthogonal to a weighted homogeneous polynomial of weighted degree j/M in $\mathcal{Q}_p(t)$ for all $t \in \mathbb{R}$.

Now, we show that polynomials are dense in $\mathcal{Q}_p(t)$. Suppose $f \in \mathcal{Q}_p(t)$ is a function which is orthogonal to every polynomial. Then, we wish to show that $f = 0$, almost everywhere in $\mathbb{B}_p$. By Lemma II.8 we may expand f in a series of weighted homogeneous polynomials, i.e., we may write

$$f(\zeta) = \sum_{k=0}^{\infty} f_k(\zeta),$$

where f_k is a weighted homogeneous polynomial of weighted degreee k/M as in (II.4). The series above converges to f uniformly on compact subsets of $\mathbb{B}_p$. Since f is orthogonal to all weighted homogeneous polynomials f_j of weighted degree j/M, for all $j \in \mathbb{N}$, we have

$$\begin{aligned} 0 &= \langle f, f_j \rangle_{\mathcal{Q}_p(t)} \qquad \text{(II.20)} \\ &= \int_{\mathbb{B}_p} f(\zeta)\overline{f_j(\zeta)}\lambda(p(\zeta),t)\, \mathrm{d}V(\zeta) \\ &= \lim_{r\to 1^-} \int_{\{p(\zeta)<r\}} f(\zeta)\overline{f_j(\zeta)}\lambda(p(\zeta),t)\, \mathrm{d}V(\zeta) \\ &= \lim_{r\to 1^-} \int_{\{p(\zeta)<r\}} \sum_{k=0}^{\infty} f_k(\zeta)\overline{f_j(\zeta)}\lambda(p(\zeta),t)\, \mathrm{d}V(\zeta) \\ &= \lim_{r\to 1^-} \sum_{k=0}^{\infty} \int_{\{p(\zeta)<r\}} f_k(\zeta)\overline{f_j(\zeta)}\lambda(p(\zeta),t)\, \mathrm{d}V(\zeta) \\ &= \lim_{r\to 1^-} \int_{p(\zeta)<r} \left|f_j(\zeta)\right|^2 \lambda(p(\zeta),t)\, \mathrm{d}V(\zeta) \end{aligned}$$

which shows that $f_j = 0$ almost everywhere in $\mathbb{B}_p$. Since $j \in \mathbb{N}$ was arbitrary, it follows that $f = 0$ almost everywhere in $\mathbb{B}_p$. This shows that polynomials are dense in $\mathcal{Q}_p(t)$ for all $t \in \mathbb{R}$.

Suppose $f \in \mathcal{X}_p$. We then show that $f \in \int^{\oplus} \mathcal{Q}_p(t)\,\mathrm{d}t$. Since $f \in \mathcal{X}_p$, we have

$$\|f\|_{\mathcal{X}_p}^2 = \int_{\mathbb{R}} \int_{\mathbb{B}_p} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t < \infty.$$

Thus, by Fubini's theorem we see that for almost all $t \in \mathbb{R}$,

$$\int_{\mathbb{B}_p} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}V(\zeta) < \infty$$

i.e., $f(t,\cdot) \in L^2(\mathbb{B}_p, \lambda(p,t))$. We now show that $f(t,\cdot)$ is holomorphic in $\mathbb{B}_p$ for almost all $t \in \mathbb{R}$. To show this it suffices to show that $f(t,\cdot)$ is holomorphic in each variable $\zeta_1,\dots,\zeta_n$ by Hartogs's theorem on separate analyticity. To show that $f(t,\cdot)$ is holomorphic in the variable ζ_j, by Weyl's lemma it suffices to show that

$$\partial f(t,\cdot)/\partial \overline{\zeta}_j = 0 \text{ in the sense of distributions.}$$

Let $\varphi \in \mathscr{C}_c^\infty(\mathbb{R} \times \mathbb{B}_p)$ be of the form $\varphi(t,\zeta) = \varphi_1(t)\varphi_2(\zeta)$, where $\varphi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$, and $\varphi_2 \in \mathscr{C}_c^\infty(\mathbb{B}_p)$. Since $f \in \mathcal{X}_p$, $\frac{\partial f}{\partial \overline{\zeta}_j} = 0$ in sense of distributions, the pairing of the distributional derivative of f with respect to $\overline{\zeta}_j$ with the test function φ above gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_{\mathbb{B}_p} -f(t,\zeta) \frac{\partial \varphi}{\partial \overline{\zeta}_j}\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \varphi_1(t) \left(\int_{\mathbb{B}_p} -f(t,\zeta) \frac{\partial \varphi_2}{\partial \overline{\zeta}_j}\,\mathrm{d}V(\zeta) \right) \mathrm{d}t. \end{aligned}$$

Since this holds for all $\varphi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{B}_p} -f(t,\zeta) \frac{\partial \varphi_2}{\partial \overline{\zeta}_j}\,\mathrm{d}V(\zeta) = 0,$$

for almost all $t \in \mathbb{R}$ and all $\varphi_2 \in \mathscr{C}_c^\infty(\mathbb{B}_p)$.This shows that for almost all $t \in \mathbb{R}$, we have $\frac{\partial f(t,\cdot)}{\partial \overline{\zeta}_j} = 0$ in sense of distributions and hence $f(t,\cdot) \in \mathcal{Q}_p(t)$.

Moreover we have ,

$$\begin{aligned}\int_{\mathbb{R}} |f(t,\cdot)|^2_{\mathcal{Q}_p(t)}\,\mathrm{d}t &= \int_{\mathbb{R}}\int_{\mathbb{B}_p} |f(t,\zeta)|^2\,\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\ &= \|f\|^2_{\mathcal{X}_p} < \infty. \end{aligned} \tag{II.21}$$

This shows that $f \in \int^{\oplus} \mathcal{Q}_p(t)\,\mathrm{d}t$.

Now suppose that $f \in \int^{\oplus} \mathcal{Q}_p(t)\,\mathrm{d}t$. In view of (II.21) above, it follows that $\|f\|_{\mathcal{X}_p} < \infty$. Moreover, it follows from equation (II.21) that $f \in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{C}^n)$, and since $f(t,\cdot) \in \mathcal{O}(\mathbb{B}_p)$ for almost every $t \in \mathbb{R}$, it follows that

$$\frac{\partial f}{\partial \overline{\zeta}_j} = 0, \text{ in the sense of distributions for } j = 1,\ldots,n,$$

and consequently $f \in \mathcal{X}_p$. □

CHAPTER III

FOURIER REPRESENTATIONS

In this chapter, we provide proofs of Theorems 2, 3 and 4. In order to obtain Fourier representations of $A^2(\mathcal{E}_p)$ with respect to the rotation group, and those of $A^2(\mathcal{U}_p)$ with respect to the translation and scaling groups we obtain analogous results for domains in $\mathbb{C}$ having rotation, translation and scaling symmetries, namely discs, strips and sectors respectively.

III.1. Maximal Compact Subgroup

Let $\mathbb{D}(r) = \{z \in \mathbb{C} \mid |z| < r\}$ be the disk of radius r having center at the origin in $\mathbb{C}$. Let ℓ_r^2 be the Hilbert space of complex sequences $a = (a_k)_{k=0}^{\infty}$ such that

$$\|a\|_{\ell_r^2}^2 := \pi \sum_{k=0}^{\infty} \frac{r^{2k+2}}{k+1} |a_k|^2 < \infty.$$

The following proposition included for completeness is a well known result.

Proposition III.1. *The map $T : \ell_r^2 \to A^2(\mathbb{D}(r))$ given by*

$$Ta(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \textit{for all } z \in \mathbb{D}(r) \tag{III.1}$$

is an isometric isomorphism of Hilbert spaces.

Proof. First we show that T is an isometry of ℓ_r^2 with $A^2(\mathbb{D}(r))$. Note that the monomial z^j is orthogonal to the monomial z^k in $A^2(\mathbb{D}(r))$, whenever $j \neq k$. For $a \in \ell_r^2$, we have

$$\begin{aligned} \|Ta\|_{A^2(\mathbb{D}(r))}^2 &= \sum_{k=0}^{\infty} \left\| a_k z^k \right\|_{A^2(\mathbb{D}(r))}^2 \quad \text{(By Parseval's formula)} \\ &= \pi \sum_{k=0}^{\infty} \frac{r^{2+2}}{k+1} |a_k|^2 \\ &= \|a\|_{\ell_r^2}^2 . \end{aligned}$$

Thus, T is an isometry into $L^2(\mathbb{D}(r))$ and consequently injective. Moreover, this shows that the partial sums $z \mapsto \sum_{k=0}^{N} a_k z^k$ converge to Ta in $L^2(\mathbb{D}(r))$. Since $A^2(\mathbb{D}(r))$ is closed in $L^2(\mathbb{D}(r))$ we see that $Ta \in A^2(\mathbb{D}(r))$.

To show that T is surjective, we show that the set $\{z^k\}_{k=0}^{\infty}$ in $A^2(\mathbb{D}(r))$ is a complete orthogonal system. Since we already know that z^j is orthogonal to z^k when $j \neq k$, it suffices to show that $\{z^k\}_{k=0}^{\infty}$ is complete. To see this, let $f \in A^2(\mathbb{D}(r))$ be a function that is orthogonal to z^k for all $k \in \mathbb{N}$. Since f is holomorphic in the disk $\mathbb{D}(r)$, it has a series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{III.2}$$

that converges uniformly on compact subsets of $\mathbb{D}(r)$. Then for all $k \in \mathbb{N}$, we have

$$\begin{aligned}
0 &= \langle f, a_k z^k \rangle_{A^2(\mathbb{D}(r))} \\
&= \int_{\mathbb{D}(r)} f(z)\overline{a}_k \overline{z}^k \, \mathrm{d}V(z) \\
&= \lim_{s \to r^-} \int_{\mathbb{D}(s)} f(z)\overline{a}_k \overline{z}^k \, \mathrm{d}V(z) \quad \text{(By Dominated Convergence Theorem)} \\
&= \lim_{s \to r^-} \sum_{j=0}^{\infty} \int_{\mathbb{D}(s)} a_j z^j \overline{a}_k \overline{z}^k \, \mathrm{d}V(z) \quad \text{(Uniform Convergence)} \\
&= \pi \frac{r^{2k+2}}{k+1} |a_k|^2 .
\end{aligned}$$

This shows that $a_k = 0$ for all $k \in \mathbb{N}$. Thus, $f \equiv 0$ on $\mathbb{D}(r)$. This shows that $\{z^k\}_{k=0}^{\infty}$ is a complete orthogonal system in $A^2(\mathbb{D}(r))$, which completes the proof. □

III.1.1. A Fourier Representation of $A^2(\mathcal{E}_p)$

We recall the notation necessary for obtaining the Fourier representation of $A^2(\mathcal{E}_p)$ with respect to the rotation group, where $\mathcal{E}_p$ is the domain in $\mathbb{C}^{n+1}$ given by

$$\mathcal{E}_p = \{(z, w) \in \mathbb{C} \times \mathbb{C}^n \mid |z|^2 + p(w) < 1\}.$$

For a weighted homogeneous balanced polynomial p, the domain $\mathbb{B}_p \subset \mathbb{C}^n$ is given by

$$\mathbb{B}_p = \{w \in \mathbb{C}^n \mid p(w) < 1\}.$$

For $k \in \mathbb{N}$, $\mathcal{W}_p(k)$ is the Bergman space on $\mathbb{B}_p$ with respect to the weight $w \mapsto (1-p(w))^{k+1}$, i.e.,

$$\mathcal{W}_p(k) = A^2\left(\mathbb{B}_p, (1-p)^{k+1}\right)$$

The Hilbert space $\mathcal{Y}_p$ is the of sequences $a = (a_k)_{k=0}^{\infty}$ such that $a_k \in \mathcal{W}_p(k)$ for each $k \in \mathbb{N}$, and

$$\|a\|_{\mathcal{Y}_p}^2 = \pi \sum_{k=0}^{\infty} \frac{1}{k+1} \|a_k\|_{\mathcal{W}_p(k)}^2 < \infty.$$

We make use of the following fact when integrating over $\mathcal{E}_p$: for each $w \in \mathbb{B}_p$ such that $(z,w) \in \mathcal{E}_p$, we have $z \in \mathbb{D}(\sqrt{1-p(w)})$.

Proof of Theorem 2. We first show that T is an isometry into $L^2(\mathcal{E}_p)$, and hence injective. Since $a \in \mathcal{Y}_p$ we have

$$\|a\|_{\mathcal{Y}_p}^2 = \pi \sum_{k=0}^{\infty} \frac{1}{k+1} \|a_k\|_{\mathcal{W}_p(k)}^2 = \int_{\mathbb{B}_p} \pi \sum_{k=0}^{\infty} \frac{1}{k+1} |a_k(w)|^2 (1-p(w))^{k+1} \, \mathrm{d}V(w) < \infty,$$

where we were allowed to interchange summation and integration by Tonelli's theorem. Thus, for almost all $w \in \mathbb{B}_p$ we have

$$\pi \sum_{k=0}^{\infty} \frac{(1-p(w))^{k+1}}{k+1} |a_k(w)|^2 < \infty. \tag{III.3}$$

Recall from (I.13) that for $(z,w) \in \mathcal{E}_p$, the function Ta is given by

$$Ta(z,w) = \sum_{k=0}^{\infty} a_k(w) z^k.$$

If $w \in \mathbb{B}_p$ is such that (III.3) holds, by Proposition III.1 the function $Ta(\cdot, w)$ is in $A^2(\mathbb{D}(\sqrt{1-p(w)}))$ and

$$\pi \sum_{k=0}^{\infty} \frac{(1-p(w))^{k+1}}{k+1} |a_k(w)|^2 = \int_{\mathbb{D}(\sqrt{1-p(w)})} |Ta(z,w)|^2 \, \mathrm{d}V(z). \tag{III.4}$$

Integrating (III.4) over $\mathbb{B}_p$ we get

$$\|Ta\|_{L^2(\mathcal{E}_p)}^2 = \int_{\mathbb{B}_p} \int_{\mathbb{D}(\sqrt{1-p(w)})} |Ta(z,w)|^2 \, \mathrm{d}V(z) \, \mathrm{d}V(w)$$

$$= \pi \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{\mathbb{B}_p} |a_k(w)|^2 (1-p(w))^{k+1} \, \mathrm{d}V(w)$$

$$= \|a\|_{\mathcal{Y}_p}^2 . \tag{III.5}$$

We now show that the image of T is contained in $A^2(\mathcal{E}_p)$. Let

$$T_N a(z,w) = \sum_{k=0}^{N} a_k(w) z^k. \tag{III.6}$$

For each $k \in \mathbb{N}$, the function $(z,w) \mapsto a_k(w)z^k$ is in $A^2(\mathcal{E}_p)$. Consequently $T_N a$ is in $A^2(\mathcal{E}_p)$ for all $N > 0$. Since $a = (a_0, a_1, \dots) \in \mathcal{Y}_p$, we know that the partial sums

$$\pi \sum_{k=0}^{N} \frac{1}{k+1} \|a_k\|_{\mathcal{W}_{p(k)}}^2$$

converge to $\|a\|_{\mathcal{Y}_p}^2$. Since T is an isometry, it follows that $\|T_N a - Ta\|_{L^2(\mathcal{E}_p)} \to 0$. Since $A^2(\mathcal{E}_p)$ is closed in $L^2(\mathcal{E}_p)$, it follows that $Ta \in A^2(\mathcal{E}_p)$. Thus, T is an isometry of $\mathcal{Y}_p$ into $A^2(\mathcal{E}_p)$.

Now we show that the map T is in fact surjective. Suppose $F \in A^2(\mathcal{E}_p)$. Then we have

$$\|F\|_{A^2(\mathcal{E}_p)}^2 = \int_{\mathbb{B}_p} \int_{\mathbb{D}(\sqrt{1-p(w)})} |F(z,w)|^2 \, \mathrm{d}V(z) \, \mathrm{d}V(w) < \infty.$$

Thus, for almost all $w \in \mathbb{B}_p$, we have

$$\int_{\mathbb{D}(\sqrt{1-p(w)})} |F(z,w)|^2 \, \mathrm{d}V(z) < \infty.$$

For such a $w \in \mathbb{B}_p$, by Proposition III.1, there is an $a(w) \in \ell^2_{\sqrt{1-p(w)}}$ such that

$$F(z,w) = \sum_{k=0}^{\infty} a_k(w) z^k, \quad \text{for all } z \in \mathbb{D}(\sqrt{1-p(w)})$$

and

$$\int_{\mathbb{D}(\sqrt{1-p(w)})} |F(z,w)|^2 \, \mathrm{d}V(z) = \pi \sum_{k=0}^{\infty} \frac{(1-p(w))^{k+1}}{k+1} |a_k(w)|^2 . \tag{III.7}$$

By Cauchy's integral formula applied to $F(\cdot,w)$ in the disk $\mathbb{D}(\sqrt{1-p(w)})$ we obtain a formula for the coefficients $a_k(w)$, for each $k\in\mathbb{N}$ as

$$a_k(w) = \frac{k!}{2\pi i}\int_{|\zeta|=r}\frac{F(\zeta,w)}{\zeta^{k+1}}\,\mathrm{d}\zeta, \quad \text{for } 0<r<\sqrt{1-p(w)}. \tag{III.8}$$

Differentiating under the integral in (III.8) with respect to w_j, where $1\le j\le n$, shows that a_k is holomorphic in the variables $w_1,\dots,w_n$. Thus, by Hartogs's theorem on separate analyticity it follows that $a_k\in\mathcal{O}(\mathbb{B}_p)$ for each $k\in\mathbb{N}$.

Finally integrating equation (III.7) on $\mathbb{B}_p$ on both sides yields

$$\begin{aligned}
\|a\|^2_{\mathcal{Y}_p} &= \pi\sum_{k=0}^{\infty}\frac{1}{k+1}\int_{\mathbb{B}_p}|a_k(w)|^2\,(1-p(w))^{k+1}\,\mathrm{d}V(w)\\
&= \int_{\mathbb{B}_p}\pi\sum_{k=0}^{\infty}\frac{(1-p(w))^{k+1}}{k+1}\,|a_k(w)|^2\,\mathrm{d}V(w)\\
&= \int_{\mathbb{B}_p}\int_{\mathbb{D}(\sqrt{1-p(w)})}|F(z,w)|^2\,\mathrm{d}V(z)\,\mathrm{d}V(w)<\infty.
\end{aligned}$$

This shows that $a\in\mathcal{Y}_p$ and that T is surjective. □

III.2. Translation Subgroup

III.2.1. The case of the strip $S(a,b)$

For $-\infty\le a<b\le\infty$, let $S(a,b)$ be the strip

$$S(a,b)=\{z\in\mathbb{C}\mid a<\operatorname{Im}z<b\}. \tag{III.9}$$

For $a,b\in\mathbb{R}$, let $\omega_{a,b}:\mathbb{R}\to(0,\infty)$ be given by

$$\omega_{a,b}(t)=\frac{e^{-4\pi at}-e^{-4\pi bt}}{4\pi t}, \quad \text{for all } t\in\mathbb{R}. \tag{III.10}$$

Let $L^2(\mathbb{R},\omega_{a,b})$ be the Hilbert space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|^2_{L^2(\mathbb{R};\omega_{a,b})}:=\int_{\mathbb{R}}|f(t)|^2\omega_{a,b}(t)\,\mathrm{d}t<\infty.$$

Theorem 10 (Paley-Wiener theorem for Bergman space of the strip)**.** *The mapping $T_S : L^2(\mathbb{R}; \omega_{a.b}) \to A^2(S(a,b))$ given by*

$$T_S f(z) = \int_{\mathbb{R}} f(t) e^{i2\pi zt} \, \mathrm{d}t, \quad \text{for all } z \in S(a,b) \tag{III.11}$$

is an isometric isomorphism of Hilbert spaces. Let $T_S^{-1} : A^2(S(a,b)) \to L^2(\mathbb{R}, \omega_{a,b})$ be the inverse of T_S. Then, for each function $F \in A^2(S(a,b))$ and for $t \in \mathbb{R}$ we have

$$T_S^{-1} F(t) = \int_{\mathbb{R}} F(x+ic) e^{2\pi ct} e^{-i2\pi xt} \, \mathrm{d}x,$$

for any $c \in (a,b)$.

Proof. We first show that for all $z \in S(a,b)$ and all $f \in L^2(\mathbb{R}, \omega_{a,b})$, the integral in (III.11) converges. We begin by noting that whenever $a < y < b$, i.e., when $z \in S(a,b)$,

$$\int_{\mathbb{R}} \left| \frac{e^{i2\pi zt}}{\omega_{a,b}(t)} \right|^2 \omega_{a,b}(t) \, \mathrm{d}t = \int_{\mathbb{R}} \frac{4\pi t e^{-4\pi yt}}{e^{-4\pi at} - e^{-4\pi bt}} \, \mathrm{d}t \tag{III.12}$$

$$\leq \int_{-\infty}^{0} -4\pi t e^{4\pi(b-y)t} \, \mathrm{d}t + \int_{0}^{\infty} 4\pi t e^{-4\pi(y-a)t} \, \mathrm{d}t \tag{III.13}$$

$$< \infty,$$

since both integrals in (III.13) clearly converge. Thus for all $f \in L^2(\mathbb{R}, \omega_{a,b})$ and all $z \in S(a,b)$, by the Cauchy-Schwarz inequality we have

$$\int_{\mathbb{R}} \left| f(t) e^{i2\pi zt} \right| \mathrm{d}t \leq \left(\int_{\mathbb{R}} |f(t)|^2 \, \omega_{a,b}(t) \right)^{1/2} \left(\int_{\mathbb{R}} \left| \frac{e^{i2\pi zt}}{\omega_{a,b}(t)} \right|^2 \omega_{a,b}(t) \, \mathrm{d}t \right)^{1/2} \tag{III.14}$$

which shows that the integral in (III.11) converges. Let $K \subset S(a,b)$ be a compact set. It is easy to see that the integral in (III.12) is uniformly bounded for all $z \in K$. Now integrating (III.14) with respect to z on K shows that the function $(t,z) \mapsto f(t) e^{i2\pi zt} \in L^1(\mathbb{R} \times K)$.

Now we show that for all $f \in L^2(\mathbb{R}, \omega_{a,b})$, the function $T_S f$ given by (III.11) is holomorphic in the strip $S(a,b)$. To see this, consider a closed triangle $\Gamma \subset S(a,b)$. Since $\partial\Gamma$ is a compact set it follows that $(t,z) \mapsto f(t) e^{i2\pi zt}$ belongs to $L^1(\mathbb{R} \times \partial\Gamma)$. Thus, we may use Fubini's theorem to interchange order of integration below

$$\int_{\partial\Gamma} T_S f(z)\,\mathrm{d}z = \int_{\partial\Gamma}\int_{\mathbb{R}} f(t)e^{i2\pi zt}\,\mathrm{d}t\,\mathrm{d}z = \int_{\mathbb{R}} f(t)\int_{\partial\Gamma} e^{i2\pi zt}\,\mathrm{d}z\,\mathrm{d}t = \int_{\mathbb{R}} f(t)\cdot 0\,\mathrm{d}t = 0.$$

This shows that $T_S f$ is holomorphic in $S(a,b)$ by Morera's theorem. We now compute the A^2 norm of $T_S f$:

$$\begin{aligned}
\|T_S f\|^2_{A^2(S(a,b))} &= \int_a^b \|T_S f(\cdot + iy)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&= \int_a^b \|e^{-2\pi y(\cdot)} f\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \quad \text{(By Plancherel's Theorem)} \\
&= \int_{\mathbb{R}} |f(t)|^2 \int_a^b e^{-4\pi yt}\,\mathrm{d}y\,\mathrm{d}t \quad \text{(By Tonelli's theorem)} \\
&= \int_{\mathbb{R}} |f(t)|^2 \frac{e^{-4\pi at} - e^{-4\pi bt}}{4\pi t}\,\mathrm{d}t \\
&= \|f\|^2_{L^2(\mathbb{R};\omega_{a,b})} < \infty
\end{aligned}$$

This shows that $T_S f \in A^2(S(a,b))$ and also that T_S is an isometry of $L^2(\mathbb{R},\omega_{a,b})$ with a subspace of $A^2(S(a,b))$.

Let $z \in S(a,b)$ be written as $z = x+iy$. Given a function $F \in A^2(S(a,b))$ and $c \in (a,b)$, let $F_c : \mathbb{R} \to \mathbb{C}$ be given by $F_c(x) = F(x+ic)$ for all x on the horizontal line $y = c$. To show that T_S is surjective, we show that for each F in $A^2(S(a,b))$, we may choose a $c \in (a,b)$ and obtain

$$F(z) = T_S(\widehat{F_c}e^{2\pi c(\cdot)})(z) = \int_{\mathbb{R}} \widehat{F_c}(t)e^{2\pi ct}e^{i2\pi zt}\,\mathrm{d}t = \int_{\mathbb{R}} \widehat{F_c}(t)e^{i2\pi(z-ic)t}\,\mathrm{d}t,$$

where $\widehat{F_c}$ is the L^2 Fourier transform of the function F_c, and such that $\widehat{F_c}e^{2\pi c(\cdot)} \in L^2(\mathbb{R};\omega_{a,b})$. To see this, we note that

$$\|F\|^2_{A^2(S(a,b))} = \int_a^b \int_{\mathbb{R}} |F(x+iy)|^2\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left\|F_y\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y < \infty,$$

which shows that $F_y \in L^2(\mathbb{R})$ for almost all $y \in (a,b)$. We choose a $c \in (a,b)$ such that $F_c \in L^2(\mathbb{R})$ and let

$$G(z) = T_S(\widehat{F_c}e^{i2\pi c(\cdot)})(z) = \int_{\mathbb{R}} \widehat{F_c}(t)e^{i2\pi(z-ic)t}\,dt. \tag{III.15}$$

First suppose that $\widehat{F_c}$ is compactly supported. Since $F_c \in L^2(\mathbb{R})$, by Plancherel's theorem it follows that $\widehat{F_c} \in L^2(\mathbb{R})$. Since the function $(t,z) \mapsto e^{i2\pi(z-ic)t}$ is continuous, it follows that the function $(t,z) \mapsto \widehat{F_c}(t)e^{i2\pi(z-ic)t} \in L^1(\mathbb{R}\times K)$ for any compact K contained in $S(a,b)$, as $\widehat{F_c} \in L^2(\mathbb{R})$ and is compactly supported in $\mathbb{R}$.

We claim that the function G in (III.15) is holomorphic in $S(a,b)$. To see this consider a closed triangle $\Gamma \subset S(a,b)$. Since $\partial\Gamma$ is a compact set it follows that $(t,z) \mapsto \widehat{F_c}(t)e^{i2\pi(z-ic)t}$ belongs to $L^1(\mathbb{R}\times\partial\Gamma)$. Thus, we may use Fubini's theorem to interchange order of integration below

$$\int_{\partial\Gamma} G(z)\,\mathrm{d}z = \int_{\partial\Gamma}\int_{\mathbb{R}} \widehat{F_c}(t)e^{i2\pi(z-ic)t}\,\mathrm{d}t\,\mathrm{d}z = \int_{\mathbb{R}} \widehat{F_c}(t)\int_{\partial\Gamma} e^{i2\pi(z-ic)t}\,\mathrm{d}z\,\mathrm{d}t = 0,$$

and by Morera's theorem, G is holomorphic in $S(a,b)$.

By the Fourier inversion formula, $G_c = F_c$ on the line $y = c$, and, by the identity theorem for holomorphic functions, we must have $G = F$ on $S(a,b)$. Thus we see that

$$F = T_S(\widehat{F_c}e^{2\pi c(\cdot)}) \quad \text{and} \quad \left\|\widehat{F_c}e^{2\pi c(\cdot)}\right\|_{L^2(\mathbb{R};\omega_{a,b})} = \|F\|_{A^2(S(a,b))}, \tag{III.16}$$

where we used Plancherel's theorem to arrive at the second equality in (III.16).

When $\widehat{F_c}$ is not compactly supported, for each $\varepsilon > 0$, we construct a function $G^\varepsilon \in A^2(S(a,b))$ with the following properties:

1. The measurable function $t \mapsto e^{2\pi yt}\widehat{G^\varepsilon}_y(t)$ is independent of the choice of $y \in (a,b)$.
2. For each $y \in (a,b)$, the function $\widehat{G^\varepsilon_y} \to \widehat{F_y}$ uniformly on compact subsets of $\mathbb{R}$ as $\varepsilon \to 0$.

For then, by choosing a $c \in (a,b)$ such that $G^\varepsilon \in L^2(\mathbb{R})$ it follows from property (1) of G^ε that for almost all $t \in \mathbb{R}$, we get

$$e^{2\pi ct}\widehat{G^\varepsilon_c}(t) = e^{2\pi yt}\widehat{G^\varepsilon_y}(t),$$

for all $y \in (a,b)$. Letting $\varepsilon \searrow 0$, and using property (2) of G^ε, we get

$$\widehat{F_y}(t) = e^{-2\pi(y-c)t}\widehat{F_c}(t), \quad \text{for almost all}\, t \in \mathbb{R}$$

$$\implies \mathcal{F}^{-1}\left(\widehat{F_y}\right) = \mathcal{F}^{-1}\left(e^{-2\pi(y-c)(\cdot)}\widehat{F_c}\right)$$
$$\implies F_y = T_S\left(e^{2\pi c(\cdot)}\widehat{F_c}\right),$$

which is precisely what we wanted. Here $\mathcal{F}^{-1}$ denotes the inverse L^2 Fourier transform.

We now construct such a family of functions G^ε for each $\varepsilon > 0$. Let $\phi \in \mathscr{C}^\infty(\mathbb{R})$ be a function such that $\widehat{\phi} \in \mathscr{C}_c^\infty(\mathbb{R})$ and $\widehat{\phi} \equiv 1$ on the interval $[-1,1]$. Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, so that $\widehat{\phi_\varepsilon}(t) = \widehat{\phi}(\varepsilon t)$. Note that $\int_{\mathbb{R}} \phi_\varepsilon(x)\,dx = \|\phi\|_{L^1(\mathbb{R})}$. Finally for each $\varepsilon > 0$, and $z \in S(a,b)$ let G^ε be given by

$$G^\varepsilon(z) = G^\varepsilon_y(x) = (\phi_\varepsilon * F_y)(x).$$

It is apparent that property (2) above is satisfied. Indeed consider any compact set $K \subset \mathbb{R}$. Since $G^\varepsilon_y(x) = (\phi_\varepsilon * F_y)(x)$, we have $\widehat{G^\varepsilon_y} = \widehat{\phi_\varepsilon}\widehat{F_y}$. Now choose ε_0 small enough so that $K \subset [-1/\varepsilon_0, 1/\varepsilon_0]$. For $\varepsilon < \varepsilon_0$, $\widehat{\phi_\varepsilon} \equiv 1$ so $\widehat{G^\varepsilon_y} \to \widehat{F_y}$ uniformly K.

It remains to be shown that for each $\varepsilon > 0$, property (1) holds and that $G^\varepsilon \in A^2(S(a,b))$.

First we show that G^ε is holomorphic in $S(a,b)$ for every $\varepsilon > 0$. For $N > 0$, let $\psi_N \in \mathscr{C}_c^\infty(\mathbb{R})$ be a function such that $\psi_N \equiv 1$ on $[-N,N]$ and $0 \le \psi_N < 1$ outside $[-N,N]$. For $z \in S(a,b)$ let $G^{\varepsilon,N}$ be given by

$$G^{\varepsilon,N}(z) = G^{\varepsilon,N}_y(x) = (\psi_N\phi_\varepsilon * F_y)(x) = \int_{\mathbb{R}} F_y(x-t)\psi_N(t)\phi_\varepsilon(t)\,\mathrm{d}t = \int_{\mathbb{R}} F(z-t)\psi_N(t)\phi_\varepsilon(t)\,\mathrm{d}t.$$

Since the function $(z,t) \mapsto \frac{\partial F}{\partial z}(z-t)\psi_N(t)\phi_\varepsilon(t)$ is continuous on $S(a,b)\times\mathbb{R}$, we may differentiate under the integral sign to see that $G^{\varepsilon,N}$ is holomorphic in $S(a,b)$. Note that by Young's inequality for convolution, we get the following estimate

$$\begin{aligned}\left\|\psi_N\phi_\varepsilon * F_y\right\|_{L^2(\mathbb{R})} &\le \|\psi_N\phi_\varepsilon\|_{L^1(\mathbb{R})}\left\|F_y\right\|_{L^2(\mathbb{R})} \le \|\phi_\varepsilon\|_{L^1(\mathbb{R})}\left\|F_y\right\|_{L^2(\mathbb{R})} \\ &= \|\phi\|_{L^1(\mathbb{R})}\left\|F_y\right\|_{L^2(\mathbb{R})}. \end{aligned} \tag{III.17}$$

Now we compute the A^2 norm of $G^{\varepsilon,N}$:

$$\begin{aligned}
\|G^{\varepsilon,N}\|^2_{A^2(S(a,b))} &= \int_a^b \|G_y^{\varepsilon,N}\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&= \int_a^b \|\psi_N\phi_\varepsilon * F_y\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&\le \|\phi\|^2_{L^1(\mathbb{R})}\int_a^b \|F_y\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \qquad \text{(By estimate (III.17))} \\
&= \|\phi\|^2_{L^1(\mathbb{R})}\int_{S(a,b)} |F(z)|^2\,\mathrm{d}V(z) \\
&= \|\phi\|^2_{L^1(\mathbb{R})}\|F\|^2_{A^2(S(a,b))} < \infty.
\end{aligned}$$

This shows that $G^{\varepsilon,N} \in A^2(S(a,b))$ for all $\varepsilon, N > 0$. We now show that $G^{\varepsilon,N} \to G^\varepsilon$ in $L^2(S(a,b))$. Note, that by Young's inequality for convolution, we have

$$\|(\phi_\varepsilon - \psi_N\phi_\varepsilon) * F_y\|_{L^2(\mathbb{R})} \le \|(1-\psi_N)\phi_\varepsilon\|_{L^1(\mathbb{R})}\|F_y\|_{L^2(\mathbb{R})}. \tag{III.18}$$

Thus, we have

$$\begin{aligned}
\|G^\varepsilon - G^{\varepsilon,N}\|^2_{L^2(S(a,b))} &= \int_a^b \|G_y^\varepsilon - G_y^{\varepsilon,N}\|_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&= \int_a^b \|(\phi_\varepsilon - \psi_N\phi_\varepsilon) * F_y\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&\le \|(1-\psi_N)\phi_\varepsilon\|^2_{L^1(\mathbb{R})}\int_a^b \|F_y\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y \\
&= \|(1-\psi_N)\phi_\varepsilon\|^2_{L^1(\mathbb{R})}\int_{S(a,b)} |F(z)|^2\,\mathrm{d}z \\
&= \|(1-\psi_N)\phi_\varepsilon\|_{L^1(\mathbb{R})}\|F\|^2_{A^2(S(A,b))}.
\end{aligned}$$

Note that $(1-\psi_N)\phi_\varepsilon \to 0$ pointwise on $\mathbb{R}$ and $|(1-\psi_N)\phi_\varepsilon| \le |\phi_\varepsilon|$. Since $\phi_\varepsilon \in L^1(\mathbb{R})$, by the dominated convergence theorem we see that $\|(1-\psi_N)\phi_\varepsilon\|_{L^1(\mathbb{R})} \to 0$ as $N \to \infty$. Thus, $G^{\varepsilon,N} \to G^\varepsilon$ in $L^2(S(a,b))$ as claimed. Consequently, $G^\varepsilon \in A^2(S(a,b))$ since $A^2(S(a,b))$ is closed in $L^2(S(a,b))$.

Since

$$\|G^\varepsilon\|^2_{A^2(S(a,b))} = \int_a^b \|G_y^\varepsilon\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y < \infty,$$

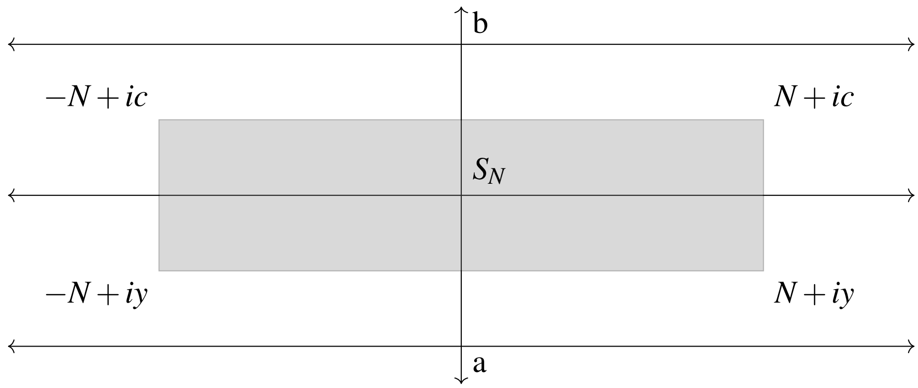

Figure 4. The Rectangle S_N

we may choose a $c \in (a,b)$ such that $\|G_c^\varepsilon\|_{L^2(\mathbb{R})} < \infty$. Since $\widehat{G_c^\varepsilon} = \widehat{\phi}_\varepsilon \widehat{F}_c$, the function $\widehat{G_c^\varepsilon}$ is compactly supported and (III.16) holds for such a G^ε in $A^2(S(a,b))$ we obtain by our choice of c,

$$G^\varepsilon(z) = T\left(\widehat{G_c^\varepsilon} e^{2\pi c(\cdot)}\right)(z) = \int_{\mathbb{R}} \widehat{G_c^\varepsilon}(t) e^{2\pi ct} e^{i2\pi zt}\,\mathrm{d}t,$$

and

$$\|\widehat{G_c^\varepsilon} e^{2\pi c(\cdot)}\|_{L^2(\mathbb{R})} = \|G^\varepsilon\|_{A^2(S(a,b))}.$$

Thus $e^{2\pi c(\cdot)}\widehat{G_c^\varepsilon} \in L^2(\mathbb{R})$, and since $\widehat{G_c^\varepsilon}$ is compactly supported $\widehat{G_c^\varepsilon} e^{-2\pi(y-c)(\cdot)}$ belongs to $L^1(\mathbb{R})$, for all $y \in (a,b)$. Thus, an application of the Riemann-Lebesgue lemma shows that

$$\lim_{|N|\to\infty} G^\varepsilon(N+iy) = \lim_{|N|\to\infty} G_y^\varepsilon(N) = \lim_{|N|\to\infty} \int_{\mathbb{R}} \widehat{G_c^\varepsilon}(t) e^{-2\pi(y-c)t} e^{i2\pi Nt}\,\mathrm{d}t = 0,$$

for all $y \in (a,b)$.

We now show that property (1) holds. For $N > 0$, let S_N be the rectangle with vertices $-N+ic, N+ic, N+iy$, and $-N+iy$ where $y \in (a,b)$.

Let ∂S_N be the boundary of S_N oriented clockwise. Then for all $t \in \mathbb{R}$ we have by Cauchy's theorem,

$$\begin{aligned} 0 &= \int_{\partial S_N} G^\varepsilon(z) e^{-i2\pi zt}\,\mathrm{d}z \qquad \text{(III.19)} \\ &= \int_{-N}^{N} G^\varepsilon(x+ic) e^{2\pi ct} e^{-i2\pi xt}\,\mathrm{d}x + \int_c^y G^\varepsilon(N+i\eta) e^{2\pi\eta t} e^{-i2\pi Nt}\,\mathrm{d}\eta \end{aligned}$$

$$+\int_{N}^{-N} G^{\varepsilon}(x+iy)e^{2\pi yt}e^{-i2\pi xt}\,\mathrm{d}x+\int_{y}^{c} G^{\varepsilon}(-N+i\eta)e^{2\pi\eta t}e^{i2\pi Nt}\,\mathrm{d}\eta$$

Since $\lim_{N\to\infty} G^{\varepsilon}(N+i\eta)=0$ there exists $N_0>0$ such that

$$|G^{\varepsilon}(N+i\eta)|<1 \quad \text{for all}\, N>N_0 \quad \text{and for all}\, \eta\in(a,b).$$

Thus, we have the following estimate

$$|G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}e^{-i2\pi Nt}|\le|G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}|<e^{2\pi\eta t}.$$

Note that $e^{2\pi\eta t}$ belongs to $L^1([c,y])$, since it is a continuous function on a compact set. Then by the dominated convergence theorem, we have

$$\begin{aligned}\lim_{N\to\infty}\left|\int_c^y G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}e^{-i2\pi Nt}\,\mathrm{d}\eta\right| &\le \lim_{N\to\infty}\int_c^y \left|G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}e^{-i2\pi Nt}\right|\mathrm{d}\eta\\ &=\int_c^y \lim_{N\to\infty}\left|G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}e^{-i2\pi Nt}\right|\mathrm{d}\eta\\ &=\int_c^y \lim_{N\to\infty}|G^{\varepsilon}(N+i\eta)|e^{2\pi\eta t}\,\mathrm{d}\eta\\ &=\int_c^y 0\,\mathrm{d}\eta=0.\end{aligned}$$

Similarly $\lim_{N\to\infty}\left|\int_y^c G^{\varepsilon}(-N+i\eta)e^{2\pi\eta t}e^{i2\pi Nt}\,\mathrm{d}\eta\right|=0$. Letting $N\to\infty$, we get larger and larger S_N. Using the fact that

$$\lim_{|N|\to\infty}\left|\int_y^c G^{\varepsilon}(N+i\eta)e^{2\pi\eta t}e^{i2\pi Nt}\,\mathrm{d}\eta\right|=0,$$

in (III.19) gives

$$\begin{aligned}0&=\lim_{N\to\infty}\int_{-N}^{N} G^{\varepsilon}(x+ic)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x+\lim_{N\to\infty}\int_{N}^{-N} G^{\varepsilon}(x+iy)e^{2\pi yt}e^{-i2\pi xt}\,\mathrm{d}x\\ &=\lim_{N\to\infty}\int_{-N}^{N} G^{\varepsilon}(x+ic)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x-\lim_{N\to\infty}\int_{-N}^{N} G^{\varepsilon}(x+iy)e^{2\pi yt}e^{-i2\pi xt}\,\mathrm{d}x\\ &=\lim_{N\to\infty}\int_{-N}^{N} G_c^{\varepsilon}(x)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x-\lim_{N\to\infty}\int_{-N}^{N} G_y^{\varepsilon}(x)e^{2\pi yt}e^{-i2\pi xt}\,\mathrm{d}x\\ &=\int_{\mathbb{R}} G_c^{\varepsilon}(x)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x-\int_{\mathbb{R}} G_y^{\varepsilon}(x)e^{2\pi yt}e^{-i2\pi xt}\,\mathrm{d}x\\ &=e^{2\pi ct}\widehat{G_c^{\varepsilon}}(t)-e^{2\pi yt}\widehat{G_y^{\varepsilon}}(t).\end{aligned}$$

Thus, property (1) holds as claimed and this completes the proof. ☐

III.2.2. A Fourier Representation of $A^2(\mathcal{U}_p)$

Let $\mathcal{L}_{\text{Trans}}(p)$ be the Hilbert space of measurable functions $g : (0,\infty) \times \mathbb{C}^n \to \mathbb{C}$ such that

$$\|g\|^2_{\mathcal{L}_{\text{Trans}}(p)} := \int_{\mathbb{C}^n} \int_0^\infty |g(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \, \mathrm{d}V(w) \, \mathrm{d}t < \infty.$$

Recall that $\mathcal{H}_p$ is the closed subspace of $\mathcal{L}_{\text{Trans}}(p)$ consisting of functions $g \in \mathcal{L}_{\text{Trans}}(p)$ such that

$$\frac{\partial g}{\partial \overline{w}_j} = 0 \, \text{in the sense of distributions}, \, 1 \leq j \leq n.$$

We make use of the following fact when integrating over $\mathcal{U}_p$: for each $w \in \mathbb{C}^n$ such that $(z,w) \in \mathcal{U}_p = \{(z,w) \in \mathbb{C} \times \mathbb{C}^n \mid \operatorname{Im} z > p(w)\}$, we have $z \in S(p(w),\infty)$. where we set $S(p(w),\infty) = \{z \in \mathbb{C} \mid \operatorname{Im} z > p(w)\}$ as in (III.9).

Proof of Theorem 3. We first show that the map T_S is an isometry from $\mathcal{H}_p$ into $L^2(\mathcal{U}_p)$. Since $f \in \mathcal{H}_p$, we have

$$\|f\|^2_{\mathcal{H}_p} = \int_{\mathbb{C}^n} \int_0^\infty |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \, \mathrm{d}t \, \mathrm{d}V(w) < \infty.$$

Thus, for almost all $w \in \mathbb{C}^n$, we have

$$\|f(\cdot,w)\|^2_{L^2(\mathbb{R},\omega_{p(w),\infty})} = \int_0^\infty |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \, \mathrm{d}t < \infty.$$

For such a $w \in \mathbb{C}^n$, by Theorem 10, the function $T_S f(\cdot,w)$ given by

$$T_S f(z,w) = \int_0^\infty f(t,w) e^{i2\pi zt} \, \mathrm{d}t, \quad \text{for all } z \in S(p(w),\infty)$$

is well defined and

$$\|T_S f(\cdot,w)\|^2_{A^2(S(p(w),\infty))} = \int_{S(p(w),\infty)} |T_S f(z,w)|^2 \, \mathrm{d}V(z) = \int_0^\infty |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t} \, \mathrm{d}t. \quad \text{(III.20)}$$

Integrating equation (III.20) over $\mathbb{C}^n$, we get

$$\int_{\mathbb{C}^n}\int_{S(p(w),\infty)} |T_S f(z,w)|^2 \,\mathrm{d}V(z)\,\mathrm{d}V(w) = \int_{\mathbb{C}^n}\int_0^\infty |f(t,w)|^2 \frac{e^{-4\pi p(w)t}}{4\pi t}\,\mathrm{d}t\,\mathrm{d}V(w)$$
$$\text{i.e.,}\quad \|T_S f\|^2_{L^2(\mathcal{U}_p)} = \|f\|^2_{\mathcal{H}_p}.$$

We now show that the image of T_S lies in $A^2(\mathcal{U}_p)$. Let $\varphi_N \in \mathscr{C}_c^\infty([0,\infty))$ be such that $\varphi_N \equiv 1$ on $[0,N]$ and $0 \le \varphi \le 1$ on $[0,\infty)$. Then $\varphi_N f$ is a measurable function on $(0,\infty)\times\mathbb{C}^n$ and for $(z,w)\in\mathcal{U}_p$, the function $T_S(\varphi_N f)$ is given by

$$T_S(\varphi_N f)(z,w) = \int_0^\infty \varphi_N(t) f(t,w) e^{i2\pi zt}\,\mathrm{d}t. \tag{III.21}$$

Note that for $N>0$, we have $\|\varphi_N f\|_{\mathcal{H}_p} \le \|f\|_{\mathcal{H}_p}$ and consequently $\varphi_N f \in \mathcal{H}_p$, since it is clear that

$$\frac{\partial(\varphi_N f)}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions for } j=1,\dots,n.$$

Thus, we see that for almost all $w\in\mathbb{C}^n$,

$$\|\varphi_N f(\cdot,w)\|_{L^2(\mathbb{R},\omega_{p(w),\infty})} \le \|f(\cdot,w)\|_{L^2(\mathbb{R},\omega_{p(w),\infty})} < \infty$$

and by Theorem 10 we see that for almost all $w\in\mathbb{C}^n$, the function $T_S(\varphi_N f)(\cdot,w)$ belongs to $A^2(S(p(w),\infty))$ which shows that $T_S(\varphi_N f)$ is holomorphic in the variable z. Since f is square integrable on $L^2((0,\infty)\times\mathbb{C}^n)$ with respect to the weight $(t,w)\mapsto e^{-4\pi p(w)t}/4\pi t$, it follows that $f\in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{C}^n)$. Let $\alpha\in\mathscr{C}_c^\infty(\mathbb{C}^n)$, then the action of the distributional derivative of $T_S(\varphi_N f)$ with respect to $\overline{w}_j$ on the test function α above gives

$$\begin{aligned}\int_{\mathbb{C}^n} -T_S(\varphi_N f)(z,w)\frac{\partial\alpha}{\partial\overline{w}_j}\,\mathrm{d}V(w) &= \int_{\mathbb{C}^n}\left(\int_0^\infty -\varphi_N(t)f(t,w)e^{i2\pi zt}\,\mathrm{d}t\right)\frac{\partial\alpha}{\partial\overline{w}_j}\,\mathrm{d}V(w)\\ &= \int_0^\infty\int_{\mathbb{C}^n} -f(t,w)\varphi_N(t)\frac{\partial\alpha}{\partial\overline{w}_j}e^{i2\pi zt}\,\mathrm{d}V(w)\,\mathrm{d}t \qquad \text{(III.22)}\\ &= 0,\end{aligned}$$

since

$$\varphi_N(t)\frac{\partial\alpha}{\partial\overline{w}_j}e^{i2\pi zt}\in\mathscr{C}_c^\infty((0,\infty)\times\mathbb{C}^n),$$

and the integral in (III.22) is the pairing of the distributional derivative of f with respect to $\overline{w}_j$ with the test function above. By Weyl's lemma we see that $T_S(\varphi_N f)$ is holomorphic in the variable w_j for $1\leq j\leq n$. Thus, by Hartogs's theorem on separate analyticity we see that $T_S(\varphi_N f)\in\mathcal{O}(\mathcal{U}_p)$ and consequently $T_S(\varphi_N f)\in A^2(\mathcal{U}_p)$. We claim that $T_S(\varphi_N f)\to T_S f$ in $L^2(\mathcal{U}_p)$. Since T_S is an isometry, to show that $T_S(\varphi_N f)\to T_S f$ in $L^2(\mathcal{U}_p)$ it suffices to show that $\varphi_N f\to f$ in $\mathcal{H}_p$. Indeed, we have $|f-\varphi_N f|<|f|$ on $(0,\infty)\times\mathbb{C}^n$ and $|f-\varphi_N f|\to 0$, as $N\to\infty$. Thus, by dominated convergence theorem we obtain $\|f-\varphi_N f\|_{\mathcal{H}_p}\to 0$, as $N\to\infty$, which proves the claim. Since $A^2(\mathcal{U}_p)$ is closed in $L^2(\mathcal{U}_p)$ it follows that $T_S f\in A^2(\mathcal{U}_p)$.

We now show that the map T is surjective. Suppose $F\in A^2(\mathcal{U}_p)$. We have

$$\|F\|^2_{A^2(\mathcal{U}_p)}=\int_{\mathbb{C}^n}\int_{S(p(w),\infty)}|F(z,w)|^2\,\mathrm{d}V(z)\,\mathrm{d}V(w)<\infty.$$

By Fubini's theorem, for almost all $w\in\mathbb{C}^n$, we have

$$\int_{S(p(w),\infty)}|F(z,w)|^2\,\mathrm{d}V(z)<\infty, \tag{III.23}$$

where $S(p(w),\infty)=\{z\in\mathbb{C}\mid \operatorname{Im}z>p(w)\}$. Thus, by Theorem 10, for such a $w\in\mathbb{C}^n$, there is a measurable function $f(\cdot,w)\in L^2(\mathbb{R},\omega_{p(w),\infty})$

$$F(z,w)=\int_0^\infty f(t,w)e^{i2\pi zt}\,\mathrm{d}t$$

and

$$\begin{aligned}\|F(\cdot,w)\|^2_{A^2(S(p(w),\infty))}&=\int_{S(p(w),\infty)}|F(z,w)|^2\,\mathrm{d}V(z)\\&=\int_0^\infty|f(t,w)|^2\,\frac{e^{-4\pi p(w)t}}{4\pi t}\,\mathrm{d}t.\end{aligned} \tag{III.24}$$

Using notation of Theorem 10, $f(\cdot,w)$ is given by

$$f(t,w) = T_S^{-1}F(\cdot,w) = \int_{\mathbb{R}} F(x+ic,w)e^{i2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x,$$

for any $c > p(w)$. Integrating (III.24) over $\mathbb{C}^n$, we get

$$\begin{aligned}
\|F\|^2_{A^2(\mathcal{U}_p)} &= \int_{\mathbb{C}^n} \|F(\cdot,w)\|^2_{A^2(S(p(w),\infty))} \\
&= \int_{\mathbb{C}^n}\int_0^{\infty} |f(t,w)|^2\,\frac{e^{-4\pi p(w)t}}{4\pi t}\,\mathrm{d}t \\
&= \|f\|^2_{\mathcal{L}_{\mathrm{Trans}}(p)}\,. \qquad \text{(III.25)}
\end{aligned}$$

This, shows that $T_S^{-1} : A^2(\mathcal{U}_p) \to \mathcal{L}_{\mathrm{Trans}}(p)$ is an isometry. Let $\phi_N \in \mathscr{C}_c^{\infty}(\mathbb{R})$ be such that $\phi \equiv 1$ on $[-N,N]$ and $0 \le \phi \le 1$. For $t \in (0,\infty)$ and $w \in \mathbb{C}^n$, let f_N be given by

$$f_N(t,w) = T_S^{-1}(\phi_N F) = \int_{\mathbb{R}} \phi_N(x)F(x+ic,w)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x, \qquad \text{(III.26)}$$

Note that $(x,t,w) \mapsto \phi_N(x)F(x+ic,w)e^{2\pi ct}e^{-i2\pi xt}$ is a smooth function in each of the variables. Consequently, the function $(x,t,w) \mapsto \phi_N(x)F(x+ic,w)e^{2\pi ct}e^{-i2\pi xt}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)\times\mathbb{C}^n)$, and by Fubini's theorem we may interchange the order of integration below. For $(t,w) \in (0,\infty)\times\mathbb{C}^n$, let $\alpha \in \mathscr{C}_c^{\infty}((0,\infty)\times\mathbb{C}^n)$ be of the form

$$\alpha(t,w) = \alpha_1(t)\alpha_2(w), \text{ where } \alpha_1 \in \mathscr{C}_c^{\infty}(\mathbb{R}), \alpha_2 \in \mathscr{C}_c^{\infty}(\mathbb{C}^n).$$

Then, the action of the distributional derivative with respect to $\overline{w}_j$ of f_N as in (III.26) on the test function α gives

$$\begin{aligned}
&\int_0^{\infty}\int_{\mathbb{C}^n} f_N(t,w)\alpha_1(t)\left(-\frac{\partial \alpha_2}{\partial \overline{w}_j}(w)\right)\mathrm{d}V(w)\,\mathrm{d}t \\
&= \int_0^{\infty}\int_{\mathbb{C}^n}\left(\int_{\mathbb{R}} -\phi_N(x)F(x+ic,w)e^{-i2\pi xt}\,\mathrm{d}x\right)e^{2\pi ct}\alpha_1(t)\frac{\partial \alpha_2}{\partial \overline{w}_j}(w)\,\mathrm{d}V(w)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \phi_N(x)\int_0^{\infty}\left(\int_{\mathbb{C}^n} -F(x+ic,w)\frac{\partial \alpha_2}{\partial \overline{w}_j}(w)\,\mathrm{d}V(w)\right)e^{2\pi ct}e^{-i2\pi xt}\alpha_1(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= 0,
\end{aligned}$$

since F is holomorphic in the variable w_j. This shows that $\frac{\partial f_N}{\partial \overline{w}_j} = 0$, in the sense of distributions for all $1 \leq j \leq n$.

Note that $|F - \phi_N F| \leq |F|$ on $\mathcal{U}_p$. Since $|F - \phi_N F| \to 0$ as $N \to \infty$, by dominated convergence theorem it follows that $\phi_N F \to F$ in $A^2(\mathcal{U}_p)$. Since T_S^{-1} is an isometry it follows that, $f_N \to f$ in $\mathcal{L}_{\text{Trans}}(p)$ and consequently $\frac{\partial f_N}{\partial \overline{w}_j} \to \frac{\partial f}{\partial \overline{w}_j}$ as distributions for $1 \leq j \leq n$. This shows that $f \in \mathcal{H}_p$ and completes the proof. □

III.3. Scaling Subgroup

III.3.1. The case of the sector $V(a,b)$

For $0 \leq a < b \leq \pi$, let $V(a,b)$ be the *sector*

$$V(a,b) = \{z \in \mathbb{C} \mid a < \arg z < b\}. \tag{III.27}$$

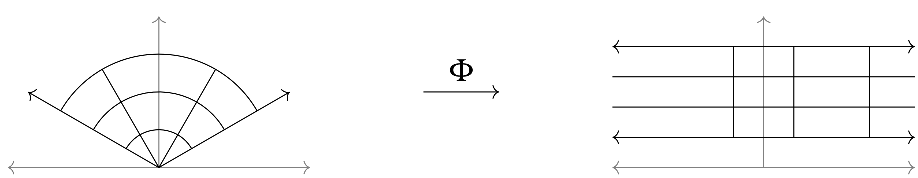

Figure 5. Conformal Equivalence of a Sector with a Strip

Recall from (III.10) that for a,b as above, $\omega_{a,b} : \mathbb{R} \to (0,\infty)$ is given by

$$\omega_{a,b}(t) = \frac{e^{-4\pi at} - e^{-4\pi bt}}{4\pi t}, \text{ for } t \in \mathbb{R}.$$

Theorem 11 (Paley-Wiener Theorem for the Bergman Space of the Sector)**.** *The mapping $T_V : L^2(\mathbb{R}, \omega_{a,b}) \to A^2(V(a,b))$ given by*

$$T_V f(z) = \int_{\mathbb{R}} f(t) \frac{z^{i2\pi t}}{z} \,\mathrm{d}t, \quad \textit{for all } z \in V(a,b) \tag{III.28}$$

is an isometric isomorphism of Hilbert spaces.

Proof. The mapping $\Phi : V(a,b) \to S(a,b)$ given by $\Phi(z) = \log|z| + i\arg z$ where $a < \arg z < b$ for all $z \in V(a,b)$ is a conformal equivalence. By Lemma II.3, this conformal equivalence establishes an isometric isomorphism

$$\Phi^* : A^2(S(a,b)) \to A^2(V(a,b)) \quad \text{given by} \quad \Phi^*(F) = (F \circ \Phi) \cdot (\Phi'). \tag{III.29}$$

By Theorem 10, we know that $T_S : L^2(\mathbb{R}; \omega_{a,b}) \to A^2(S(a,b))$ given by

$$T_S(f) = \int_{\mathbb{R}} f(t) e^{i2\pi zt}\,\mathrm{d}t$$

is an isometric isomorphism.

Thus, $\Phi^* \circ T_S$ is the desired isometric isomorphism of $L^2(\mathbb{R}; \omega_{a,b})$ with $A^2(V(a,b))$, which we compute below

$$\begin{aligned}
\Phi^* \circ T_S(f)(z) &= \Phi^* \left(\int_{\mathbb{R}} f(t) e^{i2\pi zt}\,\mathrm{d}t \right) \\
&= \left(\int_{\mathbb{R}} f(t) e^{i2\pi \Phi(z) t}\,\mathrm{d}t \right) \cdot \Phi'(z) \\
&= \left(\int_{\mathbb{R}} f(t) e^{(\log z) i2\pi t}\,\mathrm{d}t \right) \cdot \frac{1}{z} \\
&= \left(\int_{\mathbb{R}} f(t) z^{i2\pi t}\,\mathrm{d}t \right) \cdot \frac{1}{z} \\
&= \int_{\mathbb{R}} f(t) \frac{z^{i2\pi t}}{z}\,\mathrm{d}t.
\end{aligned} \tag{III.30}$$

This completes the proof. □

Corollary III.2. *Let* $(\Phi^*)^{-1} : A^2(V(a,b)) \to A^2(S(a,b))$ *be the inverse of the isometric isomorphism* Φ^*, *where* Φ^* *is as in* (III.29). *Then, for* $F \in A^2(V(a,b))$, *the function* $f = T_V^{-1} F \in L^2(\mathbb{R}; \omega_{a,b})$ *is given by*

$$f(t) = T_V^{-1} F(t) := \int_{\mathbb{R}} (\Phi^*)^{-1} F(x + ic) e^{2\pi ct} e^{-i2\pi xt}\,\mathrm{d}x, \quad \textit{for all } t \in \mathbb{R}$$

Moreover, we have

$$\|f\|_{L^2(\mathbb{R},\omega_{a,b})} = \left\|(\Phi^*)^{-1} F\right\|_{A^2(S(a,b))} = \|F\|_{A^2(V(a,b))}.$$

Proof. It follows from equation (III.30) that given $f \in L^2(\mathbb{R},\omega_{a,b})$ such that $T_V f = F$ is given by

$$f = (T_S^{-1} \circ (\Phi^*)^{-1})F = T_S^{-1}\left((\Phi^*)^{-1} F\right) = \int_{\mathbb{R}} (\Phi^*)^{-1} F(x+ic) e^{2\pi ct} e^{-i2\pi xt}\,\mathrm{d}x,$$

where the last equality follows from Theorem 10. Since T_S^{-1} and $(\Phi^*)^{-1}$ are isometries, we obtain

$$\|f\|_{L^2(\mathbb{R},\omega_{a,b})} = \left\|T_S^{-1}(\Phi^*)^{-1} F\right\|_{L^2(\mathbb{R},\omega_{a,b})} = \left\|(\Phi^*)^{-1} F\right\|_{A^2(S(a,b))} = \|F\|_{A^2(V(a,b))}.$$

□

We now introduce a domain $\mathcal{C}_p \subset \mathbb{C}^{n+1}$ which is biholomorphically equivalent to $\mathcal{U}_p$ (see Lemma III.3 below), whose properties we use in the proof of Theorem 3. Let $\mathcal{C}_p$ be the domain

$$\mathcal{C}_p = \{(\gamma,\zeta) \in \mathbb{C}\times\mathbb{C}^n \mid \operatorname{Im}\gamma > p(\zeta)\,|\gamma|\}, \tag{III.31}$$

and recall that $\mathbb{B}_p = \{\zeta \in \mathbb{C}^n \mid p(\zeta) < 1\}$.

Since $(z,w) \in \mathcal{U}_p$, we have $\operatorname{Im} z > p(w) \geq 0$, so there exists a branch of logarithm defined on $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ independently of $w \in \mathbb{C}^n$. We take this branch of logarithm to be the one that coincides with the natural logarithm on the real line and define the powers $z^{1/2m_1},\cdots,z^{1/2m_n}$ with respect to this branch.

Lemma III.3. *The map $\Psi : \mathcal{U}_p \to \mathcal{C}_p$ given by*

$$\Psi(z,w) = \left(z, \frac{w_1}{z^{1/2m_1}},\cdots,\frac{w_n}{z^{1/2m_n}}\right) = \left(z,\widehat{\rho}_{1/z}(w)\right), \quad \textit{for all } (z,w) \in \mathcal{U}_p,$$

where $\widehat{\rho}_{1/z}$ is as in Lemma II.4, is a biholomorphic equivalence.

Proof. First, we show that $\Psi(z,w) \in \mathcal{C}_p$. Since $(z,w) \in \mathcal{U}_p$ we have $\operatorname{Im} z > p(w)$. We then get

$$\operatorname{Im} z > p(w) = p\left(\widehat{\rho}_z(\widehat{\rho}_{1/z}(w))\right) = p\left(\widehat{\rho}_{1/z}(w)\right)|z|$$

where we used Lemma II.4 to arrive at the last equality. This then shows that $\Psi(z,w) \in \mathcal{C}_p$. It is easy to see that Ψ is one-to-one. If $\Psi(z,w) = \Psi(z',w')$ we get

$$\left(z, \frac{w_1}{z^{1/2m_1}}, \cdots, \frac{w_n}{z^{1/2m_n}}\right) = \left(z', \frac{w'_1}{z'^{1/2m_1}}, \cdots, \frac{w'_n}{z'^{1/2m_n}}\right) \implies (z,w) = (z',w').$$

We now show that the function Ψ is onto. Note that for $(\gamma,\zeta) \in \mathcal{C}_p$, we have $p(\zeta)\,|\gamma| < \operatorname{Im}\gamma$, which gives

$$p(\zeta) < \frac{\operatorname{Im}\gamma}{|\gamma|} < 1,$$

Thus, for all $\zeta \in \mathbb{C}^n$ such that $(\gamma,\zeta) \in \mathcal{C}_p$, we have $p(\zeta) < 1$, i.e., $\zeta \in \mathbb{B}_p$. For each $\zeta \in \mathbb{B}_p$, such that $(\gamma,\zeta) \in \mathcal{C}_p$, we have

$$\operatorname{Im}\gamma > p(\zeta)\,|\gamma| \ \text{ i.e. } \ \gamma \in V_\zeta = \left\{\gamma \in \mathbb{C} \mid \sin^{-1} p(\zeta) < \arg z < \pi - \sin^{-1} p(\zeta)\right\}.$$

V_ζ is a subset of the upper half plane, and hence we define the powers $\gamma^{1/2m_1}, \cdots, \gamma^{1/2m_n}$ using the branch of the logarithm that coincides with the natural logarithm on the positive real line, independent of $\zeta \in \mathbb{B}_p$. Let $(\gamma,\zeta) \in \mathcal{C}_p$ be given. Let $z = \gamma$ and $w_j = \gamma^{1/2m_j}\zeta_j$, for $j = 1,\ldots,n$. Then

$$\Psi(z,w) = \Psi(\gamma, \gamma^{1/2m_1}\zeta_1, \ldots, \gamma^{1/2m_n}\zeta_n) = \left(\gamma, \frac{\gamma^{1/2m_1}\zeta_1}{\gamma^{1/2m_1}}, \ldots, \frac{\gamma^{1/2m_n}\zeta_n}{\gamma^{1/2m_n}}\right) = (\gamma,\zeta).$$

This shows that Ψ is a biholomorphism between $\mathcal{U}_p$ and $\mathcal{C}_p$. □

By Lemma II.3 Ψ induces an isomorphism $\Psi^* : A^2(\mathcal{C}_p) \to A^2(\mathcal{U}_p)$. For $f \in A^2(\mathcal{C}_p)$ the function $\Psi^* f \in A^2(\mathcal{U}_p)$ is given by

$$\Psi^* f(z,w) = f(\Psi(z,w)) \cdot \det \Psi'(z,w), \text{ for } (z,w) \in \mathcal{U}_p. \qquad \text{(III.32)}$$

III.3.2. Another Fourier Representation of $A^2(\mathcal{U}_p)$

Recall from equation (I.18) that for $\zeta \in \mathbb{B}_p$, $\lambda(p(\zeta),t)$ is given by

$$\lambda(p(\zeta),t) = \begin{cases} \dfrac{1}{4\pi t}\left(e^{-4\pi t \sin^{-1} p(\zeta)} - e^{-4\pi t(\pi - \sin^{-1} p(\zeta))}\right) & \text{if } t \neq 0 \\ \pi - 2\sin^{-1}(p(\zeta)) & \text{if } t = 0. \end{cases}$$

Let $\mathcal{L}_{\mathrm{Scal}}(p)$ be the Hilbert space of measurable functions $f : \mathbb{R} \times \mathbb{B}_p \to \mathbb{C}$ such that

$$\|f\|^2_{\mathcal{L}_{\mathrm{Scal}}(p)} := \int_{\mathbb{R}} \int_{\mathbb{B}_p} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t < \infty.$$

Then $\mathcal{X}_p$ is the closed subspace of $\mathcal{L}_{\mathrm{Scal}}(p)$ consisting of functions f such that

$$\frac{\partial f}{\partial \overline{w}_j} = 0 \text{ in the sense of distributions, } 1 \leq j \leq n.$$

We use the following fact about $\mathcal{C}_p$: if $\zeta \in \mathbb{B}_p$ is such that $(\gamma,\zeta) \in \mathcal{C}_p$ then γ belongs to the domain V_ζ given by

$$V_\zeta = \{\gamma \in \mathbb{C} \mid \sin^{-1} p(\zeta) < \arg\gamma < \pi - \sin^{-1} p(\zeta)\}.$$

Proof of Theorem 4. We first show that the map $\widetilde{T}_V : \mathcal{X}_p \to A^2(\mathcal{C}_p)$ given by

$$\widetilde{T}_V f(\gamma,\zeta) = \int_{\mathbb{R}} f(t,\zeta) \frac{\gamma^{2\pi i t}}{\gamma}\,\mathrm{d}t, \quad \text{for all } (\gamma,\zeta) \in \mathcal{C}_p,$$

is an isometric isomorphism from $\mathcal{X}_p$ into $A^2(\mathcal{C}_p)$. We then show that composing this isomorphism with $\Psi^* : A^2(\mathcal{C}_p) \to A^2(\mathcal{U}_p)$ in (III.32) gives the isometric isomorphism (I.20).

We now show that the map $\widetilde{T}_V$ as given above is an isometry from $\mathcal{X}_p$ to $L^2(\mathcal{C}_p)$. Since $f \in \mathcal{X}_p$ we have

$$\|f\|^2_{\mathcal{X}_p} = \int_{\mathbb{B}_p} \int_{\mathbb{R}} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}t\,\mathrm{d}V(\zeta) < \infty,$$

where we were allowed to interchange the order of integration by Tonelli's theorem. Thus, for almost all $\zeta \in \mathbb{B}_p$, we have

$$\int_{\mathbb{R}} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}t < \infty.$$

Let $a(\zeta) = \sin^{-1} p(\zeta)$ and $b(\zeta) = \pi - \sin^{-1} p(\zeta)$. Recall that V_ζ is the planar sector

$$V_\zeta := V\left(a(\zeta), b(\zeta)\right) = \{\gamma \in \mathbb{C} \mid \sin^{-1} p(\zeta) < \arg\gamma < \pi - \sin^{-1} p(\zeta)\}.$$

For such a $\zeta \in \mathbb{B}_p$, by Theorem 11, the function $\widetilde{T}_V f(\cdot, \zeta)$ given by

$$\widetilde{T}_V f(\gamma, \zeta) = \int_{\mathbb{R}} f(t, \zeta) \frac{\gamma^{2i\pi t}}{\gamma} \,\mathrm{d}t, \quad \text{for all } \gamma \in V_\zeta$$

is well defined and

$$\begin{aligned} \left\| \widetilde{T}_V f(\cdot, w) \right\|^2_{A^2(V_\zeta)} = \int_{V_\zeta} \left| \widetilde{T}_V f(\gamma, \zeta) \right|^2 \mathrm{d}V(z) &= \int_{\mathbb{R}} |f(t, w)|^2 \,\omega_{a(\zeta), a(\zeta)}(t) \,\mathrm{d}t \\ &= \int_{\mathbb{R}} |f(t, w)|^2 \,\lambda(p(\zeta), t) \,\mathrm{d}t, \end{aligned} \tag{III.33}$$

where the last equality follows from definitions (III.10) and (I.18) of $\omega_{a(\zeta), b(\zeta)}(t)$ and $\lambda(p(\zeta), t)$ respectively. Integrating equation (III.33) over $\mathbb{B}_p$, we get

$$\begin{aligned} \left\| \widetilde{T}_V f \right\|^2_{L^2(\mathcal{C}_p)} &= \int_{\mathbb{B}_p} \int_{V_\zeta} \left| \widetilde{T}_V f(\gamma, \zeta) \right|^2 \mathrm{d}V(\gamma) \,\mathrm{d}V(\zeta) \\ &= \int_{\mathbb{B}_p} \int_{\mathbb{R}} |f(t, w)|^2 \,\lambda(p(\zeta), t) \,\mathrm{d}t \,\mathrm{d}V(\zeta) \\ &= \|f\|^2_{\mathcal{X}_p}. \end{aligned}$$

We now show that the image of $\widetilde{T}_V$ lies in $A^2(\mathcal{C}_p)$. Let $\phi_N \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that $\phi_N \equiv 1$ on $[-N, N]$ and $0 \le \phi \le 1$ on $\mathbb{R}$. Then, $\phi_N f$ is a measurable function on $\mathbb{R} \times \mathbb{C}^n$ and for $(\gamma, \zeta) \in \mathcal{C}_p$ we have

$$\widetilde{T}_V(\phi_N f)(\gamma, \zeta) = \int_{\mathbb{R}} \phi_N(t) f(t, \zeta) \frac{\gamma^{i2\pi t}}{\gamma} \,\mathrm{d}t. \tag{III.34}$$

Note that for $N > 0$, $\|\phi_N f\|_{\mathcal{X}_p} \le \|f\|_{\mathcal{X}_p}$ and thus, $\phi_N f \in \mathcal{X}_p$ since it is clear that

$$\frac{\partial(\phi_N f)}{\partial \overline{\zeta}_j} = 0 \text{ in the sense of distributions for } j = 1, \ldots, n.$$

Thus we see that for almost all $\zeta \in \mathbb{B}_p$,

$$\|\phi_N f(\cdot, \zeta)\|_{L^2(\mathbb{R}, \omega_{a(\zeta), b(\zeta)})} \le \|f(\cdot, \zeta)\|_{L^2(\mathbb{R}, \omega_{a(\zeta), b(\zeta)})} < \infty,$$

and by Theorem 11 we see that for almost all $\zeta \in \mathbb{B}_p$, the function $\widetilde{T}_V(\phi_N f)(\cdot,\zeta) \in A^2(V_\zeta)$ which shows that $\widetilde{T}_V(\phi_N f)$ is holomorphic in the variable γ. Since f is square integrable on $\mathbb{R}\times\mathbb{B}_p$ with a weight λ, it follows that $f \in L^1_{\text{loc}}(\mathbb{R}\times\mathbb{B}_p)$.Let $\alpha \in \mathscr{C}^\infty_c(\mathbb{B}_p)$, then the action of the distributional derivative of $\widetilde{T}_V(\phi_N f)$ with respect to $\overline{\zeta}_j$ on the test function α gives

$$\begin{aligned}\int_{\mathbb{B}_p} -\widetilde{T}_V(\phi_N f)(\gamma,\zeta)\left(\frac{\partial\alpha}{\partial\overline{\zeta}_j}\right)\mathrm{d}V(\zeta) &= \int_{\mathbb{B}_p}\left(\int_{\mathbb{R}} -\phi_N(t)f(t,\zeta)\frac{\gamma^{i2\pi t}}{\gamma}\,\mathrm{d}t\right)\left(\frac{\partial\alpha}{\partial\overline{\zeta}_j}\right)\mathrm{d}V(\zeta)\\ &= \int_{\mathbb{R}}\int_{\mathbb{B}_p} -f(t,\zeta)\phi_N(t)\frac{\partial\alpha}{\partial\overline{\zeta}_j}\frac{\gamma^{i2\pi t}}{\gamma}\,\mathrm{d}V(\zeta)\,\mathrm{d}t \qquad \text{(III.35)}\\ &= 0,\end{aligned}$$

since

$$\phi_N(t)\frac{\partial\alpha}{\partial\overline{\zeta}_j}\frac{\gamma^{i2\pi t}}{\gamma} \in \mathscr{C}^\infty_c(\mathbb{R}\times\mathbb{B}_p),$$

and since the integral in (III.35) is the pairing of the distributional derivative of f with respect to $\overline{\zeta}_j$ with the test function above. By Weyl's lemma we see that $\widetilde{T}_V(\phi_N f)$ is holomorphic in the variable ζ_j for $1 \le j \le n$. Thus, by Hartogs's theorem on separate analyticity we see that $\widetilde{T}_V(\phi_N f) \in \mathcal{O}(\mathcal{C}_p)$, and consequently $\widetilde{T}_V(\phi_N f) \in A^2(\mathcal{C}_p)$. We claim that $\widetilde{T}_V(\phi_N f) \to \widetilde{T}_V f$ in $L^2(\mathcal{C}_p)$. Since $\widetilde{T}_V$ is an isometry, to show that $\widetilde{T}_V(\phi_N f) \to \widetilde{T} g$ in $L^2(\mathcal{C}_p)$ it suffices to show that $(\phi_N f) \to f$ in $\mathcal{X}_p$. Indeed $|f-\phi_N f| < |f|$ on $\mathbb{R}\times\mathbb{B}_p$, and $|f-\phi_N f| \to 0$ as $N \to \infty$. Thus, by dominated convergence theorem we see that $\|f-\phi_N f\|_{\mathcal{X}_p} \to 0$, as $N \to \infty$. Since $A^2(\mathcal{C}_p)$ is closed in $L^2(\mathcal{C}_p)$, this shows that $\widetilde{T}_V f$ is in $A^2(\mathcal{C}_p)$.

Now we show that the map $\widetilde{T}_V$ is surjective. Suppose $F \in A^2(\mathcal{C}_p)$. Then we have

$$\|F\|^2_{A^2(\mathcal{C}_p)} = \int_{\mathbb{B}_p}\int_{V_\zeta} |F(\gamma,\zeta)|^2\,\mathrm{d}V(\gamma)\,\mathrm{d}V(\zeta) < \infty,$$

where $V_\zeta = \{\gamma \in \mathbb{C} \mid \sin^{-1}(p(\zeta)) < \arg\gamma < \pi - \sin^{-1}(p(\zeta))\}$. Thus, for almost all $\zeta \in \mathbb{B}_p$, we have

$$\int_{V_\zeta} |F(\gamma,\zeta)|^2\,\mathrm{d}V(\gamma) < \infty. \qquad \text{(III.36)}$$

Thus, by Theorem 11, for such a $\zeta \in \mathbb{B}_p$, there exists an $f(\cdot,\zeta) \in L^2(\mathbb{R}, \omega_{\alpha(\zeta),\beta(\zeta)})$ such that

$$F(\gamma,\zeta) = \int_{\mathbb{R}} f(t,\zeta) \frac{\gamma^{i2\pi t}}{\gamma}\,\mathrm{d}t, \quad \text{for all } \gamma \in V_\zeta$$

and

$$\int_{V_\zeta} |F(\gamma,\zeta)|^2\,\mathrm{d}V(\gamma) = \int_{\mathbb{R}} |f(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}t. \tag{III.37}$$

Using the notation of Theorem 11, by Corollary III.2, the measurable function $f(\cdot,\zeta)$ is given by

$$f(t,\zeta) = \int_{\mathbb{R}} (\Phi^*)^{-1} F(x+ic,\zeta) e^{2\pi ct} e^{-i2\pi xt}\,\mathrm{d}x := \widetilde{T}_V^{-1} F(t,\zeta), \text{ for } t \in \mathbb{R}, \tag{III.38}$$

where $c \in (\sin^{-1} p(\zeta), \pi - \sin^{-1} p(\zeta))$.

We now show that the distributional derivative with respect to $\overline{\zeta}_j$ of the function f above vanishes, for $1 \le j \le n$. Let $\phi_N \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that $\phi \equiv 1$ on $[-N,N]$ and $0 \le \phi \le 1$. For every $t \in \mathbb{R}$ and every $\zeta \in \mathbb{B}_p$, let $f_N(t,\zeta)$ be given by

$$f_N(t,\zeta) = \int_{\mathbb{R}} \phi_N(x) (\Phi^*)^{-1} F(x+ic,\zeta) e^{2\pi ct} e^{-i2\pi xt}\,\mathrm{d}x. \tag{III.39}$$

The function $(x,t,\zeta) \mapsto \phi_N(x)(\Phi^*)^{-1}F(x+ic,\zeta)e^{2\pi ct}e^{-i2\pi xt}$ is a smooth function in each of the variables, and consequently is locally integrable on $\mathbb{R}\times\mathbb{R}\times\mathbb{B}_p$. For $(t,\zeta) \in \mathbb{R}\times\mathbb{B}_p$, let $\alpha \in \mathscr{C}_c^\infty(\mathbb{R}\times\mathbb{B}_p)$ be of the form

$$\alpha(t,\zeta) = \alpha_1(t)\alpha_2(\zeta), \text{ where } \alpha_1 \in \mathscr{C}_c^\infty(\mathbb{R}), \alpha_2 \in \mathscr{C}_c^\infty(\mathbb{B}_p).$$

Then the action of the distributional derivative with respect to $\overline{\zeta}_j$ of f_N as in equation (III.39) on the test function α gives

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{B}_p} f_N(t,\zeta)\alpha_1(t) \frac{-\partial \alpha_2}{\partial \overline{\zeta}_j}(\zeta)\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{B}_p} \left(\int_{\mathbb{R}} \phi_N(x) F(e^{x+ic},\zeta) e^{x+ic} e^{-i2\pi xt}\,\mathrm{d}x \right) e^{2\pi ct} \alpha_1(t) \frac{-\partial \alpha_2}{\partial \overline{\zeta}_j}(\zeta)\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \phi_N(x) \int_0^\infty \left(\int_{\mathbb{C}^n} F(x+ic,w) \frac{-\partial \alpha_2}{\partial \overline{w}_j}\,\mathrm{d}V(w) \right) e^{x+ic} e^{-i2\pi xt} e^{2\pi ct} \alpha_1(t)\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

$$= 0,$$

since F is holomorphic in the variable ζ_j. This shows that $\frac{\partial f_N}{\partial \bar{\zeta}_j} = 0$ in the sense of distributions for all $1 \le j \le n$ and thus $f_N \in \mathcal{X}_p$ for all $N > 0$.

Since $\left\|(\Phi^*)^{-1}F(\cdot,\zeta) - \phi_N(\Phi^*)^{-1}F(\cdot,\zeta)\right\|_{A^2(S(a,b))} \to 0$ as $N \to \infty$, it follows from Corollary III.2 that $\|f(\cdot,\zeta) - f_N(\cdot,\zeta)\|_{L^2(\mathbb{R},\omega_{\alpha(\zeta),\beta(\zeta)})} \to 0$ as $N \to \infty$, for almost every $\zeta \in \mathbb{B}_p$. Consequently we obtain

$$\begin{aligned}
\|f - f_N\|^2_{\mathcal{L}_{\mathrm{Scal}(p)}} &= \int_{\mathbb{B}_p}\int_{\mathbb{R}} |f(t,\zeta) - f_N(t,\zeta)|^2 \lambda(p(\zeta),t)\,\mathrm{d}t\,\mathrm{d}V(\zeta) \\
&= \int_{\mathbb{B}_p}\int_{\mathbb{R}} |f(t,\zeta) - f_N(t,\zeta)|^2 \omega_{\alpha(\zeta),\beta(\zeta)}(t)\,\mathrm{d}t\,\mathrm{d}V(\zeta) \\
&= \int_{\mathbb{B}_p} \|f(\cdot,\zeta) - f_N(\cdot,\zeta)\|^2_{L^2(\mathbb{R},\omega_{\alpha(\zeta),\beta(\zeta)})}\,\mathrm{d}V(\zeta) \to 0,
\end{aligned}$$

as $N \to \infty$. This shows that $f_N \to f$ in $\mathcal{L}_{\mathrm{Scal}(p)}$ and consequently, $\frac{\partial f_N}{\partial \bar{\zeta}_j} \to \frac{\partial f}{\partial \bar{\zeta}_j}$ in sense of distributions for $1 \le j \le n$. This shows that $\widetilde{T}_V f \in \mathcal{X}_p$ and that $\widetilde{T}_V$ is surjective and thus $\widetilde{T}_V : \mathcal{X}_p \to A^2(\mathcal{C}_p)$ is an isometric isomorphism. The map $\widetilde{T}_V^{-1}$ given by equation (III.38) is the inverse of $\widetilde{T}_V$.

Recall that $\Psi^* : A^2(\mathcal{C}_p) \to A^2(\mathcal{U}_p)$ as in (III.32) is an isometric isomorphism. Then $\Psi^* \circ \widetilde{T}_V$ is an isometric isomorphism from $\mathcal{X}_p$ to $A^2(\mathcal{U}_p)$. We now show that $T_V = \Psi^* \circ \widetilde{T}_V$. For $f \in \mathcal{X}_p$ and $(z,w) \in \mathcal{U}_p$ we have

$$\begin{aligned}
\Psi^*(\widetilde{T}_V f)(z,w) &= \widetilde{T}_V f(\Psi(z,w)) \cdot \det\, \Psi'(z,w) \\
&= \left(\int_{\mathbb{R}} f\left(t, \frac{w_1}{z^{1/2m_1}}, \ldots, \frac{w_n}{z^{1/2m_n}}\right) \frac{z^{2\pi i t}}{z}\,\mathrm{d}t \right) \frac{1}{z^{1/2\mu}} \\
&= T_V f(z,w),
\end{aligned}$$

where we used the fact that for $(z,w) \in \mathcal{U}_p$,

$$\det\, \Psi'(z,w) = \frac{1}{z^{1/2\mu}}.$$

This completes the proof of the theorem. □

Remark. The biholomorphism $\Psi : \mathcal{U}_p \to \mathcal{C}_p$ of Lemma III.3 has an inverse $\Psi^{-1} : \mathcal{C}_p \to \mathcal{U}_p$ given by

$$\Psi^{-1}(\gamma, \zeta) = \left(\gamma, \gamma^{1/2m_1}\zeta_1, \ldots, \gamma^{1/2m_n}\zeta_n\right), \quad \text{for } (\gamma, \zeta) \in \mathcal{C}_p.$$

The biholomorphic map Ψ^{-1} induces an isometric isomorphism induces an isometric isomorphism $(\Psi^{-1})^* : A^2(\mathcal{U}_p) \to A^2(\mathcal{C}_p)$. Since $\Psi \circ \Psi^{-1}$ is the identity automorphism of $\mathcal{U}_p$, it follows that the unitary automorphism $(\Psi^{-1})^* \circ \Psi^*$ of $A^2(\mathcal{U}_p)$ induced by $\Psi \circ \Psi^{-1}$ is the identity map. This shows that $(\Psi^*)^{-1} = (\Psi^{-1})^*$. Since $T_V = \Psi^* \circ \widetilde{T}_V$, it follows that

$$T_V^{-1} = \widetilde{T}_V^{-1} \circ (\Psi^*)^{-1} = \widetilde{T}_V^{-1} \circ (\Psi^{-1})^*.$$

Thus, for $F \in A^2(\mathcal{U}_p)$ we have

$$\begin{aligned}
T_V^{-1}F(t,\zeta) &= \widetilde{T}_V^{-1}\left((\Psi^{-1})^*F(\gamma,\zeta)\right) \\
&= \widetilde{T}_V^{-1}\left(F(\gamma, \gamma^{1/2m_1}\zeta_1, \ldots, \gamma^{1/2m_n}\zeta_n)\gamma^{1/2\mu}\right) \\
&= \int_{\mathbb{R}} e^{(x+ic)/2\mu}(\Phi^*)^{-1}F(x+ic,\zeta)e^{2\pi ct}e^{-i2\pi xt}\,\mathrm{d}x
\end{aligned}$$

where the map Φ^* is as in the proof of Theorem 11. Here we used the formula (III.38) for $\widetilde{T}_V^{-1}$.

CHAPTER IV

BERGMAN KERNEL OF POLYNOMIAL HALF SPACES

In this chapter we first recall the notion of a Reproducing Kernel Hilbert space, and also that a weighted Bergman space on a domain in $\mathbb{C}^n$ is a reproducing kernel Hilbert space. We then describe a method in Proposition IV.2 to compute the reproducing kernel of a reproducing kernel Hilbert space. Finally we compute the well known formula for Bergman kernel of Siegel Upper Half space in $\mathbb{C}^{n+1}$.

IV.1. Reproducing Kernel Hilbert Spaces

IV.1.1. Reproducing Kernel Hilbert Spaces

Let H be a Hilbert space. By the Riesz representation theorem, for each bounded linear functional $\varphi \in H^*$, there is an element $R_H(\varphi) \in H$ such that

$$\varphi(f) = \langle f, R_H(\varphi) \rangle_H, \quad f \in H. \tag{IV.1}$$

We call the map $R_H : H^* \to H$ the *Riesz map*. The map R_H is a conjugate linear isometry of Hilbert spaces.

Given a set Ω, let H be a Hilbert space of functions on Ω. Recall that H is called a *reproducing kernel Hilbert space* on Ω if, for each z in Ω, the evaluation map $e_z : H \to \mathbb{C}$ given by $e_z f = f(z)$ is a bounded linear functional on H. The function $K : \Omega \times \Omega \to \mathbb{C}$ given by

$$K(z, Z) = \overline{R_H(e_z)(Z)}, \tag{IV.2}$$

is called the *reproducing kernel* for H, where R_H is as in (IV.1). For each $z \in \Omega$ and $f \in H$ we have,

$$\left\langle f, \overline{K(z,\cdot)} \right\rangle_H = \langle f, R_H(e_z) \rangle_H = f(z),$$

which is why K is called the reproducing kernel for H. The following proposition gives the most important example of a reproducing kernel Hilbert space for the purposes of this thesis.

Proposition IV.1. *Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega,\lambda)$ be the weighted Bergman space on Ω with weight the continuous function $\lambda > 0$. Then $A^2(\Omega,\lambda)$ is a reproducing kernel Hilbert space.*

Proof. Note that $A^2(\Omega,\lambda)$ is a Hilbert space of functions on Ω. To show that $A^2(\Omega,\lambda)$ is a reproducing kernel Hilbert space, it suffices to show that the evaluation map $e_z : A^2(\Omega) \to \mathbb{C}$ given by

$$e_z(f) = f(z), \quad f \in A^2(\Omega,\lambda)$$

is a bounded linear functional on $A^2(\Omega,\lambda)$. It is clear that e_z is a linear functional. To see that it is bounded, note that for each $f \in A^2(\Omega,\lambda)$ we have

$$|e_z(f)| = |f(z)| \leq C_z \|f\|_{A^2(\Omega,\lambda)},$$

where the last inequality follows from the Bergman inequality (II.1) applied to the compact set $\{z\}$. □

In particular, $A^2(\Omega)$ the Bergman space on $\Omega \subset \mathbb{C}^n$ is a reproducing kernel Hilbert space. The reproducing kernel for $A^2(\Omega)$ is called the *Bergman kernel* of Ω.

IV.1.2. Computing Reproducing Kernels

Proposition IV.2. *(cf. [8, Lemma IX.3.5]) Let H be a reproducing kernel Hilbert space on a set Ω. Suppose L is another Hilbert space such that there is an isometric isomorphism $T : L \to H$. Then the function $K : \Omega \times \Omega \to \mathbb{C}$ given by*

$$K(z,Z) = \langle R_L(e_Z \circ T), R_L(e_z \circ T)\rangle_L, \tag{IV.3}$$

is the reproducing kernel for H.

Proof. The map $e_z \circ T : L \to \mathbb{C}$ is given by

$$(e_z \circ T)f = e_z(Tf) = Tf(z). \tag{IV.4}$$

It is clear that $e_z \circ T$ is bounded and linear since each of e_z and T are bounded and linear. Consequently, $e_z \circ T$ is a bounded linear funtional on L. Thus, we have

$$\begin{aligned} Tf(z) &= (e_z \circ T)f && \text{(from equation } (IV.4)) \\ &= \langle f, R_L(e_z \circ T)\rangle_L && \text{(by (IV.1))} \\ &= \langle Tf, TR_L(e_z \circ T)\rangle_H. && (T \text{ is an isometry}) \end{aligned}$$

where R_L is the Riesz map for the Hilbert space L. That is for every $g \in H$ and each $z \in \Omega$, we have

$$g(z) = \langle g, TR_L(e_z \circ T)\rangle_H, \quad \text{i.e.,} \quad TR_L(e_z \circ T) = R_H(e_z) \tag{IV.5}$$

Thus, the reproducing kernel K for the Hilbert space H is given by

$$\begin{aligned} K(z,Z) &= \overline{R_H(e_z)(Z)} && \text{(By (IV.2))} \\ &= \overline{TR_L(e_z \circ T)(Z)} && \text{(By equation (IV.5))} \\ &= \overline{e_Z(TR_L(e_z \circ T))} \\ &= \overline{\langle TR_L(e_z \circ T), TR_L(e_Z \circ T)\rangle_H} && \text{(By equation } (IV.1)) \\ &= \langle TR_L(e_Z \circ T), TR_L(e_z \circ T)\rangle_H \\ &= \langle R_L(e_Z \circ T), R_L(e_z \circ T)\rangle_L. && (T \text{ is an isometry}) \end{aligned}$$

□

If we take $T : H \to H$ to be the identity map in Proposition IV.2, we have the alternative representation of the reproducing kernel

$$K(z,Z) = \langle R_H(e_Z), R_H(e_z)\rangle_H. \tag{IV.6}$$

It immediately follows that the reproducing kernel has the Hermitian symmetry

$$K(z,Z) = \overline{K(Z,z)}. \tag{IV.7}$$

Let K denote the reproducing kernel for the weighted Bergman space $A^2(\Omega,\lambda)$ on $\Omega\subset\mathbb{C}^n$ with weight λ. Then the following identity is an immediate consequence of (IV.6)

$$K(z,Z)=\int_\Omega K(z,\zeta)\overline{K(Z,\zeta)}\lambda(\zeta)\,\mathrm{d}V(\zeta). \tag{IV.8}$$

Indeed, for each z and each w in Ω, by (IV.6), we have

$$\begin{aligned}K(z,Z)&=\left\langle R_{A^2(\Omega,\lambda)}(e_Z),R_{A^2(\Omega,\lambda)}(e_z)\right\rangle_{A^2(\Omega,\lambda)}\\&=\int_\Omega R_{A^2(\Omega,\lambda)}(e_Z)(\zeta)\overline{R_{A^2(\Omega,\lambda)}(e_z)(\zeta)}\lambda(\zeta)\,\mathrm{d}V(\zeta)\\&=\int_\Omega \overline{K(Z,\zeta)}K(z,\zeta)\lambda(\zeta)\,\mathrm{d}V(\zeta). \qquad \text{(By equation (IV.2))}\end{aligned}$$

We now show that Proposition IV.2 can be used to recapture the familiar formula for the reproducing kernel of a weighted Bergman space.

Corollary IV.3. *Let Ω be a domain in $\mathbb{C}^n$ and let $\{\phi_j\}_{j=1}^\infty$ be an orthonormal basis for $A^2(\Omega,\lambda)$. Then K, the reproducing kernel for $A^2(\Omega,\lambda)$ is given by*

$$K(z,Z)=\sum_{j=1}^\infty \phi_j(z)\overline{\phi_j(Z)}. \tag{IV.9}$$

Proof. Let ℓ^2 be the Hilbert space of square summable sequences, and consider the map $T:\ell^2\to A^2(\Omega,\lambda)$ given by

$$a:=(a_j)_{j=1}^\infty\mapsto\sum_{j=1}^\infty a_j\phi_j:=Ta.$$

The map T is an isometry, for we have

$$\|a\|_{\ell^2}^2=\sum_{j=1}^\infty\left|a_j\right|^2=\|Ta\|_{A^2(\Omega)}^2,$$

where the last equality follows from Parseval's formula. Thus, T is injective. Moreover T is surjective since $\{\phi_j\}_{j=1}^\infty$ is a basis for $A^2(\Omega,\lambda)$ and consequently T is an isometric isomorphism.

Let $z \in \Omega$ be fixed. The map $e_z \circ T : \ell^2 \to \mathbb{C}$ is given for $a \in \ell^2$ by

$$(e_z \circ T)a = \sum_{j=1}^{\infty} a_j \phi_j(z).$$

Denoting the sequence of complex numbers $\left(\overline{\phi_j(z)}\right)_{j=1}^{\infty}$ in ℓ^2 by $\overline{\phi(z)}$, we see that for $a \in \ell^2$,

$$(e_z \circ T)a = \left\langle a, \overline{\phi(z)} \right\rangle_{\ell^2}.$$

Thus, by definition (IV.1) of the Riesz map, we have

$$R_{\ell^2}(e_z \circ T) = \overline{\phi(z)}.$$

Applying Proposition IV.2 to $T : \ell^2 \to A^2(\Omega, \lambda)$ gives

$$K(z,Z) = \langle R_{\ell^2}(e_Z \circ T), R_{\ell^2}(e_z \circ T) \rangle_{\ell^2} = \left\langle \overline{\phi(Z)}, \overline{\phi(z)} \right\rangle_{\ell^2} = \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(Z)}.$$

□

The well known transformation formula for Bergman kernel under biholomorphic maps can also be obtained by using Proposition IV.2 as we show below.

Corollary IV.4 (Transformation of the Bergman Kernel)**.** *Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic map between two domains Ω_1 and Ω_2 in $\mathbb{C}^n$. Then, the Bergman kernels K_1 of Ω_1 and K_2 of Ω_2 are related as*

$$K_1(z,Z) = \det \Phi'(z) \, K_2(\Phi(z), \Phi(Z)) \, \overline{\det \Phi'(Z)}.$$

Proof. Recall from Lemma II.3, that the biholomorphic map $\Phi : \Omega_1 \to \Omega_2$ induces an isometric isomorphism $\Phi^* : A^2(\Omega_2) \to A^2(\Omega_1)$ given by

$$\Phi^* F(z) = F(\Phi(z)) \cdot \det \Phi'(z).$$

Thus, the map $e_z \circ \Phi^* : A^2(\Omega_2) \to \mathbb{C}$ is given by

$$(e_z \circ \Phi^*)F = F(\Phi(z)) \cdot \det \Phi'(z) = \left\langle F, R_{A^2(\Omega_2)}(e_{\Phi(z)}) \overline{\det \Phi'(z)} \right\rangle_{A^2(\Omega_2)}, \tag{IV.10}$$

for each $F \in A^2(\Omega_2)$. Comparing equation (IV.10) with definition (IV.1) of Riesz map, we see that

$$R_{A^2(\Omega_2)}(e_z \circ \Phi^*) = R_{A^2(\Omega_2)}(e_{\Phi(z)}) \cdot \overline{\det \Phi'(z)}.$$

Applying Proposition IV.2 to $\Phi^* : A^2(\Omega_2) \to A^2(\Omega_1)$, we get

$$\begin{aligned}
K_1(z,Z) &= \left\langle R_{A^2(\Omega_2)}(e_{\Phi(Z)})\overline{\det \Phi'(Z)}, R_{A^2(\Omega_2)}(e_{\Phi(z)})\overline{\det \Phi'(z)} \right\rangle_{A^2(\Omega_2)} \\
&= \overline{\det \Phi'(Z)} \cdot \left\langle R_{A^2(\Omega_2)}(e_{\Phi(Z)}), R_{A^2(\Omega_2)}(e_{\Phi(z)}) \right\rangle_{A^2(\Omega_2)} \cdot \det \Phi'(z) \\
&= \overline{\det \Phi'(Z)} \cdot \overline{\left\langle R_{A^2(\Omega_2)}(e_{\Phi(z)}), R_{A^2(\Omega_2)}(e_{\Phi(Z)}) \right\rangle}_{A^2(\Omega_2)} \cdot \det \Phi'(z) \\
&= \overline{\det \Phi'(Z)} \cdot \overline{R_{A^2(\Omega_2)}(e_{\Phi(z)})(\Phi(Z))} \cdot \det \Phi'(z) \\
&= \overline{\det \Phi'(Z)} \cdot K_2(\Phi(z), \Phi(Z)) \cdot \det \Phi'(z).
\end{aligned}$$

□

IV.2. Asymptotic Behavior of the Bergman Kernel

We first prove a result that is needed in the proof of Theorem 5.

Proposition IV.5. *Let $g \in \mathscr{C}^1(\mathbb{R}^{N-1})$ and $\mathcal{U}$ be a domain in $\mathbb{R}^N$ given by*

$$x_N - g(x') > 0, \textit{ for } (x_1, \ldots, x_{N-1}, x_N) = (x', x_N) \in \mathbb{R}^N.$$

Suppose that $g \geq 0$ on $\mathbb{R}^N$, $g(0) = 0$ and $\triangledown g(0) = 0$. If $d(x, \partial\mathcal{U})$ denotes the distance of a point $x \in \mathbb{R}^N$ to the boundary $\partial\mathcal{U}$ of $\mathcal{U}$, then we have

$$\lim_{\substack{x \to 0 \\ x \in \mathcal{U}}} \frac{x_N}{d(x, \partial\mathcal{U})} = 1.$$

Proof. For $x \in \mathcal{U}$, let $y(x) \in \partial\mathcal{U}$ be the point on the boundary of $\mathcal{U}$ closest to x. First we show that the set of points in $\partial\mathcal{U}$ closest to x is non-empty. Let $\delta = \inf\{|z - x| \mid z \in \partial\mathcal{U}\}$. Then, there is a sequence $\{z_n\}_{n=1}^{\infty} \in \partial\mathcal{U}$ such that $|z_n - x| \to \delta$ as $n \to \infty$. Then there is an $N > 0$ such that for all

$n > N$ we have $|z_n - x| < \delta + 1$. Then for such n, we have

$$|z_n| \leq |z_n - x| + |x| \leq |x| + \delta + 1,$$

which shows that the sequence $\{z_n\}_{n=1}^{\infty}$ is bounded. Thus, there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ that converges to a point $y(x) \in \partial\mathcal{U}$, since $\partial\mathcal{U}$ is closed. This shows that there is at least one point $y(x)$ in $\partial\mathcal{U}$ that is closest to $x \in \mathcal{U}$. Then we have

$$\begin{aligned}
|y(x)| &\leq |y(x) - x| + |x| \\
&= \left|(y'(x), y_N(x)) - (x', x_N)\right| + |x| \\
&\leq \left|(x', g(x')) - (x', x_N)\right| + |x| \quad \text{(since } (x', g(x')) \in \partial\mathcal{U}) \\
&= \left|x_N - g(x')\right| + |x| \\
&\leq |x_N| + |x| \qquad\qquad\qquad\qquad \text{(since } g \geq 0) \\
&\leq 2\,|x|\,.
\end{aligned}$$

Since the vector $x - y(x)$ is normal to the boundary $\partial\mathcal{U}$ at $y(x)$, there is a $\lambda \in \mathbb{R}$ for which we have

$$\begin{aligned}
x_j - y_j(x) &= -\lambda \frac{\partial g}{\partial x_j}(y'),\ 1 \leq j \leq N-1 \\
x_N - y_N(x) &= \lambda.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
d(x, \partial\mathcal{U}) &= |x - y(x)| \\
&= \sqrt{\lambda^2 + \lambda^2 \left|\bigtriangledown g(y')\right|^2} \\
&= |x_N - y_N(x)| \sqrt{1 + \left|\bigtriangledown g(y')\right|^2}.
\end{aligned}$$

Since $|y(x)| \leq 2\,|x|$, we get $y(x) \to 0$ as $x \to 0$. Thus, we obtain

$$\lim_{\substack{x \to 0 \\ x \in \mathcal{U}}} \frac{x_N}{d(x, \partial\mathcal{U})} = \lim_{\substack{x \to 0 \\ x \in \mathcal{U}}} \frac{x_N}{|x_N - y(x)| \sqrt{1 + \left|\bigtriangledown g(y')\right|^2}}$$

$$= 1,$$

as $y(0) = 0$ and $\bigtriangledown g(0) = 0$. □

Corollary IV.6. *Let $\mathcal{U}_p$ be a polynomial half space. For $(z,w) \in \mathcal{U}_p$, let $d(z,w)$ be the distance of (z,w) to the boundary of $\mathcal{U}_p$. Then we have*

$$\lim_{(z,w)\to 0} \frac{\operatorname{Im} z}{d(z,w)} = 1.$$

Proof. The domain $\mathcal{U}_p \subset R^{2n+2}$ is given by $y - p(u_1, v_1, \ldots, u_n, v_n) > 0$, where $z = x + iy$ and $w_j = u_j + iv_j$ for $1 \leq j \leq n$.

Since p is a positive polynomial with no constant terms, we have $p \geq 0$, $p(0) = 0$ and $p \in \mathscr{C}^1(\mathbb{R}^{2n+1})$, and the result of previous proposition applies. □

Recall that for $c > 0$, the domain Γ_c is the subset of $\mathcal{U}_p$ given by

$$\Gamma_c\{(z,w) \in \mathcal{U}_p \,|\, |z| > c\,|w|\}.$$

Proof of Theorem 5. For a point $(z,w) \in \mathcal{U}_p$, write $z = x + iy$. Recall that for $(z,w) \in \mathcal{U}_p$, the translation $\tau_{-x} : \mathcal{U}_p \to \mathcal{U}_p$ given by

$$\tau_{-x}(z,w) = (z - x, w)$$

is an automorphism of $\mathcal{U}_p$. Thus, an application of Corollary IV.4 now shows that

$$\begin{aligned} K_{\text{diag}}(z,w) &= \left|\det \tau'_{-x}(z,w)\right|^2 K_{\text{diag}}(\tau_{-x}(z,w)) \\ &= K_{\text{diag}}(iy,w), \end{aligned} \tag{IV.11}$$

where we used the fact that $\det \tau'_{-x}(z,w) = 1$ to get the last equality. Also recall that $\rho_y : \mathcal{U}_p \to \mathcal{U}_p$, for $(z,w) \in \mathcal{U}_p$ given by

$$\rho_y(z, w_1, \cdots, w_n) = \left(yz, y^{1/2m_1} w_1, \cdots, y^{1/2m_n} w_n\right)$$

is an automorphism of $\mathcal{U}_p$, where $y = \operatorname{Im} z$. Now, applying Corollary IV.4 we get

$$\begin{aligned} K_{\text{diag}}(iy, w) &= \left|\det \rho_y(iy, w)\right|^2 K_{\text{diag}}(\rho_y(iy, w)) \\ &= y^{2+1/\mu} K_{\text{diag}}(\rho_y(iy, w)) \end{aligned} \tag{IV.12}$$

because $\det \rho_y(iy, w) = y^{1+1/2\mu}$ where $1/\mu = \sum_{j=1}^{n} 1/m_j$. From equation (IV.12), we get

$$K_{\text{diag}}(i, \rho_y^{-1}(w)) = y^{2+1/\mu} K_{\text{diag}}(iy, w). \tag{IV.13}$$

Note that $\rho_y^{-1}(w) = \rho_{1/y}(w)$, so we have

$$\begin{aligned} \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} \left|\rho_{1/y}(w_1, \ldots, w_n)\right|^2 &= \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} \frac{|w_1|^2}{y^{1/m_1}} + \cdots \frac{|w_n|^2}{y^{1/m_n}} \\ &\leq \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} \frac{|w|^2}{y} \\ &= \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} y \cdot \frac{|w|^2}{y^2} \\ &\leq \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} y \cdot \frac{1}{c^2} \quad \text{(Since } (iy, w) \in \Gamma_c, \text{ we get } |iy| > c\,|w|) \\ &= 0. \end{aligned}$$

Using this and equations (IV.11) and (IV.13) we may write

$$\begin{aligned} K_{\text{diag}}(i, 0) &= \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} K_{\text{diag}}(i, \rho_y^{-1}(w)) \\ &= \lim_{\substack{(iy,w)\to(0,0) \\ (iy,w)\in\Gamma_c}} y^{2+1/\mu} K_{\text{diag}}(iy, w) \\ &= \lim_{\substack{(z,w)\to(0,0) \\ (z,w)\in\Gamma_c}} y^{2+1/\mu} K_{\text{diag}}(z, w). \end{aligned}$$

Combining this with Corollary IV.6, we obtain

$$\lim_{\substack{(z,w)\to(0,0)\\(z,w)\in\Gamma_c}} (d(z,w))^{2+1/\mu} K_{\text{diag}}(z,w) = K_{\text{diag}}(i,0),$$

which completes the proof. □

IV.3. A Formula for the Bergman Kernel of $\mathcal{E}_p$

Recall that for $k \in \mathbb{N}$, $\mathcal{W}_p(k)$ is the weighted Bergman space on $\mathbb{B}_p$ with respect to the weight $w \mapsto (1-p(w))^{k+1}$, i.e.,

$$\mathcal{W}_p(k) = A^2\left(\mathbb{B}_p, (1-p)^{k+1}\right).$$

Also recall that $\mathcal{Y}_p$ is the Hilbert space of sequences $a = (a_k)_{k=0}^{\infty}$ such that $a_k \in \mathcal{W}_p(k)$ for each $k \in \mathbb{N}$, and

$$\|a\|_{\mathcal{Y}_p}^2 = \pi \sum_{k=0}^{\infty} \frac{1}{k+1} \|a_k\|_{\mathcal{W}_p(k)}^2 < \infty.$$

Proof of Theorem 6. Let $T : \mathcal{Y}_p \to A^2(\mathcal{E}_p)$ be the map as in Theorem 2, which for a sequence $a = (a_0, a_1, \dots) \in \mathcal{Y}_p$ is given by

$$Ta(z,w) = \sum_{k=0} a_k(w) z^k, \ \text{ for } (z,w) \in \mathcal{E}_p.$$

Let $e_{(z,w)}$ be the evaluation functional at a point $(z,w) \in \mathcal{E}_p$. Then, for $a \in \mathcal{Y}_p$ the map $e_{(z,w)} \circ T : \mathcal{Y}_p \to \mathbb{C}$ is given by

$$\begin{aligned}(e_{(z,w)} \circ T)a &= Ta(z,w) \\ &= \sum_{k=0}^{\infty} a_k(w) z^k \\ &= \sum_{k=0}^{\infty} \left\langle a_k, \overline{Y_p(k; w, \cdot)} \right\rangle_{\mathcal{W}_p(k)} \\ &= \sum_{k=0}^{\infty} \frac{\pi}{k+1} \left(\int_{\mathbb{B}_p} a_k(\zeta) Y_p(k; w, \zeta) \frac{(k+1)z^k}{\pi} (1-p(\zeta))^{k+1}\, dV(\zeta) \right), \qquad \text{(IV.14)}\end{aligned}$$

where the last equality follows from the reproducing property of the kernel $Y_p(k;\cdot,\cdot)$. On the other hand, the image $R_{\mathcal{Y}_p}\left(e_{(z,w)}\circ T\right)$ of the functional $e_{(z,w)}\circ T\in\mathcal{Y}_p^*$ under the Riesz map $R_{\mathcal{Y}_p}:\mathcal{Y}_p^*\to\mathcal{Y}_p$ is given by

$$\begin{aligned}(e_{(z,w)}\circ T)a &= \left\langle a, R_{\mathcal{Y}_p}(e_{(z,w)}\circ T)\right\rangle_{\mathcal{Y}_p}, \quad a\in\mathcal{Y}_p\\ &= \sum_{k=0}^{\infty}\frac{\pi}{k+1}\int_{\mathbb{B}_p} a_k(\zeta)\overline{R_{\mathcal{Y}_p}(e_{(z,w)}\circ T)(k,\zeta)}(1-p(\zeta))^{k+1}\,\mathrm{d}V(\zeta) \end{aligned}\tag{IV.15}$$

For every $k\in\mathbb{N}$ and every $\zeta\in\mathbb{B}_p$ we claim that

$$Y_p(k;w,\zeta)\frac{(k+1)z^k}{\pi}=\overline{R_{\mathcal{Y}_p}(e_{(z,w)}\circ T)(k,\zeta)}.$$

Let $\phi:\mathbb{N}\times\mathbb{B}_p\to\mathbb{C}$ be the difference of the two quantities above, i.e.,

$$\phi(k,\zeta)=Y_p(k;w,\zeta)\frac{(k+1)z^k}{\pi}-\overline{R_{\mathcal{Y}_p}(e_{(z,w)}\circ T)(k,\zeta)}.$$

It then follows from equations (IV.14) and (IV.15) that

$$\sum_{k=0}^{\infty}\frac{\pi}{k+1}\int_{\mathbb{B}_p} a_k(\zeta)\phi(k,\zeta)(1-p(\zeta))^{k+1}\,\mathrm{d}V(\zeta)=0 \tag{IV.16}$$

for all $a\in\mathcal{Y}_p$. Let $a=(a_0,a_1,\dots,)\in\mathcal{Y}_p$ be such that $a_k\in\mathcal{W}_p(k)$ for a $k\in\mathbb{N}$, and $a_j\equiv 0$ on $\mathbb{B}_p$, for $j\in\mathbb{N}$ and $j\neq k$. For this choice of the sequence a in $\mathcal{Y}_p$, equation (IV.16) reduces to

$$\pi\int_{\mathbb{B}_p} a_k(\zeta)\phi(k,\zeta)(1-p(\zeta))^{k+1}\,\mathrm{d}V(\zeta)=0,$$

for any $a_k\in\mathcal{W}_p(k)$. Thus, we see that $\phi(k,\zeta)=0$ for almost all $\zeta\in\mathbb{B}_p$. Since k was arbitrary we see that $\phi(k,\zeta)=0$ for all $k\in\mathbb{N}$ and almost all $\zeta\in\mathbb{B}_p$. This proves the claim, and we have

$$R_{\mathcal{Y}_p}\left(e_{(z,w)}\circ T\right)(k,\zeta)=\overline{Y_p(k;w,\zeta)}\frac{(k+1)\bar{z}^k}{\pi},\quad (k,\zeta)\in\mathbb{N}\times\mathbb{B}_p.$$

Applying Proposition IV.2 to isometric isomorphism $T:\mathcal{Y}_p\to A^2(\mathcal{E}_p)$, we get

$$B(z,w;Z,W)=\left\langle R_{\mathcal{Y}_p}(e_{(Z,W)}\circ T), R_{\mathcal{Y}_p}(e_{(z,w)}\circ T)\right\rangle_{\mathcal{Y}_p}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\pi}{k+1} \int_{\mathbb{B}_p} \left(Y_p(k;w,\zeta) \frac{(k+1)z^k}{\pi} \right) \left(\overline{Y_p(k;W,\zeta)} \frac{(k+1)\overline{Z}^k}{\pi} \right) (1-p(\zeta))^{k+1} \, \mathrm{d}V(\zeta) \\
&= \sum_{k=0}^{\infty} \frac{k+1}{\pi} \left(\int_{\mathbb{B}_p} Y_p(k;w,\zeta) \overline{Y_p(k;W,\zeta)} (1-p(\zeta))^{k+1} \, \mathrm{d}V(\zeta) \right) z^k \overline{Z}^k \\
&= \sum_{k=0}^{\infty} \frac{k+1}{\pi} Y_p(k;w,W) z^k \overline{Z}^k,
\end{aligned}
$$

where we used (IV.8) to arrive at the last equality. □

IV.4. A Formula for the Bergman Kernel of $\mathcal{U}_p$

To prove Theorem 7, we will need the following lemma.

Lemma IV.7. *For $t > 0$, let $\mathcal{S}_p(t)$ be the weighted Bergman space on $\mathbb{C}^n$ with weight $w \mapsto e^{-4\pi p(w)t}$. Let the reproducing kernel for $\mathcal{S}_p(t)$ be denoted by $H_p(t;\cdot,\cdot)$. Let $\widehat{\rho}_t : \mathbb{C}^n \to \mathbb{C}^n$ be given by*

$$
\widehat{\rho}_t(w_1,\dots,w_n) = \left(t^{1/2m_1} w_1, \cdots, t^{1/2m_n} w_n \right), \quad \text{for } w \in \mathbb{C}^n.
$$

Then for all $w, W \in \mathbb{C}^n$, we have

$$
H_p(t;w,W) = t^{1/\mu} H_p\left(1; \widehat{\rho}_t(w), \widehat{\rho}_t(W)\right), \tag{IV.17}
$$

where for the multi-index $m = (m_1,\dots,m_n)$ we let $1/\mu = \sum_{j=1}^{n} 1/m_j$.

Proof. We first show that the map $D_t : \mathcal{S}_p(1) \to \mathcal{S}_p(t)$ given by

$$
D_t f(\zeta) = t^{1/2\mu} f\left(\widehat{\rho}_t(\zeta)\right) \tag{IV.18}
$$

is an isometric isomorphism. Let $w = \widehat{\rho}_t(\zeta)$. Then by Lemma II.4 we have $p(w) = p(\widehat{\rho}_t(\zeta)) = t p(\zeta)$ and

$$
\mathrm{d}V(w) = t^{1/\mu} \, \mathrm{d}V(\zeta).
$$

Further we have

$$
\|f\|^2_{\mathcal{S}_p(1)} = \int_{\mathbb{C}^n} |f(w)|^2 e^{-4\pi p(w)} \, \mathrm{d}V(w)
$$

$$
\begin{aligned}
&= \int_{\mathbb{C}^n} |f(\widehat{\rho}_t(\zeta))|^2 e^{-4\pi p(\zeta)t} t^{1/\mu}\, \mathrm{d}V(\zeta) \\
&= \int_{\mathbb{C}^n} \left| t^{1/2\mu} f(\widehat{\rho}_t(\zeta))\right|^2 e^{-4\pi p(\zeta)t}\, \mathrm{d}V(\zeta). \\
&= \|D_t f\|^2_{\mathcal{S}_p(t)} \qquad \text{(IV.19)}
\end{aligned}
$$

It follows from equation (IV.19) that $f \in \mathcal{S}_p(1)$ if and only if $D_t f \in \mathcal{S}_p(t)$. Moreover, if $f \in \mathcal{S}_p(1)$, then it follows from equation (IV.19) that

$$
\|f\|_{\mathcal{S}_p(1)} = \|D_t f\|_{\mathcal{S}_p(t)},
$$

which shows that D_t is an isometry and hence injective. For $F \in \mathcal{S}_p(t)$ let the function $f : \mathbb{C}^n \to \mathbb{C}$ be given by $f(\zeta) = t^{-1/2\mu} F\left(\widehat{\rho}_{1/t}(\zeta)\right)$. We have

$$
D_t f(\zeta) = t^{1/2\mu} \cdot f(\widehat{\rho}_t(\zeta)) = t^{1/2\mu} \cdot t^{-1/2\mu} F\left(\widehat{\rho}_{1/t}(\widehat{\rho}_t(\zeta))\right) = F(\zeta).
$$

Since $D_t f = F \in \mathcal{S}_p(t)$ it follows that $f \in \mathcal{S}_p(1)$. This shows that D_t is surjective, and hence an isometric isomorphism.

We now apply Lemma IV.2 to the isometric isomorphism D_t in (IV.18) to obtain the equation (IV.17). Applying equation (IV.1) (which defines the Riesz map) to the functional $e_\zeta \circ D_t : \mathcal{S}_p(1) \to \mathbb{C}$ we obtain

$$
\begin{aligned}
\left\langle f, R_{\mathcal{S}_p(1)}(e_\zeta \circ D_t)\right\rangle_{\mathcal{S}_p(1)} &= (e_\zeta \circ D_t) f \\
&= e_\zeta(D_t f) \\
&= D_t f(\zeta) \\
&= f(\widehat{\rho}_t(\zeta)) t^{1/2\mu} \\
&= \left\langle f, R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(\zeta)})\right\rangle_{\mathcal{S}_p(1)} t^{1/2\mu} \\
&= \left\langle f, t^{1/2\mu} R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(\zeta)})\right\rangle_{\mathcal{S}_p(1)}.
\end{aligned}
$$

This shows that

$$R_{\mathcal{S}_p(1)}(e_\zeta \circ D_t) = t^{1/2\mu} R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(\zeta)}).$$

Applying Proposition IV.2 to the isometric isomorphism D_t in equation (IV.18), we see that the reproducing kernels $H_p(1;\cdot,\cdot)$ and $H_p(t;\cdot,\cdot)$ are related as

$$\begin{aligned} H_p(t;w,W) &= \left\langle R_{\mathcal{S}_p(1)}(e_W \circ D_t), R_{\mathcal{S}_p(1)}(e_w \circ D_t) \right\rangle_{\mathcal{S}_p(1)} \\ &= \left\langle t^{1/2\mu} R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(W)}), t^{1/2\mu} R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(w)}) \right\rangle_{\mathcal{S}_p(1)} \\ &= t^{1/\mu} \left\langle R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(W)}), R_{\mathcal{S}_p(1)}(e_{\widehat{\rho}_t(w)}) \right\rangle_{\mathcal{S}_p(1)} \\ &= t^{1/\mu} H_p\left(1; \widehat{\rho}_t(w), \widehat{\rho}_t(W)\right), \end{aligned}$$

where the last equality follows from (IV.6). □

Proof of Theorem 7. Let $T_S : \mathcal{H}_p \to A^2(\mathcal{U}_p)$ be the isometric isomorphism given by

$$T_S f(z,w) = \int_0^\infty f(t,w) e^{i2\pi zt}\, \mathrm{d}t, \quad \text{for } (z,w) \in \mathcal{U}_p$$

as in Theorem 7. We begin by showing that the image $R_{\mathcal{H}_p}\left(e_{(z,w)} \circ T_S\right)$ of the functional $e_{(z,w)} \circ T_S \in \mathcal{H}_p^*$ under the Riesz map $R_{\mathcal{H}_p} : \mathcal{H}_p^* \to \mathcal{H}_p$ is given by

$$R_{\mathcal{H}_p}\left(e_{(z,w)} \circ T_S\right)(t,\zeta) = 4\pi t \overline{H_p(t;w,\zeta)} e^{-i2\pi \bar{z} t}.$$

For each $(z,w) \in \mathcal{U}_p$, consider the map $\chi_{z,w} : (0,\infty) \times \mathbb{C}^n$ given by

$$\chi_{z,w}(t,\zeta) = 4\pi t H_p(t;w,\zeta) e^{i2\pi zt}, \quad \text{for } (t,w) \in (0,\infty) \times \mathbb{C}^n.$$

Letting $z = x + iy$, we obtain

$$\begin{aligned} \|\chi_{z,w}\|_{\mathcal{H}_p} &= \int_0^\infty \int_{\mathbb{C}^n} |\chi_{z,w}(t,\zeta)|^2 \frac{e^{-4\pi p(\zeta)t}}{4\pi t}\, \mathrm{d}V(\zeta)\, \mathrm{d}t \\ &= \int_0^\infty 4\pi t e^{-4\pi yt} \int_{\mathbb{C}^n} \left|H_p(t;w,\zeta)\right|^2 e^{-4\pi p(\zeta)t}\, \mathrm{d}V(\zeta)\, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= 4\pi \int_0^\infty t e^{-4yt} \int_{\mathbb{C}^n} \left| H_p(1; \widehat{\rho}_t(w), \widehat{\rho}_t(\zeta) \right|^2 t^{2/\mu} e^{-4\pi p(\widehat{\rho}_t(\zeta))} \, \mathrm{d}V(\zeta) \, \mathrm{d}t \text{ (From lemma } IV.7) \\
&= 4\pi \int_0^\infty t^{1+1/\mu} e^{-4\pi yt} \int_{\mathbb{C}^n} \left| H_p(1; \widehat{\rho}_t(w), \widehat{\rho}_t(\zeta) \right|^2 e^{-4\pi p(\widehat{\rho}_t(\zeta))} t^{1/\mu} \, \mathrm{d}V(\zeta) \, \mathrm{d}t \\
&= 4\pi \int_0^\infty t^{1+1/\mu} e^{-4\pi yt} \int_{\mathbb{C}^n} \left| H_p(1; \widehat{\rho}_t(w), \widehat{\rho}_t(\zeta) \right|^2 e^{-4\pi p(\widehat{\rho}_t(\zeta))} \, \mathrm{d}V(\widehat{\rho}_t(\zeta)) \, \mathrm{d}t \\
&= 4\pi \int_0^\infty t^{1+1/\mu} e^{-4\pi yt} \left\| H_p(1; \widehat{\rho}_t(w), \cdot) \right\|^2_{\mathcal{S}_p(1)} \, \mathrm{d}t \\
&= 4\pi \left\| H_p(1; \widehat{\rho}_t(w), \cdot) \right\|^2_{\mathcal{S}_p(1)} \int_0^\infty t^{1+1/\mu} e^{-4\pi yt} \, \mathrm{d}t \\
&< \infty
\end{aligned}
$$

because $H_p(1; \widehat{\rho}_t(w), \cdot) \in \mathcal{S}_p(1)$ and the integral $\int_0^\infty t^{1+1/\mu} e^{-4\pi yt} \, \mathrm{d}t$ converges (as $y > 0$). This shows that $\chi_{z,w} \in \mathcal{H}_p$ and we have

$$
\begin{aligned}
\langle f, \overline{\chi_{z,w}} \rangle_{\mathcal{H}_p} &= \int_0^\infty \int_{\mathbb{C}^n} f(t,\zeta) \left(4\pi t H_p(t,w,\zeta) e^{2\pi i z t} \right) \frac{e^{-4\pi p(\zeta) t}}{4\pi t} \, \mathrm{d}V(\zeta) \, \mathrm{d}t && \text{(IV.20)} \\
&= \int_0^\infty \left(\int_{\mathbb{C}^n} f(t,\zeta) H_p(t; w, \zeta) e^{-4\pi p(\zeta) t} \, \mathrm{d}V(\zeta) \right) e^{i2\pi z t} \, \mathrm{d}t \\
&= \int_0^\infty f(t,w) e^{i2\pi z t} \, \mathrm{d}t. && \text{(IV.21)}
\end{aligned}
$$

Here the integral on the left hand side of (IV.20) is finite, as it is the inner product of two functions in $\mathcal{H}_p$. Thus, by Fubini's theorem we were justified in evaluating the double integral on the left hand side of (IV.20) iteratively. The equation (IV.21) follows by the reproducing property of the kernel $H_p(t; w, \zeta)$.

This shows that the map $R_{\mathcal{H}_p}\left(e_{(z,w)} \circ T_S\right)$ is given by

$$
R_{\mathcal{H}_p}\left(e_{(z,w)} \circ T_S\right)(t,\zeta) = \overline{\chi_{z,w}(t,\zeta)} = 4\pi t \overline{H_p(t; w, \zeta)} e^{-2\pi i \bar{z} t}.
$$

We now apply Proposition IV.2 to the isometric isomorphism T_S above to obtain equation (I.25). By Lemma (IV.2), we get

$$
\begin{aligned}
K(z,w; Z,W) &= \left\langle R_{\mathcal{H}_p}\left(e_{(Z,W)} \circ T_S\right), R_{\mathcal{H}_p}\left(e_{(z,w)} \circ T_S\right) \right\rangle_{\mathcal{H}_p} \\
&= \int_0^\infty \int_{\mathbb{C}^n} \left(4\pi t H_p(t; w, \zeta) e^{i2\pi z t} \right) \left(4\pi t \overline{H_p(t; W, \zeta)} e^{-2\pi i \overline{Z} t} \right) \frac{e^{-4\pi p(\zeta) t}}{4\pi t} \, \mathrm{d}V(\zeta) \, \mathrm{d}t
\end{aligned}
$$

$$= 4\pi \int_0^\infty t e^{i2\pi(z-\overline{Z})t} \int_{\mathbb{C}^n} H_p(t;w,\zeta)\overline{H_p(t;W,\zeta)} e^{-4\pi p(\zeta)t}\, \mathrm{d}V(\zeta)\, \mathrm{d}t$$
$$= 4\pi \int_0^\infty t H_p(t;w,W) e^{i2\pi(z-\overline{Z})t}\, \mathrm{d}t,$$

where we used (IV.8) to obtain the last equality. □

IV.5. Another Formula for the Bergman Kernel of $\mathcal{U}_p$

Proof of Theorem 8. Let $T_V : \mathcal{X}_p \to A^2(\mathcal{U}_p)$ be the isometric isomorphism given by

$$T_V f(z,w) = \int_{\mathbb{R}} f\left(t, \frac{w_1}{z^{1/2m_1}}, \cdots, \frac{w_n}{z^{1/2m_n}}\right) \frac{z^{2\pi it}}{z^{1+1/2\mu}}\, \mathrm{d}t,$$

as in Theorem 4, where $z \in \mathbb{C}$ and $w \in \mathbb{C}^n$. For each (z,w) in $\mathcal{U}_p$, we first show that the image of the functional $e_{(z,w)} \circ T_V \in \mathcal{X}_p^*$ under the Riesz map $R_{\mathcal{X}_p}$ is given by

$$R_{\mathcal{X}_p}\left(e_{(z,w)} \circ T_V\right)(t,\zeta) = \overline{X_p(t;w,\zeta)} \frac{\overline{z}^{-2\pi it}}{\overline{z}^{1+1/2\mu}}.$$

For T_V as above, we represent the map $e_{(z,w)} \circ T_V : \mathcal{X}_p \to \mathbb{C}$ in two different ways. One one hand,

$$\begin{aligned}\left(e_{(z,w)} \circ T_V\right) f &= T_V f(z,w)\\ &= \int_{\mathbb{R}} f\left(t, \frac{w_1}{z^{1/2m_1}}, \cdots, \frac{w_n}{z^{1/2m_n}}\right) \frac{z^{2\pi it}}{z^{1+1/2\mu}}\, \mathrm{d}t\\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{B}_p} f\left(t, \frac{\zeta_1}{z^{1/2m_1}}, \cdots, \frac{\zeta_n}{z^{1/2m_n}}\right) X_p(t;w,\zeta)\lambda(p(\zeta),t)\, \mathrm{d}V(\zeta)\right) \frac{z^{i2\pi t}}{z^{1+1/2\mu}}\, \mathrm{d}t,\end{aligned} \tag{IV.22}$$

where the last equality follows from the reproducing property of the kernel $X_p(t;\cdot,\cdot)$. On the other hand $R_{\mathcal{X}_p}\left(e_{(z,w)} \circ T_V\right)$, the image of the functional $e_{(z,w)} \circ T_V \in \mathcal{X}_p^*$ under the Riesz map $R_{\mathcal{X}_p} : \mathcal{X}_p^* \to \mathcal{X}_p$ is given by

$$\begin{aligned}\left(e_{(z,w)} \circ T_V\right) f &= \left\langle f, R_{\mathcal{X}_p}\left(e_{(z,w)} \circ T_V\right)\right\rangle_{\mathcal{X}_p}\\ &= \int_{\mathbb{R}} \int_{\mathbb{B}_p} f\left(t, \frac{\zeta_1}{z^{1/2m_1}}, \ldots, \frac{\zeta_n}{z^{1/2m_n}}\right) \overline{R_{\mathcal{X}_p}\left(e_{(z,w)} \circ T_V\right)(t,\zeta)} \lambda(p(\zeta),t)\, \mathrm{d}V(\zeta)\, \mathrm{d}t.\end{aligned} \tag{IV.23}$$

The integral in equation (IV.23) is finite as it represents an inner product of two functions in $\mathcal{X}_p$ and hence maybe evaluated iteratively.

We claim that

$$X_p(t;w,\zeta)\frac{z^{2\pi it}}{z^{1+1/2\mu}} = \overline{R_{\mathcal{X}_p}\left(e_{(z,w)}\circ T_V\right)(t,\zeta)}.$$

Let $\phi : \mathbb{R}\times\mathbb{B}_p \to \mathbb{C}$ be the difference of the two quantities above, i.e.,

$$\phi(t,\zeta) = X_p(t;w,\zeta)\frac{z^{2\pi it}}{z^{1+1/2\mu}} - \overline{R_{\mathcal{X}_p}\left(e_{(z,w)}\circ T_V\right)(t,\zeta)}.$$

It then follows from equations (IV.22) and (IV.23) that the iterated integral

$$\int_{\mathbb{R}}\int_{\mathbb{B}_p} f\left(t,\frac{\zeta_1}{z^{1/2m_1}},\cdots,\frac{\zeta_n}{z^{1/2m_n}}\right)\phi(t,\zeta)\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t = 0, \tag{IV.24}$$

for all $f\in\mathcal{X}_p$.

Let $g(t,\zeta) = g_1(t)q(\zeta)$, where $g_1\in\mathscr{C}_c(\mathbb{R})$ and q is a polynomial. Since g_1 is compactly supported and q is a continuous function on $\overline{\mathbb{B}}_p$, $g\in\mathcal{X}_p$ and it follows from equation (IV.24) that

$$\begin{aligned} 0 &= \int_{\mathbb{R}}\int_{\mathbb{B}} g(t,\zeta)\phi(t,\zeta)\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} g_1(t)\int_{\mathbb{B}_p} q(\zeta)\phi(t,\zeta)\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t. \end{aligned} \tag{IV.25}$$

Since equation (IV.25) holds for all continuous compactly supported g_1, we must have for almost every $t\in\mathbb{R}$,

$$\int_{\mathbb{B}_p} q(\zeta)\phi(t,\zeta)\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta) = 0,$$

Since q is a polynomial and since polynomials are dense in $\mathcal{Q}_p(t)$ for all $t\in\mathbb{R}$, (by Proposition II.9) it follows that for almost all $t\in\mathbb{R}$,

$$\phi(t,\zeta)\equiv 0 \text{ on } \mathbb{B}_p,$$

i.e., for almost all $t,\zeta\in\mathbb{R}\times\mathbb{B}_p$ we have

$$X_p(t;w,\zeta)\frac{z^{i2\pi t}}{z^{1+1/2\mu}} = \overline{R_{\mathcal{X}_p}\left(e_{(z,w)}\circ T_V\right)(t,\zeta)}.$$

Applying Proposition (IV.2) to the isometric isomorphism T_V above, we get

$$\begin{aligned}
K(z,w;Z,W) &= \left\langle R_{\mathcal{X}_p}\left(e_{(Z,W)}\circ T_V\right), R_{\mathcal{X}_p}\left(e_{(z,w)}\circ T_V\right)\right\rangle_{\mathcal{X}_p} \\
&= \int_{\mathbb{R}}\int_{\mathbb{B}_p}\left(X_p(t;w,\zeta)\frac{z^{2i\pi t}}{z^{1+1/2\mu}}\right)\left(\overline{X_p(t;W,\zeta)\frac{\overline{Z}^{-2\pi it}}{\overline{Z}^{1+1/2\mu}}}\right)\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{B}_p} X_p(t;w,\zeta)\overline{X_p(t;W,\zeta)}\lambda(p(\zeta),t)\,\mathrm{d}V(\zeta)\right)\frac{z^{2\pi it}\overline{Z}^{-2\pi it}}{(z\overline{Z})^{1+1/2\mu}}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} X_p(t;w,W)\frac{z^{2\pi it}\overline{Z}^{-2\pi it}}{(z\overline{Z})^{1+1/2\mu}}\,\mathrm{d}t,
\end{aligned}$$

where we used (IV.8) to obtain the last equality. □

IV.6. An Application: the Siegel Upper Half Space

In this section, we apply Theorem 7 to compute the Bergman kernel for the Siegel Upper Half space $\mathcal{U}_{n+1}$ in $\mathbb{C}^{n+1}$. We restrict our attention to the polynomial $p(w) = |w|^2$, which is a weighted homogeneous balanced polynomial with respect to the tuple $m = (1,\dots,1) \in \mathbb{N}^n$.

Lemma IV.8. *Let the weighted homogeneous balanced polynomial $p : \mathbb{C}^n \to \mathbb{C}^n$ be given by $p(w) = |w|^2$ for $w \in \mathbb{C}^n$. Let $\mathcal{S}_n(1)$ be the weighted Bergman space on $\mathbb{C}^n$ with weight $w \mapsto e^{-4\pi|w|^2}$. Let $H_n(1;\cdot,\cdot)$ be the reproducing kernel for $\mathcal{S}_n(1)$. Then for $w, W \in \mathbb{C}^n$, we have*

$$H_n(1;w,W) = 4^n e^{4\pi w\cdot W}, \tag{IV.26}$$

where $w\cdot W = \sum_{j=1}^n w_j\overline{W}_j$ is the Hermitian inner product of w,W in $\mathbb{C}^n$.

Remark. The weighted Bergman space $\mathcal{S}_n(1)$ above is called the *Segal-Bargmann* space, which arises in connection to quantum mechanics (see [9] for instance).

Proof. For a multi-index $\alpha \in \mathbb{N}^n$, and $w \in \mathbb{C}^n$, let $q_\alpha(w) = w^\alpha$. If $\alpha \neq \beta$, then the monomials q_α and q_β are orthogonal to each other in $\mathcal{S}_n(1)$. We now compute the $\mathcal{S}_n(1)$ norm of q_α.

$$\|q_\alpha\|^2_{\mathcal{S}_n(1)} = \int_{\mathbb{C}^n} w^\alpha\overline{w}^\alpha e^{-4\pi|w|^2}\,\mathrm{d}V(w)$$

$$= \left(\int_{\mathbb{C}} |w_1|^{2\alpha_1} e^{-4\pi|w_1|^2} \, \mathrm{d}V(w_1) \right) \cdots \left(\int_{\mathbb{C}} |w_n|^{2\alpha_n} e^{-4\pi|w_n|^2} \, \mathrm{d}V(w_n) \right)$$

Thus, to compute the norm of q_α it suffices to evaluate the following integral

$$\begin{aligned}
\int_{\mathbb{C}} \left|w_j\right|^{2\alpha_j} e^{-4\pi\left|w_j\right|^2} \mathrm{d}V(w_j) &= 2\pi \int_0^\infty r_j^{2\alpha_j} e^{-4\pi r_j^2} r_j \, \mathrm{d}r_j \quad \text{(Using Polar co-ordinates)} \\
&= \frac{1}{4} \int_0^\infty \left(\frac{u}{4\pi}\right)^{\alpha_j} e^{-u} \, \mathrm{d}u \quad \text{(Letting } u = -4\pi r_j^2 \text{)} \\
&= \frac{1}{4 \cdot (4\pi)^{\alpha_j}} \Gamma(\alpha_j + 1) \\
&= \frac{\alpha_j!}{4 \cdot (4\pi)^{\alpha_j}}.
\end{aligned}$$

Thus, it follows that

$$\|q_\alpha\|^2_{\mathcal{S}_n(1)} = \frac{\alpha_1! \cdot \alpha_2! \cdots \alpha_n!}{4^n \cdot (4\pi)^{\alpha_1} \cdot (4\pi)^{\alpha_2} \cdots (4\pi)^{\alpha_n}}.$$

Since polynomials are dense in $\mathcal{S}_n(1)$ by Theorem II.7, it follows that the set

$$\left\{ \frac{q_\alpha}{\|q_\alpha\|_{\mathcal{S}_n(1)}} \right\}_{\alpha \in \mathbb{N}^n}$$

is then an orthonormal basis for $\mathcal{S}_n(1)$. Then by Lemma IV.2, we see that for $w, W \in \mathbb{C}^n$, the reproducing Kernel $H_n(1;\cdot,\cdot)$ for $\mathcal{S}_n(1)$ is given by

$$\begin{aligned}
H_n(1; w, W) &= \sum_{\alpha \in \mathbb{N}^n} \frac{q_\alpha(w)}{\|q_\alpha\|_{\mathcal{S}_n(1)}} \cdot \frac{\overline{q_\alpha(W)}}{\|q_\alpha\|_{\mathcal{S}_n(1)}} \\
&= \sum_{\alpha \in \mathbb{N}^n} \frac{2^n \cdot (2\sqrt{\pi})^{\alpha_1} w_1^{\alpha_1} \cdots (2\sqrt{\pi})^{\alpha_1} w_n^{\alpha_n}}{\sqrt{\alpha_1! \cdots \alpha_n!}} \cdot \frac{2^n \cdot (2\sqrt{\pi})^{\alpha_1} \overline{W}_1^{\alpha_1} \cdots (2\sqrt{\pi})^{\alpha_1} \overline{W}_n^{\alpha_n}}{\sqrt{\alpha_1! \cdots \alpha_n!}} \\
&= 4^n \sum_{\alpha \in \mathbb{N}^n} \frac{(4\pi w_1 \overline{W}_1)^{\alpha_1} \cdots (4\pi w_n \overline{W}_n)^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} \\
&= 4^n e^{4\pi w_1 \overline{W}_1} \cdots e^{4\pi w_n \overline{W}_n} \\
&= 4^n e^{4\pi w \cdot W}.
\end{aligned}$$

□

Now using Lemma IV.7 along with Theorem 7 gives us the following well known formula for the Bergman kernel of the Siegel Upper half space $\mathcal{U}_{n+1}$.

Theorem 12. *Let $\mathcal{U}_{n+1} \subset \mathbb{C}^{n+1}$ be the Siegel Upper Half space given by*

$$\mathcal{U}_{n+1} = \{(z,w) \in \mathbb{C} \times \mathbb{C}^n \mid \operatorname{Im} z > |w|^2\},$$

and let K be its Bergman kernel. Then, for $(z,w),(Z,W) \in \mathcal{U}_{n+1}$, we have

$$K(z,w;Z,W) = \frac{(n+1)!}{4\pi^{n+1}} \left[\frac{i}{2}(\overline{Z} - z) - \sum_{j=1}^{n} w_j \overline{W}_j \right]^{-n-2}.$$

Proof. Recall that the balanced weighted homogeneous polynomial $p(w) = |w|^2$ corresponds to the tuple $m = (1,\dots,1)$. Then for $t > 0$, using the notation of Lemma IV.7, the map $\widehat{\rho}_t : \mathbb{C}^n \to \mathbb{C}^n$ becomes,

$$\widehat{\rho}_t(w) = (t^{1/2m_1} w_1, \dots, t^{1/2m_n} w_n) = (t^{1/2} w_1, \dots, t^{1/2} w_n) = \sqrt{t} w, \text{ for } w \in \mathbb{C}^n.$$

In addition to this, we see that $1/\mu = 1/m_1 + \cdots + 1/m_n = n$. Thus, by Lemma IV.7 we see that the reproducing kernels $H_n(t;\cdot,\cdot)$ and $H_n(1;\cdot,\cdot)$ for weighted Bergman spaces $\mathcal{S}_n(t)$ and $\mathcal{S}_n(1)$ respectively are related as

$$\begin{aligned} H_n(t;w,W) &= t^{1/\mu} H_n(1; \widehat{\rho}_t(w), \widehat{\rho}_t(W)) \quad (\text{for } w,W \in \mathbb{C}^n) \\ &= t^n H_n(1; \sqrt{t}w, \sqrt{t}W) \\ &= 4^n t^n e^{-4\pi t w \cdot W} \qquad \text{(By Lemma IV.8)} \end{aligned}$$

For $(z,w),(Z,W) \in \mathcal{U}_{n+1}$, Theorem 7 gives the representation for Bergman Kernel K of $\mathcal{U}_{n+1}$ as

$$\begin{aligned} K(z,w;Z,W) &= 4\pi \int_0^\infty H_n(t;w,W) e^{i2\pi(z-\overline{Z})t} \, \mathrm{d}t \\ &= 4^{n+1} \pi \int_0^\infty t^{n+1} e^{\alpha t} \, \mathrm{d}t, \end{aligned} \tag{IV.27}$$

where $\alpha = 2\pi i(z - \overline{Z}) + 4\pi w \cdot W$. Note that we have

$$\begin{aligned}
\left|e^{\alpha t}\right| &= \left|e^{2\pi i(z-\overline{Z})t}\right| \left|e^{4\pi t w \cdot W}\right| \\
&= e^{-2\pi(\operatorname{Im} z + \operatorname{Im} Z)t} e^{4\pi t \operatorname{Re}(w \cdot W)} \\
&\leq e^{-2\pi(\operatorname{Im} z + \operatorname{Im} Z)t} e^{4\pi t |w \cdot W|} \\
&\leq e^{-2\pi(\operatorname{Im} z + \operatorname{Im} Z)t} e^{4\pi t |w||W|} \qquad \text{(By Cauchy Schwarz Inequality)} \\
&\leq e^{-2\pi(\operatorname{Im} z + \operatorname{Im} Z)t} e^{2\pi t(|w|^2 + |W|^2)} \quad \text{(By AM-GM inequality)} \\
&= e^{-2\pi t(\operatorname{Im} z - |w|^2)} e^{-2\pi t(\operatorname{Im} Z - |W|^2)}.
\end{aligned}$$

Since $(z,w),(Z,W) \in \mathcal{U}_{n+1}$, we have $\operatorname{Im} z - |w|^2 > 0$ and $\operatorname{Im} Z - |W|^2 > 0$. This shows that $e^{\alpha t} \to 0$ as $t \to \infty$ and also that the integral in (IV.27) converges for all $(z,w),(Z,W) \in \mathcal{U}_{n+1}$. Thus, we obtain

$$\begin{aligned}
\int_0^\infty t^{n+1} e^{\alpha t} \, \mathrm{d}t &= \left.\frac{t^{n+1} e^{\alpha t}}{\alpha}\right|_0^\infty - \frac{(n+1)}{\alpha} \int_0^\infty t^n e^{\alpha t} \, \mathrm{d}t \\
&= -\frac{(n+1)}{\alpha} \int_0^\infty t^n e^{\alpha t} \, \mathrm{d}t,
\end{aligned}$$

where the first term vanishes because $e^{\alpha t} \to 0$ as $t \to \infty$. Integrating by parts successively in this manner, we obtain

$$\begin{aligned}
K(z,w;Z,W) &= 4^{n+1}\pi \int_0^\infty t^{n+1} e^{\alpha t} \, \mathrm{d}t \\
&= 4^{n+1}\pi \frac{(-1)^n (n+1)!}{\alpha^{n+2}} \\
&= 4^{n+1}\pi \frac{(-1)^{n+2}(n+1)!}{\left(2\pi i(z-\overline{Z}) + 4\pi w \cdot W\right)^{n+2}} \\
&= \frac{4^{n+1}\pi}{4^{n+2}\pi^{n+2}} \frac{(-1)^{n+2}(n+1)!}{\left(\frac{i}{2}(z-\overline{Z}) + w \cdot W\right)^{n+2}} \\
&= \frac{(n+1)!}{4\pi^{n+1}} \left[\frac{i}{2}(\overline{Z} - z) - \sum_{j=1}^n w_j \overline{W}_j\right]^{-n-2}.
\end{aligned}$$

□

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