

POWER SERIES AS FOURIER SERIES

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ABSTRACT. An abstract theory of Fourier series in locally convex topological vector spaces is developed. An analog of Fejér’s theorem is proved for these series. The theory is applied to distributional solutions of Cauchy-Riemann equations to recover basic results of complex analysis. Some classical results of function theory are also shown to be consequences of the series expansion.

1. INTRODUCTION

1.1. **Motivation.** Two types of series expansion ubiquitous in mathematics are the power series of an analytic function, and the Fourier series of an integrable function on the circle. In the complex domain, well-known theorems of elementary complex analysis guarantee the existence of locally uniformly convergent power series expansions of holomorphic functions in domains with ample symmetry such as disks and annuli, where “holomorphic” can be taken in the sense of Goursat, i.e., the function is complex-differentiable at each point. On the other hand, the convergence of a Fourier series is a subtle matter, and its study has led to many developments in analysis (see [Zyg02]). The close connection between these two types of outwardly different series expansions is a recurring theme in many areas of classical analysis, e.g., in the theory of Hardy spaces.

It is however not difficult to see that at a certain level of abstraction, Fourier series and power series are in fact two examples of the same phenomenon, the representation theory of the circle group. The aim of this article is to take this idea seriously, and use it to recapture some basic results of complex analysis. The take-home message is that many properties of holomorphic functions, such as the almost-supernatural regularity phenomena are profitably thought of as expressions of symmetry, more precisely the invariance of certain locally convex spaces under the Reinhardt action of the torus group on \mathbb{C}^n . It is hoped that the point of view taken here has pedagogical as well as conceptual value, and will be of interest to students of complex analysis.

1.2. **Abstract Fourier series.** We begin in Section 2 with an account of Fourier series associated to a continuous representation of the n -dimensional torus \mathbb{T}^n on locally convex topological vector spaces, using some ideas of [Joh76]. This provides the unifying language of sufficient generality to encompass both classical Fourier expansions and the power series representations of complex analysis. Even at this very general and “soft” level, one can establish a version of Fejér’s theorem on the summability in the Cesàro sense (Theorem 2.1), which can be thought of as a “completeness” statement for the holomorphic monomials.

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1.3. Analyticity of holomorphic distributions. Using the abstract framework of Section 2, we recapture in Section 3, in a novel way, some of the basic classical facts about holomorphic functions. We show that on a Reinhardt domain, a distribution which satisfies the Cauchy-Riemann equations (which we call a *holomorphic distribution*) has a complex power series representation that converges uniformly along with derivatives of all orders on compact subsets of a “relatively complete” log-convex Reinhardt domain (see Theorem 3.1 below), and thus a holomorphic distribution is a function, in fact an analytic function. Traditionally, to prove such an assertion, one would start by showing that a holomorphic distribution is actually a (smooth) function. This can be done either by ad hoc arguments for the Laplacian going back to Weyl (see [Wey40]), or in a more general way, by noticing that fundamental solution $\frac{1}{\pi z}$ of the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ is smooth, in fact real-analytic, away from the singularity at the origin. It then follows by standard arguments about convolutions (see [Hör03, Theorem 4.4.3]), that any distributional solution of $\frac{\partial}{\partial \bar{z}}u = 0$ is real-analytic. Therefore, a holomorphic distribution is a holomorphic function in the sense of Goursat, and classical results of elementary complex analysis give the power series expansion via the Cauchy integral formula. The extension of the domain of convergence to the envelope of holomorphy can be obtained by convexity arguments (see [Ran86]).

While this classical argument has the admirable advantage of placing holomorphic functions in the context of solutions of hypoelliptic equations, it also has the shortcoming that the crucial property of *complex-analyticity* (and the associated Hartogs phenomenon in several variables) of holomorphic distributions is proved not from an analysis of the action of a differential operator on distributions, but by falling back on the Cauchy integral formula. After using the real-analytic hypoellipticity of $\frac{\partial}{\partial \bar{z}}$ to conclude that holomorphic distributions are real-analytic, we discard all of this information, except that holomorphic functions are \mathcal{C}^1 and satisfy the Cauchy-Riemann equations in the classical sense. In our approach here, however, complex analyticity of holomorphic distributions is proved directly in a conceptually straightforward way, by expanding a holomorphic distribution on a Reinhardt domain in a Fourier series, and then showing that the resulting series (the Laurent series of the holomorphic function) converges in the \mathcal{C}^∞ -topology, not only on the original set where the distribution was defined, but possibly on a larger domain, thus underlining the fact that Hartogs phenomenon can be thought of as a regularity property of the solutions of the same nature as smoothness. Our proof also clearly locates the origin of the remarkable regularity of holomorphic distributions in (a) the invariance properties of the space of holomorphic distributions under Reinhardt rotations and translations, (b) symmetry and convexity properties of the Laurent monomial functions $z \mapsto z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $\alpha_j \in \mathbb{Z}$ and (c) the fact that radially symmetric holomorphic distributions are constants. It is also interesting that the fact that $\frac{1}{\pi z}$ is a fundamental solution of the Cauchy-Riemann operator, which is key to many results of complex analysis including the Cauchy integral formula, does not play any role in our approach.

The method of proving analyticity via Fourier expansion can be used in other contexts. For example, replacing the representation theory of \mathbb{T}^n by that of the special orthogonal group $SO(n)$, the method can be used to show that a harmonic distribution in a ball of \mathbb{R}^n is in fact a real analytic function and admits an expansion in solid harmonics (see, e.g., [CH53, pp. 316-317]), which converges uniformly along with all derivatives on compact subsets of the ball. Similarly, one can obtain the Taylor/Laurent expansion of a monogenic

function of a Clifford-algebra variable in “spherical monogenics”, the analogs for functions of a Clifford-algebra variable of the monomial functions $z \mapsto z^n, n \in \mathbb{Z}$ (see [BDS82]).

1.4. Analyticity of continuous holomorphic functions. While the theory of generalized functions forms the natural context for solution of linear partial differential equations such as the Cauchy-Riemann equations, for aesthetic and pedagogical reasons it is natural to ask whether it is possible to develop the Fourier approach to power series, as outlined in Section 3, using only classical notions of functions as continuous mappings, and derivatives as limits of difference quotients, and without anachronistically invoking distributions. We accomplish this in the last Section 5 of this paper. We start from the assumption that a continuous function on the complex complex plane satisfies the hypothesis of the Morera theorem, and show directly that it is complex-analytic *without any recourse to the Cauchy integral representation formula* (which, being a case of Stoke’s theorem, requires the differentiability of the function, at least at each point). Not unexpectedly, one of the steps of the proof uses the classical triangle-division used in the standard proof of the Cauchy theorem for triangles.

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2. FOURIER SERIES IN LOCALLY CONVEX SPACES

2.1. Nets, series and integrals in LCTVS. We begin by recalling some notions and facts about functional analysis in topological vector spaces. See the textbooks [Trè67, Rud91, Bou04] for more details on these matters.

Let X be a locally convex Hausdorff topological vector space (we use the standard abbreviation *LCTVS*). Recall that the topology of X can also be described by prescribing the family of *continuous seminorms* on X : a net $\{x_j\}$ in X converges to x , if and only if for each continuous seminorm p on X , we have $p(x - x_j) \rightarrow 0$ as a net of real numbers. In practice, we describe the topology of an LCTVS by specifying a *generating family* of seminorms (analogous to describing a topology by a subbasis): a collection of continuous seminorms $\{p_k : k \in K\}$ on X is said to *generate the topology* of X if for every continuous seminorm q on X , there exists a finite subset $\{k_1, \dots, k_n\} \subset K$ and a $C > 0$ such that

$$q(x) \leq C \cdot \max\{p_{k_1}(x), \dots, p_{k_n}(x)\} \quad \text{for all } x \in X, \quad (2.1)$$

and further, for every nonzero $x \in X$ there exists at least one $k \in K$ such that $p_k(x) \neq 0$ (this *separating property* ensures that the topology of X is Hausdorff). Then clearly a net $\{x_j\}$ converges in X if and only if $p_k(x_j - x) \rightarrow 0$ for each $k \in K$.

Let Γ be a directed set with order \geq . Recall that a net $\{x_\alpha\}_{\alpha \in \Gamma}$ in X is said to be *Cauchy* if for every $\epsilon > 0$ and every continuous seminorm p on X , there exists $\gamma \in \Gamma$ such that whenever $\alpha, \beta \in \Gamma$ and $\alpha, \beta \geq \gamma$, we have $p(x_\alpha - x_\beta) < \epsilon$. The space X is said to be

complete if every Cauchy net in X converges. Observe that in the above definition we can use a generating family of seminorms rather than all continuous seminorms.

If $\{S_k\}_{k \in \mathbb{N}}$ is a sequence in a vector space, we can define the sequence of the corresponding *Cesàro means* $\{C_n\}_{n \in \mathbb{N}}$ by $C_n = \frac{1}{n+1} \sum_{k=0}^n S_k$. The following is the analog for sequences in LCTVS of an elementary fact well-known for sequences of numbers:

Proposition 2.1. *Let $\{S_k\}_{k \in \mathbb{N}}$ be a convergent sequence in an LCTVS X . Then the sequence of Cesàro means $\{C_n\}_{n \in \mathbb{N}}$ is also convergent, and has the same limit.*

Proof. Let $S = \lim_{k \rightarrow \infty} S_k$. If p is a continuous seminorm on X and $\epsilon > 0$, there is N_1 such that $p(S_k - S) < \epsilon/2$ for $k \geq N_1$. Set $s = \sum_{k=0}^{N_1} p(S_k - S)$. Then for $n > N_1$, we have

$$p(C_n - S) = p\left(\frac{1}{n+1} \sum_{k=0}^n S_k - S\right) \leq \frac{1}{n+1} \sum_{k=0}^n p(S_k - S) < \frac{s}{n+1} + \frac{n - N_1}{n+1} \cdot \frac{\epsilon}{2}.$$

Therefore, if we choose $N > N_1$ so large that $\frac{s}{n+1} < \frac{\epsilon}{2}$, then for $n \geq N$ we have $p(C_n - S) < \epsilon$, so $C_n \rightarrow S$ in X . \square

For a formal series $\sum_{j=0}^{\infty} x_j$ in an LCTVS X , convergence is defined in the usual way, i.e.

the sequence of partial sums converges in X . A formal sum $\sum_{\alpha \in \mathfrak{A}} x_\alpha$ over a countable index set \mathfrak{A} , where x_α are vectors in an LCTVS X , is said to be *absolutely convergent* if there exists a bijection $\tau : \mathbb{N} \rightarrow \mathfrak{A}$ such that for every continuous seminorm p on X , the series of non-negative real numbers $\sum_{j=0}^{\infty} p(x_{\tau(j)})$ is convergent (see [KK97]). To check that

a series is absolutely convergent, we only need to check the convergence of the above series for seminorms in a fixed generating family. If X is a locally compact Hausdorff space, absolute convergence in the Fréchet space $\mathcal{C}(X)$ of continuous complex valued functions on X is what is classically called *normal convergence* (see [Rem91, pp. 104 ff.]). Absolute convergence is typical for many spaces of holomorphic functions, e.g. in the space of holomorphic functions on a Reinhardt domain smooth up to the boundary (see [Daw21]).

The following result, whose proof mimics the corresponding result for numbers, shows that absolutely convergent series in LCTVS behave very much like absolutely convergent series of numbers:

Proposition 2.2. *Let X be a complete LCTVS, and let $\sum_{\alpha \in \mathfrak{A}} x_\alpha$ be an absolutely convergent series of elements of X . Then the series is unconditionally convergent: there is an $s \in X$ such that for every bijection $\theta : \mathbb{N} \rightarrow \mathfrak{A}$, the series $\sum_{j=0}^{\infty} x_{\theta(j)}$ converges in X to s .*

The element $s \in X$ is naturally called the *sum* of the series, and we write $s = \sum_{\alpha \in \mathfrak{A}} x_\alpha$.

Proof. By definition, there exists a bijection $\sigma : \mathbb{N} \rightarrow \mathfrak{A}$, such that for each continuous seminorm p on X , the series $\sum_{j=0}^{\infty} p(x_{\sigma(j)})$ converges. Let $y_j = x_{\sigma(j)}$ and $s_k = \sum_{j=0}^k y_j$. Since $\sum_{j=0}^{\infty} p(y_j)$ converges, for $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that whenever $m, \ell \in \mathbb{N}$

with $m \geq \ell \geq N_0$, $\sum_{j=\ell+1}^m p(y_j) < \epsilon$. Therefore for $m \geq \ell \geq N_0$,

$$p(s_m - s_\ell) = p\left(\sum_{j=\ell+1}^m y_j\right) \leq \sum_{j=\ell+1}^m p(y_j) < \epsilon. \quad (2.2)$$

Therefore $\{s_k\}$ is Cauchy sequence in the complete LCTVS X , and therefore converges to an $s \in X$. In order to complete the proof, it suffices to show that for every bijection $\tau : \mathbb{N} \rightarrow \mathbb{N}$, the series $\sum_{j=0}^{\infty} y_{\tau(j)}$ converges to the same sum s . Let $s_k^\tau = \sum_{j=0}^k y_{\tau(j)}$. Choose $u \in \mathbb{N}$ such that the set of integers $\{0, 1, 2, \dots, N_0\}$ is contained in the set $\{\tau(0), \tau(1), \dots, \tau(u)\}$. Then, if $k > u$, the elements y_1, \dots, y_{N_0} get cancelled in the difference $s_k - s_k^\tau$ and we have $p(s_k - s_k^\tau) < \epsilon$ by (2.2). This proves that the sequences $\{s_k\}$ and $\{s_k^\tau\}$ converge to the same limit. So, $s_k^\tau \rightarrow s$ as $k \rightarrow \infty$. \square

By a famous theorem of Dvoretzky and Rogers, the converse of the above result fails when X is an infinite dimensional Banach space ([KK97, Chapter 4]).

We will also need the notion of the weak (or Pettis) integral of a LCTVS valued function which we will now recall (see [Bou04, p. INT III.32-39] for details). Let K be a compact Hausdorff space and let μ be a Borel measure on K , and let f be a continuous map from K to an LCTVS X . An element $x \in X$ is called a *Pettis integral* of f on K with respect to μ if for all $\phi \in X'$,

$$\phi(x) = \int_K (\phi \circ f) d\mu, \quad (2.3)$$

where X' denotes the dual space of X (the space of continuous linear functionals on X), and the right hand side of (2.3) is an integral of a continuous function. If X is complete, one can show that there exists a unique $x \in X$ such that (2.3) holds and we denote the Pettis integral of f on K with respect to μ by $\int_K f d\mu = x$. In fact, the integral exists uniquely, as soon as the space X is *quasi-complete*, i.e. if each *bounded* Cauchy net in X converges, where a net $\{x_\alpha\}_{\alpha \in \Gamma}$ is bounded if for each continuous seminorm p , the net of real numbers $\{p(x_\alpha)\}_{\alpha \in \Gamma}$ is bounded. While there are situations in which this more refined existence theorem for Pettis integrals is useful (e.g. when the space X is a dual space with weak-* topology), in this paper, we only consider integrals in complete LCTVS. Since each Hausdorff TVS has a unique Hausdorff completion, we can define Pettis integrals in any LCTVS, provided we allow the integral to have a value in the completion.

If $T : X \rightarrow Y$ is a continuous linear map of complete LCTVSs, and $f : K \rightarrow X$ is continuous then we have

$$T\left(\int_K f d\mu\right) = \int_K T \circ f d\mu, \quad (2.4)$$

since for any linear functional $\psi \in Y'$,

$$\psi\left(T\left(\int_K f d\mu\right)\right) = (\psi \circ T)\left(\int_K f d\mu\right) = \int_K (\psi \circ T)(f) d\mu$$

using the fact that $\psi \circ T \in X'$ and the definition (2.3) of the Pettis integral.

2.2. Representation of the torus on an LCTVS X . Let $\mathbb{T}^n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : |\lambda_j| = 1 \text{ for every } 1 \leq j \leq n\}$ be the n -dimensional unit torus. With the subspace topology inherited from \mathbb{C}^n and binary operation defined as $(\lambda, \xi) \mapsto \lambda \cdot \xi = (\lambda_1 \xi_1, \dots, \lambda_n \xi_n)$, is a compact abelian topological group. For a LCTVS X , and a continuous function $f : \mathbb{T}^n \rightarrow$

X , we denote the Pettis integral of f with respect to the Haar measure of \mathbb{T}^n (normalized to be a probability measure) by

$$\int_{\mathbb{T}^n} f(\lambda) d\lambda.$$

Let X be an LCTVS, and let $\lambda \mapsto \sigma_\lambda$ be a continuous representation of \mathbb{T}^n on X . Recall that this means that for each $\lambda \in \mathbb{T}^n$, the map σ_λ is an automorphism (i.e. linear self-homeomorphism) of X as a topological vector space, the map $\lambda \mapsto \sigma_\lambda$ is a group homomorphism from \mathbb{T}^n to $\text{Aut}(X)$, and the associated map

$$\sigma : \mathbb{T}^n \times X \rightarrow X, \quad \sigma(\lambda, x) = \sigma_\lambda(x)$$

is continuous.

Given a representation σ of the group \mathbb{T}^n on an LCTVS X , a continuous seminorm p on X is said to be invariant (with respect to σ) if $p(\sigma_\lambda(x)) = p(x)$ for all $x \in X$ and $\lambda \in \mathbb{T}^n$.

Proposition 2.3. *A representation σ of \mathbb{T}^n on an LCTVS X is continuous if and only if the following two conditions are both satisfied:*

- (a) *the topology of X is generated by a family of invariant seminorms, and*
- (b) *for each $x \in X$ the function from \mathbb{T}^n to X given by $\lambda \mapsto \sigma_\lambda(x)$ is continuous at the identity element of \mathbb{T}^n .*

Proof. Assume that σ is continuous, i.e., $\sigma : \mathbb{T}^n \times X \rightarrow X$ is continuous, so for $x \in X$, the function $\lambda \mapsto \sigma_\lambda(x)$ is continuous on \mathbb{T}^n , in particular at the identity. Therefore (b) follows.

For a continuous seminorm q on X , define $p(x) = \sup_{\lambda \in \mathbb{T}^n} q(\sigma_\lambda(x))$, which is finite since \mathbb{T}^n is compact, and which is easily seen to be a seminorm. To show p is invariant, we note that

$$p(\sigma_\lambda(x)) = \sup_{\mu \in \mathbb{T}^n} q(\sigma_\mu(\sigma_\lambda(x))) = \sup_{\mu \in \mathbb{T}^n} q(\sigma_{\mu\lambda}(x)) = \sup_{\xi \in \mathbb{T}^n} q(\sigma_\xi(x)) = p(x).$$

It remains to show that p is continuous. For $x, y \in X$, we have

$$p(x) = \sup_{\lambda \in \mathbb{T}^n} q(\sigma_\lambda(y) + \sigma_\lambda(x - y)) \leq \sup_{\lambda \in \mathbb{T}^n} (q(\sigma_\lambda(y)) + q(\sigma_\lambda(x - y))) \leq p(y) + p(x - y),$$

so that $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$, and it follows that the seminorm p is continuous on X if and only if it is continuous at 0. We show that for each $\epsilon > 0$, there exists a neighbourhood V of 0 in X such that for all $\lambda \in \mathbb{T}^n$, $q \circ \sigma_\lambda < \epsilon$ on V . For each $\xi \in \mathbb{T}^n$, since q and σ are continuous, there exists a neighbourhood U_ξ of ξ in \mathbb{T}^n and a neighbourhood V_ξ of 0 in X such that $q(\sigma_\lambda(x)) < \epsilon$ for all $x \in V_\xi$ and $\lambda \in U_\xi$. The collection $\{U_\xi\}_{\xi \in \mathbb{T}^n}$ forms an open cover of \mathbb{T}^n . Since \mathbb{T}^n is compact, let $\{U_{\xi_1}, \dots, U_{\xi_k}\}$ be a finite subcover of \mathbb{T}^n corresponding to the open cover. Then for all $x \in \bigcap_{j=1}^k V_{\xi_j}$ and $\lambda \in \mathbb{T}^n$, we have $q(\sigma_\lambda(x)) < \epsilon$.

Now assume the two conditions (a) and (b). Let (Γ, \geq) be a directed set and let $(\lambda_\alpha)_{\alpha \in \Gamma}$ and $(x_\alpha)_{\alpha \in \Gamma}$ be nets in \mathbb{T}^n and X respectively with $(\lambda_\alpha, x_\alpha) \rightarrow (\lambda, x)$ in $\mathbb{T}^n \times X$, we need to show $\sigma_{\lambda_\alpha}(x_\alpha) \rightarrow \sigma_\lambda(x)$ in X , i.e., $p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_\lambda(x)) \rightarrow 0$ for each invariant continuous seminorm p of X . But we have, by the invariance of p :

$$p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_\lambda(x)) = p\left(x_\alpha - \sigma_{\lambda_\alpha^{-1}\lambda}(x)\right) \leq p(x_\alpha - x) + p\left(x - \sigma_{\lambda_\alpha^{-1}\lambda}(x)\right).$$

The term $p(x_\alpha - x)$ goes to zero since $x_\alpha \rightarrow x$, and the term $p\left(x - \sigma_{\lambda_\alpha^{-1}\lambda}(x)\right)$ also goes to zero since $\lambda_\alpha \rightarrow \lambda$ and $\mu \mapsto \sigma_\mu(x)$ is continuous at $\mu = \mathbf{1}$. The result follows. \square

2.3. Abstract Fejér Theorem. Let X be a complete LCTVS and let σ be a continuous representation of \mathbb{T}^n on X . For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $x \in X$, define

$$\pi_\alpha^\sigma(x) = \int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(x) d\lambda, \quad (2.5)$$

the Pettis integral of the continuous function $\lambda \mapsto \lambda^{-\alpha} \sigma_\lambda(x)$ on \mathbb{T}^n with respect to the Haar probability measure of \mathbb{T}^n . We will say that $\pi_\alpha^\sigma(x)$ is the α -th *Fourier component* of x with respect to the representation σ . We use the standard convention with respect to multi-index powers, i.e., $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$. We will say that the subspace of X defined as

$$[X]_\alpha^\sigma = \{x \in X : \sigma_\lambda(x) = \lambda^\alpha \cdot x \text{ for all } \lambda \in \mathbb{T}^n\} \quad (2.6)$$

is the α -th *Fourier mode* of the space X , and we will call the map π_α^σ the α -th *Fourier projection*, both with respect to the representation σ . We note the following facts:

Proposition 2.4. *As above, let X be a complete LCTVS and let σ be a continuous representation of \mathbb{T}^n on X .*

- (1) *For each $\alpha \in \mathbb{Z}^n$, the α -th Fourier mode $[X]_\alpha^\sigma$ is a closed σ -invariant subspace of X , and the Fourier projection π_α^σ is a continuous linear projection from X onto $[X]_\alpha^\sigma$.*
- (2) *Let Y be another complete LCTVS, and let τ be a continuous representation of \mathbb{T}^n on Y , and let $j : Y \rightarrow X$ be a continuous linear map intertwining σ and τ , i.e., for each $\lambda \in \mathbb{T}^n$, $j \circ \tau_\lambda = \sigma_\lambda \circ j$. Then for each $\alpha \in \mathbb{Z}^n$, we have*

$$j \circ \pi_\alpha^\tau = \pi_\alpha^\sigma \circ j. \quad (2.7)$$

Proof. (1) Linearity of π_α^σ follows from the linearity of σ_λ . Recall from Proposition 2.3 that there exists a family \mathcal{P} of continuous invariant seminorms that generates the locally convex topology of X . To see the continuity of π_α^σ , observe that for each $p \in \mathcal{P}$, we have

$$p(\pi_\alpha^\sigma(x)) = p\left(\int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(x) d\lambda\right) \leq \int_{\mathbb{T}^n} p(\lambda^{-\alpha} \sigma_\lambda(x)) d\lambda = \int_{\mathbb{T}^n} p(x) d\lambda = p(x), \quad (2.8)$$

where the inequality is due to Proposition 6 in [Bou04, p. INT III.37]. Now, if $x \in [X]_\alpha^\sigma$, then

$$\pi_\alpha^\sigma(x) = \int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(x) d\lambda = \int_{\mathbb{T}^n} \lambda^{-\alpha} \cdot \lambda^\alpha x d\lambda = x \cdot \left(\int_{\mathbb{T}^n} d\lambda\right) = x,$$

so that $[X]_\alpha^\sigma \subset \pi_\alpha^\sigma(X)$. To prove that $\pi_\alpha^\sigma(X) \subset [X]_\alpha^\sigma$, notice that for each $\alpha \in \mathbb{Z}^n$, each $\lambda \in \mathbb{T}^n$ and $x \in X$,

$$\begin{aligned} \sigma_\lambda(\pi_\alpha^\sigma(x)) &= \sigma_\lambda\left(\int_{\mathbb{T}^n} \mu^{-\alpha} \sigma_\mu(x) d\mu\right) = \int_{\mathbb{T}^n} \mu^{-\alpha} \sigma_{\lambda \cdot \mu}(x) d\mu && \text{using (2.4)} \\ &= \lambda^\alpha \cdot \int_{\mathbb{T}^n} (\lambda\mu)^{-\alpha} \sigma_{\lambda\mu}(x) d\mu = \lambda^\alpha \cdot \pi_\alpha^\sigma(x) && \text{by Haar invariance.} \end{aligned}$$

- (2) For $x \in X$, we have

$$j \circ \pi_\alpha^\tau(x) = j\left(\int_{\mathbb{T}^n} \lambda^{-\alpha} \tau_\lambda(x) d\lambda\right) = \int_{\mathbb{T}^n} \lambda^{-\alpha} j \circ \tau_\lambda(x) d\lambda = \int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda \circ j(x) d\lambda = \pi_\alpha^\sigma \circ j(x),$$

where we have used (2.4) to go from the second to the third step. \square

Remark. The inequality (2.8)

$$p(\pi_\alpha^\sigma(x)) \leq p(x). \quad (2.9)$$

which holds for each α for an invariant seminorm p can be thought of as an abstract form of the familiar *Cauchy inequalities* of complex analysis.

If X is $L^1(\mathbb{T})$, the Banach space of integrable functions on \mathbb{T} , and if σ is the continuous representation of \mathbb{T} on $L^1(\mathbb{T})$ given by

$$\sigma_\lambda(f)(\mu) = f(\lambda \cdot \mu), \quad \lambda, \mu \in \mathbb{T}, f \in L^1(\mathbb{T}),$$

an easy computation shows that for $\phi \in \mathbb{R}$, and $f \in L^1(\mathbb{T})$, we have

$$\pi_\alpha^\sigma(f)(e^{i\phi}) = e^{i\alpha\phi} \cdot \hat{f}(\alpha), \quad \text{with } \hat{f}(\alpha) = \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{-i\alpha\theta} f(e^{i\theta}) d\theta,$$

the α -th term of the Fourier series of f .

It is therefore natural to define, for $x \in X$, the *Fourier series* of x with respect to σ to be the formal series

$$x \sim \sum_{\alpha \in \mathbb{Z}^n} \pi_\alpha^\sigma(x). \quad (2.10)$$

For an integer $k \geq 0$, define the k -th *square partial sum* of the Fourier series in (2.10) by

$$S_k^\sigma(x) = \sum_{|\alpha|_\infty \leq k} \pi_\alpha^\sigma(x), \quad (2.11)$$

where $|\alpha|_\infty := \max\{|\alpha_j|, 1 \leq j \leq n\}$. We are ready to state an abstract version of Fejér's theorem.

Theorem 2.1. *Let σ be a continuous representation of \mathbb{T}^n on an LCTVS X and let $x \in X$. Then the Cesàro means of the square partial sums of the Fourier series of x (with respect to σ) converge to x in the topology of X .*

Proof. Write the Cesàro means of the square partial sums of the Fourier series of x as,

$$\begin{aligned} C_N^\sigma(x) &= \frac{1}{N+1} \sum_{k=0}^N S_k^\sigma(x) = \frac{1}{N+1} \sum_{k=0}^N \sum_{|\alpha|_\infty \leq k} \pi_\alpha^\sigma(x) = \frac{1}{N+1} \sum_{k=0}^N \sum_{|\alpha|_\infty \leq k} \left(\int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(x) d\lambda \right) \\ &= \frac{1}{N+1} \int_{\mathbb{T}^n} \sum_{k=0}^N \left(\sum_{|\alpha|_\infty \leq k} \lambda^{-\alpha} \right) \sigma_\lambda(x) d\lambda = \int_{\mathbb{T}^n} F_N(\lambda) \sigma_\lambda(x) d\lambda \end{aligned}$$

where $F_N(\lambda) = \frac{1}{N+1} \sum_{k=0}^N \left(\sum_{|\alpha|_\infty \leq k} \lambda^{-\alpha} \right)$ is the classical Fejér kernel. Introducing polar coordinates, $\lambda_j = e^{i\theta_j}$ on \mathbb{T}^n , and summing, we obtain the classical representation

$$F_N(e^{i\theta_1}, \dots, e^{i\theta_n}) = \frac{1}{N+1} \prod_{j=1}^n \frac{\sin^2\left(\left(\frac{N+1}{2}\right)\theta_j\right)}{\sin^2\left(\frac{\theta_j}{2}\right)}$$

It is well-known that the Fejér kernel has the properties that (a) $F_N \geq 0$ for all N , (b) $\int_{\mathbb{T}^n} F_N(\lambda) d\lambda = 1$ and (c) For each $\delta > 0$, $F_N \rightarrow 0$ uniformly on $\mathbb{T}^n \setminus B(\mathbf{1}, \delta)$, where $B(\mathbf{1}, \delta)$ is the n -dimensional ball centered at $\mathbf{1} = (1, 1, \dots, 1)$ and radius δ .

Let p be a continuous σ -invariant seminorm on X . Then for $x \in X$, we have

$$\begin{aligned}
p(C_N(x) - x) &= p \left(\int_{\mathbb{T}^n} F_N(\lambda) \cdot \sigma_\lambda(x) d\lambda - x \cdot \int_{\mathbb{T}^n} F_N(\lambda) d\lambda \right) \\
&\quad \text{using property (b) of } F_N \\
&= p \left(\int_{\mathbb{T}^n} F_N(\lambda) \cdot \sigma_\lambda(x) d\lambda - \int_{\mathbb{T}^n} x \cdot F_N(\lambda) d\lambda \right) \\
&\quad \text{using (2.4)} \\
&= p \left(\int_{\mathbb{T}^n} (\sigma_\lambda(x) - x) \cdot F_N(\lambda) d\lambda \right) \\
&\leq \int_{\mathbb{T}^n} p(\sigma_\lambda(x) - x) \cdot F_N(\lambda) d\lambda \tag{2.12} \\
&\quad \text{using [Bou04, Prop. 6, p. INT III.37] and the positivity of } F_N(\lambda)
\end{aligned}$$

Since $\lambda \mapsto \sigma_\lambda(x)$ is continuous and p is a continuous seminorm, there exists $\delta > 0$ such that $p(\sigma_\lambda(x) - \sigma_{\mathbf{1}}(x)) = p(\sigma_\lambda(x) - x) < \frac{\epsilon}{2}$ whenever $\lambda \in \mathbb{T}^n \cap B(\mathbf{1}, \delta)$. Then on the set $\mathbb{T}^n \setminus B(\mathbf{1}, \delta)$ we have

$$p(\sigma_\lambda(x) - x) \leq p(\sigma_\lambda(x)) + p(x) = 2 \cdot p(x).$$

By property (c) of F_N above, there exists $N_1 \in \mathbb{N}$ such that whenever $\lambda \in \mathbb{T}^n \setminus B(\mathbf{1}, \delta)$, for all $N \geq N_1$ we have,

$$F_N(\lambda) < \frac{\epsilon}{4 \cdot p(x)} \tag{2.13}$$

Now from (2.12), we have

$$\begin{aligned}
p(C_N(x) - x) &\leq \int_{\mathbb{T}^n \cap B(\mathbf{1}, \delta)} F_N(\lambda) \cdot p(\sigma_\lambda(x) - x) d\lambda + \int_{\mathbb{T}^n \setminus B(\mathbf{1}, \delta)} F_N(\lambda) \cdot p(\sigma_\lambda(x) - x) d\lambda \\
&< \frac{\epsilon}{2} \int_{\mathbb{T}^n} F_n(\lambda) d\lambda + \frac{\epsilon}{4 \cdot p(x)} \cdot 2 \cdot p(x) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since by Proposition 2.3 the topology of X is generated by σ -invariant seminorms, the result follows. \square

Corollary 2.5. *Let X be a complete LCTVS and suppose that we are given a continuous representation of the group \mathbb{T}^n on X . Then if the Fourier series of an element $x \in X$ with respect to this representation is absolutely convergent in X , the sum of the Fourier series equals x .*

Proof. Since the series $\sum_{\alpha \in \mathbb{Z}^n} \pi_\alpha^\sigma(x)$ is absolutely convergent, by Proposition 2.2, there exists the sum of the series, i.e., an $\tilde{x} \in X$ such that for every bijection $\theta : \mathbb{N} \rightarrow \mathbb{Z}^n$ we have $\sum_{j=0}^{\infty} \pi_{\theta(j)}^\sigma(x) = \tilde{x}$. Let $S_N = \sum_{j=0}^N \pi_{\theta(j)}^\sigma(x)$; then the sequence of partial sums $\{S_N\}$ converges to \tilde{x} in X . By Proposition 2.1, the sequence of Cesàro means $\{C_N\}$ of the partial

sums converges to \tilde{x} as well. However, by Theorem 2.1, the Cesàro means converge to x . Therefore $\tilde{x} = x$. \square

3. RECAPTURING COMPLEX ANALYSIS

We now use the machinery developed in the previous section to give a conceptually simple account of the remarkable regularity properties of holomorphic distributions. So we will pretend that we have forgotten everything about complex analysis, but do remember the rudiments of the theory of distributions, accounts of which can be found in the classic treatises [Hör03, Trè67, Sch66]. First we clarify notations and recall a few facts.

3.1. The basic spaces. For an open $\Omega \subset \mathbb{R}^n$, the space $\mathcal{D}(\Omega)$ of test functions is the LF -space of smooth compactly supported complex valued functions, topologized as the inductive limit of the Fréchet spaces \mathcal{D}_K consisting, for a given compact $K \subset \Omega$, of those elements of $\mathcal{D}(\Omega)$ which have support in K . Recall that a subset $B \subset \mathcal{D}(\Omega)$ is bounded, if and only if there is a compact $K \subset \Omega$ such that $B \subset \mathcal{D}_K$, and for each nonnegative multi-index $\alpha \in \mathbb{N}^n$, we have

$$\sup_{\substack{\phi \in B \\ x \in K}} \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi(x) \right| < \infty, \quad (3.1)$$

where, here and later, we will use standard multi-index conventions such as

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_j \alpha_j, \quad \text{for } \alpha \in \mathbb{N}^n. \quad (3.2)$$

The space of distributions $\mathcal{D}'(\Omega)$ on Ω is the dual of $\mathcal{D}(\Omega)$, consisting of continuous linear forms on $\mathcal{D}(\Omega)$. We denote the value of a distribution $T \in \mathcal{D}'(\Omega)$ at a test function $\phi \in \mathcal{D}(\Omega)$ by $\langle T, \phi \rangle$. The space $\mathcal{D}'(\Omega)$ is endowed with the usual strong dual topology. Recall that this topology is generated by the family of seminorms

$$\{p_B : B \subset \mathcal{D}(\Omega) \text{ is bounded}\}, \quad \text{where, } p_B(T) = \sup_{\phi \in B} |\langle T, \phi \rangle|. \quad (3.3)$$

In this topology, the space $\mathcal{D}'(\Omega)$ is complete.

Given a locally integrable function $f \in L^1_{\text{loc}}(\Omega)$, we can associate to f a distribution T_f defined by

$$\langle T_f, \phi \rangle = \int_{\Omega} f \phi \, dV, \quad \text{for } \phi \in \mathcal{D}(\Omega),$$

where dV denotes the Lebesgue measure of \mathbb{R}^n . Then the *locally integrable distribution* T_f is said to be generated by f , and as usual we grant ourselves the the right to abuse language by identifying the distribution $T_f \in \mathcal{D}'(\Omega)$ with the function $f \in L^1_{\text{loc}}(\Omega)$.

We will use the abbreviations

$$\mathbf{D}_j = \frac{\partial}{\partial z_j}, \quad \overline{\mathbf{D}}_j = \frac{\partial}{\partial \bar{z}_j}, \quad (3.4)$$

for the basic constant coefficient differential operators of complex analysis, acting on functions or distributions on \mathbb{C}^n . If $\Omega \subset \mathbb{C}^n$ is an open set, define

$$\mathcal{O}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \overline{\mathbf{D}}_j f = 0, 1 \leq j \leq n\}, \quad (3.5)$$

the space of *holomorphic distributions* on Ω . The subspace $\mathcal{O}(\Omega)$ is closed in the space $\mathcal{D}'(\Omega)$ by the continuity of the operators $\overline{\mathbf{D}}_j$, therefore a complete LCTVS in the subspace topology.

3.2. The main theorem. We begin with some definitions and notational conventions. For $\lambda \in \mathbb{T}^n$ we denote by R_λ the *Reinhardt rotation* of \mathbb{C}^n by the element λ , the linear automorphism of the vector space \mathbb{C}^n given by

$$R_\lambda(z) = (\lambda_1 z_1, \dots, \lambda_n z_n). \quad (3.6)$$

A domain $\Omega \subset \mathbb{C}^n$ is defined to be *Reinhardt* if and only if $R_\lambda(\Omega) = \Omega$ for each $\lambda \in \mathbb{T}^n$. Let Z denote the union of the coordinate hyperplanes of \mathbb{C}^n :

$$Z = \bigcup_{j=1}^n \{z \in \mathbb{C}^n : z_j = 0\}. \quad (3.7)$$

Recall that a Reinhardt domain $\Omega \subset \mathbb{C}^n$ is said to be *log-convex*, if whenever $z, w \in \Omega \setminus Z$, the point $\zeta \in \mathbb{C}^n$ belongs to Ω if there is a $t \in [0, 1]$ such that for $1 \leq j \leq n$

$$|\zeta_j| = |z_j|^t |w_j|^{1-t}. \quad (3.8)$$

For $\alpha \in \mathbb{Z}^n$, let e_α be the monomial function given by

$$e_\alpha(z) = z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}. \quad (3.9)$$

For a Reinhardt subset $\Omega \subset \mathbb{C}^n$ and $1 \leq j \leq n$, let

$$\Omega^{(j)} = \{(z_1, \dots, \zeta z_j, \dots, z_n) : z \in \Omega, \zeta \in \overline{\mathbb{D}}\}, \quad (3.10)$$

where $\overline{\mathbb{D}} = \{|\zeta| \leq 1\} \subset \mathbb{C}$ is the closed disk. This can be thought of as the result of “completing” Ω in the j -th coordinate. Following [JP08], we say that the Reinhardt domain $\Omega \subset \mathbb{C}^n$ is *relatively complete* if for each $1 \leq j \leq n$, whenever we have $\Omega \cap \{z_j = 0\} \neq \emptyset$, we also have

$$\Omega^{(j)} \subset \Omega.$$

We prove the following well-known structure theorem for holomorphic distributions on Reinhardt domains, as an application of the ideas of Section 2:

Theorem 3.1. *Let Ω be a Reinhardt domain in \mathbb{C}^n and let*

$$\mathcal{S}(\Omega) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in \mathcal{C}^\infty(\Omega)\}. \quad (3.11)$$

Let $\widehat{\Omega}$ be the smallest relatively complete log-convex Reinhardt domain in \mathbb{C}^n that contains Ω . Then for each $\alpha \in \mathcal{S}(\Omega)$ there is a continuous linear functional $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ such that for each $T \in \mathcal{O}(\Omega)$, the series

$$\sum_{\alpha \in \mathcal{S}(\Omega)} a_\alpha(T) e_\alpha \quad (3.12)$$

converges absolutely in $\mathcal{C}^\infty(\widehat{\Omega})$ to a function $f \in \mathcal{C}^\infty(\widehat{\Omega})$, and $f|_\Omega$ generates the distribution T .

Remarks:

(1) For $n = 1$, all Reinhardt domains in the plane, i.e., disks and annuli, are automatically relatively complete and log-convex. For $n \geq 2$, it is easy to give examples of Reinhardt domains which are not log-convex, or not relatively complete, or perhaps both. For such a domain Ω , it follows that each holomorphic distribution $f \in \mathcal{O}(\Omega)$ extends to a holomorphic function $F \in \mathcal{O}(\widehat{\Omega})$. This is the simplest example of *Hartogs phenomenon*, the compulsory extension of all holomorphic functions from a smaller domain to a larger one, characteristic of domains in several complex variables.

(2) The functionals a_α are called the *coefficient functionals*, and the series (3.12) is of course the *Laurent series* of the function f (the *Taylor series* if $\mathcal{S}(\Omega) = \mathbb{N}^n$).

(3) It is known (by a direct construction of a plurisubharmonic exhaustion) that relatively complete log-convex Reinhardt domains are pseudoconvex. This means that such a domain Ω admits a holomorphic distribution whose Laurent expansion converges absolutely precisely on Ω .

As an immediate consequence of Theorem 3.1 we have the following:

Corollary 3.1. *Let $\Omega \subset \mathbb{C}^n$ be open. Then each distribution $T \in \mathcal{O}(\Omega)$ is complex-analytic, i.e., for each $p \in \Omega$ there is a neighborhood U of p , where the function f generating T is represented by a Taylor series centered at p .*

3.3. Holomorphic functions and maps. A holomorphic distribution $T \in \mathcal{O}(\Omega)$ will be called a *holomorphic function* if it is generated by a \mathcal{C}^∞ function f . We denote the space of holomorphic functions on Ω temporarily by $(\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$. Once Theorem 3.1 is proved, it will follow that $(\mathcal{O} \cap \mathcal{C}^\infty)(\Omega) = \mathcal{O}(\Omega)$.

Let Ω_1, Ω_2 be domains in \mathbb{C}^n . By a *holomorphic map* $\Phi : \Omega_1 \rightarrow \Omega_2$, we mean a mapping each of whose components is a holomorphic function on Ω_1 . A holomorphic map is a *biholomorphism*, if it is a bijection, and its set-theoretic inverse is also a holomorphic map. (It is of course known that the assumption of the holomorphicity of the inverse map is redundant, but this is a consequence of complex-analyticity, which is exactly what we are proving here). If $\Phi : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism, then for a distribution $T \in \mathcal{D}'(\Omega_2)$, we can define in the usual way the *pullback* distribution $\Phi^*T \in \mathcal{D}'(\Omega_1)$: if T is generated by a test function $f \in \mathcal{D}(\Omega_2)$, then Φ^*T is the distribution generated by the function $f \circ \Phi$, and for general T , we extend this definition by continuity, using the density of test functions in $\mathcal{D}'(\Omega)$, see [Hör03, Theorem 6.1.2] for details. Extending the chain rules for the complex derivative operators from test functions to distributions, we have the following relations analogous to [Hör03, formula (6.1.2)] for the Wirtinger derivatives (3.4):

$$D_j(\Phi^*T) = \sum_{k=1}^n D_j \Phi_k \cdot \Phi^*(D_k T), \quad (3.13)$$

and

$$\overline{D}_j(\Phi^*T) = \sum_{k=1}^n \overline{D}_j \overline{\Phi_k} \cdot \Phi^*(\overline{D}_k T), \quad (3.14)$$

where as above, $\Phi : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism of domains in \mathbb{C}^n , written in components as $\Phi = (\Phi_1, \dots, \Phi_n)$, and $T \in \mathcal{D}'(\Omega_2)$.

Therefore, we have the following immediate consequence of (3.14):

Proposition 3.2. *If $\Phi : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism and $T \in \mathcal{O}(\Omega_2)$, then we have $\Phi^*T \in \mathcal{O}(\Omega_1)$. If $f \in (\mathcal{O} \cap \mathcal{C}^\infty)(\Omega_2)$, then $\Phi^*f \in (\mathcal{O} \cap \mathcal{C}^\infty)(\Omega_1)$.*

Therefore the spaces of holomorphic distributions and functions are invariant under pullbacks under biholomorphic maps. In fact, in the proof of Theorem 3.1, only two simple special cases of Proposition 3.2 noted below are needed:

(1) Translation by a vector $a \in \mathbb{C}^n$ is the map $M_a : \mathbb{C}^n \rightarrow \mathbb{C}^n$

$$M_a(z) = z + a, \quad (3.15)$$

which is obviously a biholomorphic automorphism of \mathbb{C}^n . For a domain $\Omega \subset \mathbb{C}^n$, we therefore have a pullback isomorphism of spaces of holomorphic distributions $M_a^* : \mathcal{O}(M_a(\Omega)) \rightarrow$

$\mathcal{O}(\Omega)$. This can be thought of as an expression of the fact that the operator $\bar{\partial} = (\bar{D}_1, \dots, \bar{D}_n)$ is translation invariant.

(2) The Reinhardt rotations R_λ of (3.6) are clearly biholomorphic automorphisms of Reinhardt domains, and the pullback operation induces a representation of the group \mathbb{T}^n on the space of holomorphic distributions (see (3.32) below).

A domain Ω is said to be *Reinhardt centered at a* for an $a \in \mathbb{C}^n$ if there is a Reinhardt domain Ω_0 such that $\Omega = M_a(\Omega_0)$. Every open set in \mathbb{C}^n has local Reinhardt symmetry, in the sense that each point has neighborhood which is a Reinhardt domain centered at that point.

3.4. Mean value property of the monomials. It is easily verified by direct computation that for each α , we have $e_\alpha \in (\mathcal{O} \cap \mathcal{C}^\infty)(\mathbb{C}^n \setminus Z)$, where e_α is the monomial of (3.9). We now note a remarkable symmetry property (the Mean-Value Property) of the functions e_α :

Lemma 3.3. *Let $\alpha \in \mathbb{Z}^n$, let $z \in \mathbb{C}^n \setminus Z$, and let $\psi \in \mathcal{D}(\mathbb{C}^n \setminus Z)$ be a test function which has radial symmetry in each variable around the point z , i.e., there are functions $\rho_1, \dots, \rho_n \in \mathcal{D}(\mathbb{R})$ such that $\psi(\zeta) = \prod_{j=1}^n \rho_j(|\zeta_j - z_j|)$, and whose integral is 1:*

$$\int_{\mathbb{C}^n \setminus Z} \psi(\zeta) dV(\zeta) = 1. \quad (3.16)$$

Then we have

$$\langle e_\alpha, \psi \rangle = e_\alpha(z) \quad (3.17)$$

Proof. First consider the case $n = 1$. If $\alpha \geq 0$, the formula

$$\int_0^{2\pi} (z + re^{i\theta})^\alpha d\theta = 2\pi z^\alpha \quad (3.18)$$

holds for $r > 0$, by expanding the integrand using the binomial formula, and integrating the finite sum term by term. The formula (3.18) also holds for $\alpha < 0$, provided $z \neq 0$, and $0 < r < |z|$. This follows on noticing that we have an infinite series expansion

$$(z + re^{i\theta})^\alpha = z^\alpha \left(1 + \frac{r}{z} e^{i\theta}\right)^\alpha = z^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{r}{z}\right)^k e^{ik\theta},$$

where by the M -test, the convergence is uniform in θ , and then integrating the series on the right term by term.

Now $\psi(\zeta) = \rho(|\zeta - z|)$ and ψ is supported in some disc $B(z, R) \subset \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, which means $R < |z|$. Then the normalization (3.16) is equivalent to

$$2\pi \int_0^R \rho(r) r dr = 1. \quad (3.19)$$

Now

$$\begin{aligned} \langle e_\alpha, \psi \rangle &= \int_{\mathbb{C}^*} e_\alpha(\zeta) \rho(|\zeta - z|) dV(\zeta) = \int_0^R \int_0^{2\pi} (z + re^{i\theta})^\alpha \rho(r) r d\theta dr \\ &= \int_0^R \rho(r) r \left(\int_0^{2\pi} (z + re^{i\theta})^\alpha d\theta \right) dr = 2\pi z^\alpha \cdot \int_0^R \rho(r) r dr \quad \text{using (3.18)} \end{aligned}$$

$$= z^\alpha = e_\alpha(z), \quad \text{using (3.19).}$$

which establishes the result for $n = 1$. In the general case, notice that for $\alpha \in \mathbb{Z}^n$ and $\zeta \in \mathbb{C}^n \setminus Z$ we have $e_\alpha(\zeta) = \prod_{j=1}^n e_{\alpha_j}(\zeta_j)$. Therefore, since $\mathbb{C}^n \setminus Z = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$,

$$\begin{aligned} \langle e_\alpha, \psi \rangle &= \int_{\mathbb{C}^n \setminus Z} \prod_{j=1}^n e_{\alpha_j}(\zeta_j) \rho_j(|\zeta_j - z_j|) dV(\zeta) = \prod_{j=1}^n \int_{\mathbb{C}^*} e_{\alpha_j}(\zeta_j) \rho_j(|\zeta_j - z_j|) dV(\zeta_j) \\ &= \prod_{j=1}^n e_{\alpha_j}(z_j) = e_\alpha(z). \end{aligned}$$

□

3.5. The representation τ . If $\Omega \subset \mathbb{C}^n$ is a Reinhardt domain, then for each λ , the map R_λ of (3.6) is a biholomorphic automorphism of Ω . Define a representation τ of \mathbb{T}^n on the space $\mathcal{D}(\Omega)$ of test functions by

$$\tau_\lambda \psi = \psi \circ R_\lambda, \quad \psi \in \mathcal{D}(\Omega). \quad (3.20)$$

Recall that a net $\{\phi_j\}$ converges in the space $\mathcal{D}(\Omega)$, if each ϕ_j is supported in a fixed compact $K \subset \Omega$, and the net $\{\phi_j\}$ converges in the Fréchet space \mathcal{D}_K , i.e. all partial derivatives converge uniformly on K . Using this it is easily verified that τ is a continuous representation of \mathbb{T}^n in the space $\mathcal{D}(\Omega)$.

Notice that $\mathbb{C}^n \setminus Z$ is a Reinhardt domain. For a positive integer k we define the norm $\|\psi\|_k$ with respect to the polar coordinates for a function ψ in $\mathcal{D}(\mathbb{C}^n \setminus Z)$:

$$\|\psi\|_k = \max_{\substack{\beta, \gamma \in \mathbb{N}^n \\ |\beta| + |\gamma| \leq k}} \sup_{\mathbb{C}^n \setminus Z} \left| \frac{\partial^\beta}{\partial r^\beta} \frac{\partial^\gamma}{\partial \theta^\gamma} \psi \right|, \quad (3.21)$$

where the tuples $r \in (\mathbb{R}^+)^n, \theta \in \mathbb{R}^n$ are the polar coordinates on $\mathbb{C}^n \setminus Z$ specified by $z_j = r e^{i\theta_j}$, and $\frac{\partial^\beta}{\partial r^\beta}, \frac{\partial^\gamma}{\partial \theta^\gamma}$ are partial derivatives operators in the polar coordinates defined as in (3.2). From the formulas

$$\frac{\partial}{\partial x_j} = \cos \theta_j \frac{\partial}{\partial r_j} - \frac{\sin \theta_j}{r_j} \frac{\partial}{\partial \theta_j} \quad \text{and} \quad \frac{\partial}{\partial y_j} = \sin \theta_j \frac{\partial}{\partial r_j} + \frac{\cos \theta_j}{r_j} \frac{\partial}{\partial \theta_j}, \quad (3.22)$$

we see that for $0 < \varrho_1 < \varrho_2 < \infty$, there is a constant $B_k(\varrho_1, \varrho_2)$ such that for each compact set K such that

$$K \subset \{z \in \mathbb{C}^n : \varrho_1 < |z_j| < \varrho_2\},$$

we have

$$\frac{1}{B_k(\varrho_1, \varrho_2)} \cdot \|\psi\|_k \leq \|\psi\|_{\mathcal{C}^k} \leq B_k(\varrho_1, \varrho_2) \cdot \|\psi\|_k. \quad (3.23)$$

We will need the following elementary estimate:

Proposition 3.4. *Let τ be the representation of \mathbb{T}^n on $\mathbb{C}^n \setminus Z$ given by (3.20). For integers $m, k \geq 0$, and a compact $K \subset \mathbb{C}^n \setminus Z$ there is a constant $C > 0$ such that for each $\psi \in \mathcal{D}_K$ and each $\alpha \in \mathbb{Z}^n$, such that $|\alpha_j| \geq 2k$ for $1 \leq j \leq n$, we have*

$$\|\pi_\alpha^\tau \psi\|_{\mathcal{C}^k} \leq \frac{C}{\prod_{j=1}^n |\alpha_j|^m} \cdot \|\psi\|_{\mathcal{C}^{2nk+nm}}.$$

Proof. For $r \in (\mathbb{R}^+)^n$, $\alpha \in \mathbb{Z}^n$, define the α -th Fourier coefficient of ψ by

$$\widehat{\psi}(r, \alpha) = \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} \psi(re^{i\theta}) e^{-i\langle \alpha, \theta \rangle} d\theta, \quad (3.24)$$

where $re^{i\theta} = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) \in \mathbb{C}^n \setminus Z$, $\langle \alpha, \theta \rangle = \sum_{j=1}^n \alpha_j \theta_j$ and $d\theta = d\theta_1 \dots d\theta_n$ is the Lebesgue measure. We will also write $(\psi)^\wedge$ for $\widehat{\psi}$ whenever convenient. We note the following properties:

(1) We clearly have

$$\sup_{r \in (\mathbb{R}^+)^n} \left| \widehat{\psi}(r, \alpha) \right| \leq \sup_{\mathbb{C}^n \setminus Z} |\psi|. \quad (3.25)$$

(2) If $\alpha \neq 0$, by a standard integration by parts argument, for each $\ell \in \mathbb{N}^n$,

$$\widehat{\psi}(r, \alpha) = \frac{1}{(i\alpha)^\ell} \left(\frac{\partial^\ell \psi}{\partial \theta^\ell} \right)^\wedge (r, \alpha), \quad (3.26)$$

where, as usual, we set for $\zeta \in \mathbb{C}^n$ and $\beta \in \mathbb{N}^n$, $\zeta^\beta = \zeta_1^{\beta_1} \dots \zeta_n^{\beta_n}$.

(3) Differentiating under the integral sign we have, for any $\beta \in \mathbb{N}^n$,

$$\left(\frac{\partial}{\partial r} \right)^\beta \widehat{\psi}(r, \alpha) = \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} \frac{\partial^\beta \psi}{\partial r^\beta}(re^{i\theta}) e^{i\langle \beta, \theta \rangle} e^{-i\langle \alpha, \theta \rangle} d\theta = \left(\frac{\partial^\beta \psi}{\partial r^\beta} \right)^\wedge (r, \alpha - \beta),$$

and combining this with (3.26), we see that if $\beta \neq \alpha$, we have for each integer $\ell \geq 0$ that

$$\left(\frac{\partial}{\partial r} \right)^\beta \widehat{\psi}(r, \alpha) = \frac{1}{(i(\alpha - \beta))^\ell} \left(\frac{\partial^\ell}{\partial \theta^\ell} \frac{\partial^\beta}{\partial r^\beta} \psi \right)^\wedge (r, \alpha - \beta). \quad (3.27)$$

Now for $re^{i\theta} \in \mathbb{C}^n \setminus Z$ the evaluation $\mathcal{D}(\mathbb{C}^n \setminus Z) \rightarrow \mathbb{C}$, $\psi \mapsto \psi(re^{i\theta})$ is continuous, so using (2.4) we see that for each $\alpha \in \mathbb{Z}^n$ and each $\psi \in \mathcal{D}(\mathbb{C}^n \setminus Z)$, we have

$$\pi_\alpha^\tau \psi(re^{i\theta}) = \int_{\mathbb{T}} \lambda^{-\alpha} (\tau_\lambda \psi)(re^{i\theta}) d\lambda = \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} e^{-i\langle \alpha, \phi \rangle} \psi(e^{i\phi} \cdot re^{i\theta}) d\phi = e^{i\langle \alpha, \theta \rangle} \cdot \widehat{\psi}(r, \alpha), \quad (3.28)$$

where in the last step we make a change of variables in the integral from ϕ to $\phi + \theta$. Therefore, if $\beta, \gamma \in \mathbb{N}^n$ with $\beta_j \neq \alpha_j$ for $1 \leq j \leq n$, then

$$\frac{\partial^\gamma}{\partial \theta^\gamma} \frac{\partial^\beta}{\partial r^\beta} (\pi_\alpha^\tau \psi)(re^{i\theta}) = e^{i\langle \alpha, \theta \rangle} \cdot \frac{(i\alpha)^\gamma}{(i(\alpha - \beta))^\ell} \left(\frac{\partial^\ell}{\partial \theta^\ell} \frac{\partial^\beta}{\partial r^\beta} \psi \right)^\wedge (r, \alpha - \beta).$$

Now in the above formula, if we have $|\beta| + |\gamma| \leq k$, and let

$$\ell = (k + m, \dots, k + m) = (k + m)\mathbf{1},$$

where $\mathbf{1} \in \mathbb{N}^n$ is the multi-index each of whose n entries is 1, we obtain, after taking absolute values of both sides

$$\left| \frac{\partial^\gamma}{\partial \theta^\gamma} \frac{\partial^\beta}{\partial r^\beta} (\pi_\alpha^\tau \psi)(re^{i\theta}) \right| = \frac{\prod_{j=1}^n |\alpha_j|^{\gamma_j}}{\prod_{j=1}^n |\alpha_j - \beta_j|^{m+k}} \cdot \left| \left(\frac{\partial^{(m+k)\mathbf{1}}}{\partial \theta^{(m+k)\mathbf{1}}} \frac{\partial^\beta}{\partial r^\beta} \psi \right)^\wedge (r, \alpha - \beta) \right|. \quad (3.29)$$

In the first factor on the right hand side, we have by hypothesis for each j that $|\alpha_j| \geq 2k$ and $0 \leq \beta_j, \gamma_j \leq k$. Therefore, we have $|\alpha_j - \beta_j| \geq \frac{1}{2}|\alpha_j|$. The first factor can be estimated as

$$\frac{\prod_{j=1}^n |\alpha_j|^{\gamma_j}}{\prod_{j=1}^n |\alpha_j - \beta_j|^{m+k}} \leq \frac{\prod_{j=1}^n |\alpha_j|^k}{\prod_{j=1}^n \left(\frac{1}{2}|\alpha_j|^{m+k}\right)} = \frac{2^{m+k}}{\prod_{j=1}^n |\alpha_j|^m}. \quad (3.30)$$

Using (3.25), the second factor can be estimated as

$$\left| \left(\frac{\partial^{(m+k)\mathbf{1}}}{\partial \theta^{(m+k)\mathbf{1}}} \frac{\partial^\beta}{\partial r^\beta} \psi \right)^\wedge (r, \alpha - \beta) \right| \leq \sup_{\mathbb{C}^n \setminus Z} \left| \frac{\partial^{(m+k)\mathbf{1}}}{\partial \theta^{(m+k)\mathbf{1}}} \frac{\partial^\beta}{\partial r^\beta} \psi \right| \leq \|\psi\|_{2nk+nm}, \quad (3.31)$$

where in the last step we use the norm introduced in (3.21), and used the fact that

$$|(m+k)\mathbf{1} + \beta| = (m+k)n + |\beta| \leq (m+k)n + nk,$$

since each $\beta_j \leq k$. Combining (3.29), (3.30) and (3.31) we see that

$$\left| \frac{\partial^\gamma}{\partial \theta^\gamma} \frac{\partial^\beta}{\partial r^\beta} (\pi_\alpha^\tau \psi)(re^{i\theta}) \right| \leq \frac{2^{m+k}}{\prod_{j=1}^n |\alpha_j|^m} \cdot \|\psi\|_{2nk+nm},$$

from which, taking a supremum on the left hand side over $re^{i\theta} \in \mathbb{C} \setminus Z$, and remembering that $|\beta| + |\gamma| \leq k$, we conclude that

$$\|\pi_\alpha^\tau \psi\|_k \leq \frac{2^{m+k}}{\prod_{j=1}^n |\alpha_j|^m} \cdot \|\psi\|_{2nk+nm}.$$

Recall that K is a compact set in $\mathbb{C}^n \setminus Z$ such that $\psi \in \mathcal{D}_K$, i.e. , the support of ψ is contained in K . Suppose that ϱ_1, ϱ_2 are such that $K \subset A$, where A is the product of annuli $A = \{z \in \mathbb{C}^n : \varrho_1 < |z_j| < \varrho_2, 1 \leq j \leq n\}$. Formulas (3.24) and (3.28) show that the compact support of $\pi_\alpha^\tau \psi$ is also contained in the set A . Therefore, passing to the equivalent \mathcal{C}^k -norms using (3.23), we have

$$\|\pi_\alpha^\tau \psi\|_{\mathcal{C}^k} \leq B_k(\varrho_1, \varrho_2) \cdot B_{2nk+nm}(\varrho_1, \varrho_2) \cdot \frac{2^{m+k}}{\prod_{j=1}^n |\alpha_j|^m} \cdot \|\psi\|_{\mathcal{C}^{2nk+nm}},$$

which completes the proof of the result. \square

3.6. The dual representation. Let Ω be a Reinhardt domain. Since for each $\lambda \in \mathbb{T}^n$, the map R_λ maps Ω biholomorphically (and therefore diffeomorphically) to itself. Consequently, we can define a representation of \mathbb{T}^n on the space of distributions $\mathcal{D}'(\Omega)$ using the pullback operation

$$\sigma_\lambda(T) = (R_\lambda)^*(T), \quad T \in \mathcal{D}'(\Omega). \quad (3.32)$$

The representation σ_λ is closely related to the representation τ_λ introduced in (3.20). Clearly, $\mathcal{D}(\Omega)$ is an invariant (dense) subspace of σ , on which σ restricts to τ . So σ is simply the extension of the representation τ of (3.20) by continuity to the space of distributions.

The representation σ on $\mathcal{D}'(\Omega)$ is also “dual” to the representation τ on $\mathcal{D}(\Omega)$:

$$\langle \sigma_\lambda(T), \phi \rangle = \langle (\mathbf{R}_\lambda)^*(T), \phi \rangle = \left\langle T, \frac{1}{\det_{\mathbb{R}}(\mathbf{R}_\lambda)} \phi \circ \mathbf{R}_\lambda^{-1} \right\rangle = \langle T, \tau_{\lambda^{-1}} \phi \rangle, \quad (3.33)$$

so that σ_λ is the transpose of the map $\tau_{\lambda^{-1}}$. The second equality in the chain may be proved by change of variables when T is a test function, and then using density.

Proposition 3.5. *The representation σ of \mathbb{T}^n on $\mathcal{D}'(\Omega)$ defined in (3.32) is continuous.*

Proof. Let $\{\lambda_j\}$ be a sequence in \mathbb{T}^n converging to the identity element. Since pointwise convergence of a sequence in $\mathcal{D}'(\Omega)$ on each test function implies convergence in the strong dual topology (see [Trè67]), to show that $\sigma_{\lambda_j}(T) \rightarrow T$ in $\mathcal{D}'(\Omega)$ we need to show that for $\phi \in \mathcal{D}(\Omega)$ we have $\langle \sigma_{\lambda_j}(T), \phi \rangle \rightarrow \langle T, \phi \rangle$. But by (3.33), $\langle \sigma_{\lambda_j}(T), \phi \rangle = \langle T, \tau_{\lambda_j^{-1}} \phi \rangle$ and by the continuity of the representatin τ , it follows that $\sigma_{\lambda_j}(T) \rightarrow T$ in $\mathcal{D}'(\Omega)$.

To complete the proof, by Proposition 2.3, we need to show that the topology of Ω is generated by a collection of σ -invariant seminorms. Let B be a bounded subset of $\mathcal{D}(\Omega)$. Let

$$\tilde{B} = \{\sigma_\lambda(\phi) : \lambda \in \mathbb{T}^n, \phi \in B\}.$$

Then it is clear from (3.1) that \tilde{B} is also a bounded subset of $\mathcal{D}(\Omega)$. With notation as it (3.3), it is clear that for each $T \in \mathcal{D}'(\Omega)$, we have $p_B(T) \leq p_{\tilde{B}}(T)$. Therefore the topology of $\mathcal{D}'(\Omega)$ is generated by the the family of continuous invariant seminorms $\{p_{\tilde{B}}\}$. \square

3.7. Fourier series of distributions on Reinhardt domains. Let Ω be a Reinhardt domain and let $T \in \mathcal{D}'(\Omega)$ be a distribution. Then by the results of Section 2 we can expand T in a formal Fourier series with respect to the representation σ of (3.32). For simplicity of notation, whenever there is no possibility of confusion, we denote the Fourier components of T by

$$T_\alpha = \pi_\alpha^\sigma(T), \quad \alpha \in \mathbb{Z}^n, \quad (3.34)$$

so that the Fourier series of T is written as $T \sim \sum_{\alpha \in \mathbb{Z}^n} T_\alpha$. We notice the following properties of the Fourier components:

Proposition 3.6. *Let Ω and T be as above and let $\alpha \in \mathbb{Z}^n$.*

- (a) *If the distribution T lies in one of the linear subspaces $\mathcal{O}(\Omega), \mathcal{C}^\infty(\Omega), (\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$ or $\mathcal{D}(\Omega)$ of $\mathcal{D}'(\Omega)$, the Fourier component T_α also lies in the same subspace.*
- (b) *If U is another Reinhardt domain such that $U \subset \Omega$, we have*

$$T_\alpha|_U = (T|_U)_\alpha. \quad (3.35)$$

Proof. All these are consequences of Part 2 of Proposition 2.4. To see Part (a), let X be the space $\mathcal{D}'(\Omega)$ and let Y be one of the spaces $\mathcal{O}(\Omega), \mathcal{C}^\infty(\Omega)$ or $\mathcal{D}(\Omega)$. Each of these has a natural structure of a complete LCTVS (the space $\mathcal{O}(\Omega)$ as a closed subspace of $\mathcal{D}'(\Omega)$, $\mathcal{C}^\infty(\Omega)$ in its Fréchet topology and $\mathcal{D}(\Omega)$ in its LF -topology), and for each one of these topologies, the representation σ restricts to a continuous representation (in the stronger topology of the subspace), which we will call τ (this coincides with the τ introduced in (3.20)). If j is the inclusion map of any one of these subspaces into $\mathcal{D}'(\Omega)$, then clearly it is continuous (where the subspace has the natural topology). Therefore, by (2.7), we have $j(\pi_\alpha^\tau T) = \pi_\alpha^\sigma(j(T))$. By definition, the right hand side is precisely T_α , i.e. the Fourier component of T as an element of $\mathcal{D}'(\Omega)$. Notice that $\pi_\alpha^\tau T \in Y$, since it is the Fourier

component of T as an element of Y . Since j is the inclusion of Y in X , it now follows that $T_\alpha \in Y$ as well.

Since the result is true for $\mathcal{O}(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ it follows for $(\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$.

For part (b), in part 2 of Proposition 2.4, let $X = \mathcal{D}'(U)$ with representation defined in (3.32), which we call σ' for clarity, and let $Y = \mathcal{D}'(\Omega)$ with the representation σ and let $j : Y \rightarrow X$ be the restriction map of distributions, which is clearly continuous and intertwines the two representations. Then (2.7) takes the form $j \circ \pi_\alpha^\sigma = \pi_\alpha^{\sigma'} \circ j$. Applying this to the distribution T , we see that

$$T_\alpha|_U = j(\pi_\alpha^\sigma T) = \pi_\alpha^{\sigma'}(j(T)) = (T|_U)_\alpha.$$

□

We establish the following lemma:

Lemma 3.7. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain, and let*

$$\mathcal{A} = \{T \in \mathcal{O}(\Omega) : \text{each Fourier component } T_\alpha \text{ belongs to } \mathcal{C}^\infty(\Omega)\}.$$

Suppose that for each $T \in \mathcal{A}$ the Laurent series $\sum_{\alpha \in \mathbb{Z}^n} T_\alpha$ converges absolutely in the Fréchet space $\mathcal{C}(\Omega)$. Then the Laurent series of each element of \mathcal{A} converges absolutely in the space $\mathcal{C}^\infty(\Omega)$.

Proof. For a function $f \in \mathcal{C}(\Omega)$ and a compact set K , let p_K be the seminorm:

$$p_K(f) = \sup_K |f|. \quad (3.36)$$

Notice that the family $\{p_K : K \subset \Omega \text{ compact}\}$ is a generating family of seminorms for $\mathcal{C}(\Omega)$, and the hypothesis on \mathcal{A} can be expressed as

$$\sum_{\alpha \in \mathbb{Z}^n} p_K(T_\alpha) < \infty, \quad (3.37)$$

for each compact $K \subset \Omega$.

The Fréchet topology of $\mathcal{C}^\infty(\Omega)$ is generated by the seminorms $p_{K,L}$, where K ranges over compact subsets of Ω , L is a constant coefficient differential operator on \mathbb{C}^n , and

$$p_{K,L}(f) = \sup_K |Lf|.$$

We will continue to write p_K for the seminorm (3.36) corresponding to $L = 1$. From the distributional chain rule (3.13), we have for each j that

$$\mathbf{D}_j \sigma_\lambda(T) = \mathbf{D}_j(\mathbf{R}_\lambda^* T) = (\mathbf{D}_j \mathbf{R}_\lambda) \cdot \mathbf{R}_\lambda^*(\mathbf{D}_j T) = \lambda_j \cdot \sigma_\lambda(\mathbf{D}_j T).$$

Let $\beta \in \mathbb{N}^n$, and denote $\mathbf{D}^\beta = \mathbf{D}_1^{\beta_1} \dots \mathbf{D}_n^{\beta_n}$. Then, from the above we have

$$\mathbf{D}^\beta T_\alpha = \mathbf{D}^\beta \int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(T) d\lambda = \int_{\mathbb{T}^n} \lambda^{-\alpha} \mathbf{D}^\beta \sigma_\lambda(T) d\lambda = \int_{\mathbb{T}^n} \lambda^{-(\alpha-\beta)} \sigma_\lambda(\mathbf{D}^\beta T) d\lambda = (\mathbf{D}^\beta T)_{\alpha-\beta},$$

where $(\mathbf{D}^\beta T)_{\alpha-\beta} = \pi_{\alpha-\beta}^\sigma(\mathbf{D}^\beta T)$. It follows that all Fourier coefficients of the distribution $\mathbf{D}^\beta T$ are in $\mathcal{C}^\infty(\Omega)$, so $\mathbf{D}^\beta T \in \mathcal{A}$. Therefore,

$$p_{K, \mathbf{D}^\beta}(T_\alpha) = p_K(\mathbf{D}^\beta T_\alpha) = p_K((\mathbf{D}^\beta T)_{\alpha-\beta}).$$

It follows therefore that

$$\sum_{\alpha \in \mathbb{Z}^n} p_{K, \mathbf{D}^\beta}(T_\alpha) = \sum_{\gamma \in \mathbb{Z}^n} p_K \left((\mathbf{D}^\beta T)_\gamma \right) < \infty,$$

using assumption (3.37) on elements of \mathcal{A} . A partial derivative $L = \frac{\partial^\gamma}{\partial x^\gamma} \frac{\partial^\delta}{\partial y^\delta}$ on \mathbb{C}^n can be rewritten as a polynomial in the commuting differential operators $\mathbf{D}_1, \dots, \mathbf{D}_n, \overline{\mathbf{D}}_1, \dots, \overline{\mathbf{D}}_n$, and by hypothesis the derivatives $\overline{\mathbf{D}}_j$ are zero. Therefore, it follows that $\sum_\alpha p_{K, L}(T_\alpha) < \infty$. Consequently the series $\sum_\alpha T_\alpha$ is absolutely convergent in $\mathcal{C}^\infty(\Omega)$. \square

3.8. Proof of Theorem 3.1: case of annular domains. We now prove a weaker version of Theorem 3.1. A Reinhardt domain $\Omega \subset \mathbb{C}^n$ will be called *annular* if it is disjoint from the coordinate hyperplanes:

$$\Omega \cap Z = \emptyset, \quad (3.38)$$

where Z is as in (3.7). We have the following:

Proposition 3.8. *Let $\Omega \subset \mathbb{C}^n \setminus Z$ be an annular Reinhardt domain in \mathbb{C}^n . Then for each $\alpha \in \mathbb{Z}^n$ there is a continuous linear functional $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ such that for each $T \in \mathcal{O}(\Omega)$, the series of terms in $\mathcal{C}^\infty(\Omega)$ given by $\sum_{\alpha \in \mathbb{Z}^n} a_\alpha(T) e_\alpha$ converges absolutely in $\mathcal{C}^\infty(\Omega)$ to a function f , and this \mathcal{C}^∞ -smooth function generates the distribution T .*

Proof. Let $T \in \mathcal{O}(\Omega)$ and let T_α be its α -th Fourier component as in (3.34), and let $S = e_{-\alpha} \cdot T_\alpha$. The distributional Leibniz rule

$$\overline{\mathbf{D}}_j(fU) = (\overline{\mathbf{D}}_j f) \cdot U + f \cdot (\overline{\mathbf{D}}_j U), \quad f \in \mathcal{C}^\infty(\Omega) \text{ and } U \in \mathcal{D}'(\Omega), \quad (3.39)$$

shows that $S \in \mathcal{O}(\Omega)$, since $T_\alpha \in \mathcal{O}(\Omega)$ by Part (1) of Proposition 3.6. By Part (1) of Proposition 2.4, the Fourier component T_α lies in the α -th Fourier mode of $\mathcal{O}(\Omega)$, i.e., $\sigma_\lambda(T_\alpha) = \lambda^\alpha \cdot T_\alpha$, so

$$\sigma_\lambda(S) = \lambda^{-\alpha} e_{-\alpha} \cdot \lambda^\alpha T_\alpha = S.$$

Therefore, using polar coordinates $z_j = r_j e^{i\theta_j}$, we have with $\epsilon_j(h) = (1, \dots, 1, e^{ih}, \dots, 1) \in \mathbb{T}^n$ (with e^{ih} in the j -th position):

$$\frac{\partial}{\partial \theta_j} S = \lim_{h \rightarrow 0} \frac{1}{h} \left(\sigma_{\epsilon_j(h)}(S) - S \right) = 0.$$

Since $\overline{\mathbf{D}}_j S = 0$, and in polar coordinates we have

$$\overline{\mathbf{D}}_j = \frac{e^{i\theta_j}}{2} \left(\frac{\partial}{\partial r_j} + \frac{i}{r_j} \frac{\partial}{\partial \theta_j} \right), \quad (3.40)$$

we see that $\frac{\partial}{\partial r_j} S = 0$ also. Now from the representations (3.22) we have that $\frac{\partial}{\partial x_j} S = 0$

and $\frac{\partial}{\partial y_j} S = 0$ in $\mathcal{D}'(\Omega)$ for $1 \leq j \leq n$. Therefore, by a classical result in the theory of distributions (see [Sch66, Théorème VI, pp. 69ff.]) we have that S is locally constant on Ω , or more precisely, S is generated by a locally constant function. Since Ω is connected, it follows that there is a constant $a_\alpha(T) \in \mathbb{C}$ which generates the distribution S , so that $T_\alpha = a_\alpha(T) e_\alpha$. This can also be written as $a_\alpha(T) = e_{-\alpha} \cdot T_\alpha = e_{-\alpha} \cdot \pi_\alpha^\sigma(T)$. Since multiplication by the fixed \mathcal{C}^∞ -function $e_{-\alpha}$ is a continuous map on $\mathcal{D}'(\Omega)$ (and therefore on the subspace $\mathcal{O}(\Omega)$), and $\pi_\alpha^\sigma : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega)$ is continuous by part 1 of Proposition 2.4,

it follows that $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega)$ is continuous. But we know that a_α takes values in the subspace of distributions generated by constants. By a well-known theorem, on a finite-dimensional topological vector space, there is only one topology. Therefore the topology induced from $\mathcal{O}(\Omega)$ on the subspace of constants coincides with the natural topology of \mathbb{C} . Therefore $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ is continuous.

By the above, the Fourier components T_α of the Fourier series of T actually lie in $(\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$, and therefore any partial sum (i.e. sum of finitely many terms) of this series also lies in $(\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$. We will now show that the series (3.34) is absolutely convergent in the topology of $\mathcal{C}^\infty(\Omega)$, in the sense of Proposition 2.2. Let $K \subset \Omega$ be compact. Let $\delta < \text{dist}(K, \mathbb{C}^n \setminus \Omega)$, and let $\Psi \in \mathcal{D}(\mathbb{C}^n)$ be a test function such that

$$\text{support}(\Psi) \subset B(0, \delta), \quad \int_{\mathbb{C}} \Psi(\zeta) dV(\zeta) = 1, \quad \text{and } \Psi(\zeta) = \Psi(|\zeta_1|, \dots, |\zeta_n|) \text{ for } \zeta \in \mathbb{C}^n,$$

i.e. Ψ is radial around the origin in each complex coordinate. For $z \in K$, define $\psi_z(\zeta) = \Psi(\zeta - z)$, so that ψ_z is radially symmetric around z in each complex direction. Therefore, for $z \in K$, we have, by Lemma 3.3 for each α that

$$\langle T_\alpha, \psi_z \rangle = \langle a_\alpha(T) e_\alpha, \psi_z \rangle = a_\alpha(T) z^\alpha.$$

By the continuity of the mapping $T : \mathcal{D}(\Omega) \rightarrow \mathbb{C}$, there is a constant $C > 0$ and an integer $k \geq 0$ such that for all $\psi \in \mathcal{D}(\Omega)$ with support in K , we have that

$$|\langle T, \psi \rangle| \leq C \cdot \|\psi\|_{\mathcal{C}^k(\Omega)}. \quad (3.41)$$

Also notice that for any $\psi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} \langle \pi_\alpha^\sigma(T), \psi \rangle &= \left\langle \int_{\mathbb{T}^n} \lambda^{-\alpha} \sigma_\lambda(T) d\lambda, \psi \right\rangle = \int_{\mathbb{T}^n} \lambda^{-\alpha} \langle \sigma_\lambda(T), \psi \rangle d\lambda \quad \text{using (2.4)} \\ &= \int_{\mathbb{T}^n} \lambda^{-\alpha} \langle T, \tau_{\lambda^{-1}} \psi \rangle d\lambda = \left\langle T, \int_{\mathbb{T}^n} \lambda^{-\alpha} \tau_{\lambda^{-1}} \psi d\lambda \right\rangle \quad \text{using (3.33) and (2.4)} \\ &= \left\langle T, \int_{\mathbb{T}^n} \lambda^\alpha \tau_\lambda \psi d\lambda \right\rangle \\ &\quad \text{using the invariance of Haar measure of a compact group under inversion} \\ &= \langle T, \pi_{-\alpha}^\tau \psi \rangle. \end{aligned} \quad (3.42)$$

In the following estimates, let C denote a constant that depends only on the compact K and the distribution T , and may have different values at different occurrences. By combining (3.41) with (3.42), we see that for $z \in K$, $\phi \in \mathcal{D}(\Omega)$ and each α with $|\alpha| \geq 2k$, using Lemma 3.3

$$\begin{aligned} |a_\alpha(T) z^\alpha| &= |\langle T_\alpha, \psi_z \rangle| = |\langle \pi_\alpha^\sigma T, \psi_z \rangle| = |\langle T, \pi_{-\alpha}^\tau \psi_z \rangle| \\ &\leq C \cdot \|\pi_{-\alpha}^\tau \psi_z\|_{\mathcal{C}^k(\Omega)} \quad \text{using (3.41)} \\ &\leq \frac{C}{|\alpha|^2} \cdot \|\psi_z\|_{\mathcal{C}^{2nk+2n}} \quad \text{by Proposition 3.4 with } m = 2, \\ &= \frac{C}{|\alpha|^2} \cdot \|\psi_z\|_{\mathcal{C}^{2nk+2n}}, \end{aligned}$$

recalling that the local order k depends only on the distribution T and the compact set K . Now, for each z we have $\|\psi_z\|_{\mathcal{C}^{2nk+2n}} = \|\Psi\|_{\mathcal{C}^{2nk+2n}}$ by translation invariance of the

norm. Therefore, for each $\alpha \geq 2k$ we obtain the estimate for the seminorm (with the same convention as above on the constant C)

$$p_K(T_\alpha) = p_K(\pi_\alpha^\sigma T) = \sup_{z \in K} |a_\alpha(T)z^\alpha| \leq \frac{C}{|\alpha|^2} \cdot \|\Psi\|_{\mathcal{C}^{2nk+2n}} = \frac{C}{|\alpha|^2}. \quad (3.43)$$

Clearly therefore $\sum_{\alpha \in \mathbb{Z}^n} p_K(T_\alpha) < \infty$. By Lemma 3.7, the series $\sum_\alpha T_\alpha$ converges absolutely in $\mathcal{C}^\infty(\Omega)$. Let f be its sum. Since the inclusion $\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega)$ is continuous, we see easily that the Fourier series $\sum_\alpha T_\alpha$ of $T \in \mathcal{D}'(\Omega)$ converges absolutely in $\mathcal{D}'(\Omega)$. Now it follows from Corollary 2.5 that the sum of the series is T . Thus T is the distribution generated by f , and this completes the proof of the proposition. \square

3.9. Smoothness of holomorphic distributions.

Corollary 3.9. *Let $\Omega \subset \mathbb{C}^n$ be an open set. Then each holomorphic distribution on Ω is generated by a \mathcal{C}^∞ -smooth function, i.e. $\mathcal{O}(\Omega) = (\mathcal{O} \cap \mathcal{C}^\infty)(\Omega)$.*

Proof. Let $T \in \mathcal{O}(\Omega)$. It suffices to show that T is a smooth function in a neighborhood of each point $p \in \Omega$. Without loss of generality, thanks to the invariance of the space of holomorphic functions under translations (Proposition 3.2 and comments after it), we can assume that $p = 0$. Let $r > 0$ be such that the polydisc $P(r)$ given by

$$P(r) = \{|z_j| < r, 1 \leq j \leq n\} \subset \mathbb{C}^n$$

is contained in Ω . We will show that T is a smooth function on $P' = P(\frac{r}{6})$, i.e., $T \in (\mathcal{O} \cap \mathcal{C}^\infty)(P')$.

By the previous section, holomorphic distributions on an annular Reinhardt domain are smooth. By translation invariance, the same is true if the annular domain is centered at a point $a \in \mathbb{C}^n$ different from the origin, i.e. for a domain of the form $M_a(A)$ where A is an annular Reinhardt domain centered at the origin, with M_a as in (3.15) is the translation by a .

Indeed, let $a = (\frac{r}{3}, \dots, \frac{r}{3})$, and set $Q = M_a(P(\frac{r}{2}) \setminus Z)$, so that Q is the annular Reinhardt domain centered at a given by

$$Q = \left\{ z \in \mathbb{C}^n : 0 < \left| z_j - \frac{r}{3} \right| < \frac{r}{2}, 1 \leq j \leq n \right\},$$

for which it is easy to verify that $P' \subset Q \subset P(r)$. Now since $T|_Q \in (\mathcal{O} \cap \mathcal{C}^\infty)(Q)$ is a holomorphic function, the result follows. \square

3.10. Extension to the log-convex hull. In this section, we prove the following special case of Theorem 3.1 for annular Reinhardt domains:

Proposition 3.10. *Let Ω be an annular Reinhardt domain in \mathbb{C}^n . Then the Laurent series of a distribution $T \in \mathcal{O}(\Omega)$ converges absolutely in $\mathcal{C}^\infty(\hat{\Omega})$ where $\hat{\Omega}$ is the smallest log-convex annular Reinhardt domain containing Ω .*

Introduce the following notation: for a compact subset $K \subset \mathbb{C}^n \setminus Z = (\mathbb{C}^*)^n$, we let

$$\hat{K} = \left\{ z \in \mathbb{C}^n \setminus Z : |e_\alpha(z)| \leq \sup_K |e_\alpha|, \text{ for all } \alpha \in \mathbb{Z}^n \right\}. \quad (3.44)$$

Lemma 3.11. *Let Ω be an annular Reinhardt domain in \mathbb{C}^n and let $\hat{\Omega}$ be the log-convex annular Reinhardt domain containing Ω . Then for each point $p \in \hat{\Omega}$ there is a compact neighborhood M of p in $\hat{\Omega}$ and a compact subset $K \subset \Omega$ such that $M \subset \hat{K}$.*

Proof. Let $\Lambda : \mathbb{C}^n \setminus Z \rightarrow \mathbb{R}^n$ be the map

$$\Lambda(z_1, \dots, z_n) = (\log |z_1|, \dots, \log |z_n|). \quad (3.45)$$

From (3.8), it follows that an annular Reinhardt domain $V \subset \mathbb{C}^n$ is log-convex if and only if the subset $\Lambda(V)$ of \mathbb{R}^n (called the *logarithmic shadow* of V) is convex. The set $\widehat{\Omega}$ is characterized by the fact that it is the convex hull of $\Lambda(\Omega)$. Therefore, if $p \in \widehat{\Omega}$, there exist m points $q^1, \dots, q^m \in V \setminus Z$ and positive numbers t_1, \dots, t_m , with $\sum_{k=1}^m t_k = 1$ such that for $1 \leq j \leq n$, we have $\Lambda(p) = \sum_{k=1}^m t_k \Lambda(q^k)$, i.e.,

$$\log |p_j| = \sum_{k=1}^m t_k \log |q_j^k|.$$

In other words $\Lambda(p)$ lies in the convex hull of the m -element set $\{\Lambda(q^1), \dots, \Lambda(q^m)\}$. (By a theorem of Carathéodory, $m \leq n + 1$, but we do not need this fact.) We will show that we can “fatten” one the points in this set so that the convex hull of the fattened set contains a neighborhood of the point $\Lambda(p)$.

Without loss of generality, we can assume that $t_m \neq 0$. Consider the affine self-map F of \mathbb{R}^n given by

$$F(x) = t_m x + \sum_{k=1}^{m-1} t_k \Lambda(q_j^k),$$

which is clearly an affine automorphism of \mathbb{R}^n . Therefore if L is a compact neighborhood of $\Lambda(p)$ in \mathbb{R}^n , then $F^{-1}(L)$ is a compact neighborhood of $\Lambda(q^m)$, which after shrinking can be taken to be contained in $\Lambda(\Omega)$. Let $K_0 = \Lambda^{-1}(F^{-1}(L))$, and set

$$K = \{q^1, \dots, q^{m-1}\} \cup K_0.$$

We claim that \widehat{K} contains the compact neighborhood $\Lambda^{-1}(L)$ of the point p . Let $\zeta \in \Lambda^{-1}(L)$. Then there is a point $z \in K_0$ such that $\Lambda(\zeta) = \sum_{k=1}^{m-1} t_k \Lambda(q^k) + t_m \Lambda(z)$, i.e. for each $1 \leq j \leq n$ we have

$$\log |\zeta_j| = \sum_{k=1}^{m-1} t_k \log |q_j^k| + t_m \log |z_j|.$$

For simplicity of writing, introduce new notation as follows: we let $z^k = q^k$ for $1 \leq k \leq m-1$ and $z^m = z$. Then we have for a multi-index $\alpha \in \mathbb{Z}^n$ and the point $\zeta \in \Lambda^{-1}(L)$:

$$\begin{aligned} |e_\alpha(\zeta)| &= \prod_{j=1}^n |\zeta_j|^{\alpha_j} = \exp \left(\sum_{j=1}^n \alpha_j \log |\zeta_j| \right) \\ &= \exp \left(\sum_{j=1}^n \alpha_j \sum_{k=1}^m t_k \log |z_j^k| \right) = \exp \left(\sum_{k=1}^m t_k \sum_{j=1}^n \alpha_j \log |z_j^k| \right) \\ &\leq \exp \left(\max_{1 \leq k \leq m} \sum_{j=1}^n \alpha_j \log |z_j^k| \right) \\ &= \max_{1 \leq k \leq m} \exp \left(\sum_{j=1}^n \alpha_j \log |z_j^k| \right) \end{aligned} \quad (3.46)$$

$$\begin{aligned}
&= \max_{1 \leq k \leq m} |e_\alpha(z^k)| = \max \left\{ \max_{1 \leq k \leq m-1} |e_\alpha(q^k)|, |e_\alpha(z)| \right\} \\
&\leq \max_K |e_\alpha|,
\end{aligned} \tag{3.47}$$

this shows that $\zeta \in \widehat{K}$, and completes the proof. \square

Proof of Proposition 3.10. By Lemma 3.7 it is sufficient to show that the Laurent series $\sum_\alpha a_\alpha(T)e_\alpha$ (whose existence was proved in Proposition 3.8, and whose terms lie in $(\mathcal{O} \cap \mathcal{C}^\infty)(\widehat{\Omega})$) converges absolutely in $\mathcal{C}(\widehat{\Omega})$. It is sufficient to show that each point p of $\widehat{\Omega}$ has a compact neighborhood M such that we have

$$\sum_\alpha p_M(a_\alpha(T)e_\alpha) < \infty.$$

Now by Lemma 3.11, there is a compact subset $K \subset \Omega$ such that $M \subset \widehat{K}$. We observe, using the definition (3.44) that

$$\begin{aligned}
p_M(a_\alpha(T)e_\alpha) &\leq p_{\widehat{K}}(a_\alpha(T)e_\alpha) = \sup_{\widehat{K}} |a_\alpha(T)e_\alpha| = |a_\alpha(T)| \cdot \sup_{\widehat{K}} |e_\alpha| \\
&= |a_\alpha(T)| \cdot \sup_K |e_\alpha| = \sup_K |a_\alpha(T)e_\alpha| = p_K(a_\alpha(T)e_\alpha),
\end{aligned}$$

so that

$$\sum_\alpha p_M(a_\alpha(T)e_\alpha) \leq \sum_\alpha p_K(a_\alpha(T)e_\alpha) < \infty,$$

using the estimate (3.43) which holds since $K \subset \Omega$. \square

3.11. Laurent series on non-annular Reinhardt domains. We now consider the case of a general (i.e. possibly non-annular) Reinhardt domain Ω . We begin by showing that the only monomials that occur in the Laurent series are the ones smooth on Ω :

Proposition 3.12. *Let Ω be a (possibly non-annular) Reinhardt domain in \mathbb{C}^n . Then for each $\alpha \in \mathcal{S}(\Omega)$ (where $\mathcal{S}(\Omega)$ is as in (3.11)), there is a continuous linear functional $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ such that the Fourier series of a $T \in \mathcal{O}(\Omega)$ is of the form*

$$T \sim \sum_{\alpha \in \mathcal{S}(\Omega)} a_\alpha(T)e_\alpha. \tag{3.48}$$

Proof. Since $T \in \mathcal{O}(\Omega)$ we have $T|_{\Omega \setminus Z} \in \mathcal{O}(\Omega \setminus Z)$. Notice that $\Omega \setminus Z$ is an annular Reinhardt domain and therefore the proof of Proposition 3.8 shows that the Fourier components are given for $\alpha \in \mathbb{Z}^n$ by

$$(T|_{\Omega \setminus Z})_\alpha = \widehat{a}_\alpha(T|_{\Omega \setminus Z})e_\alpha,$$

where $\widehat{a}_\alpha : \mathcal{O}(\Omega \setminus Z) \rightarrow \mathbb{C}$ is the α -th coefficient functional associated to the domain $\Omega \setminus Z$ (see Proposition 3.8). Thanks to (3.35), we know that $(T|_{\Omega \setminus Z})_\alpha = (T_\alpha)|_{\Omega \setminus Z}$, so we have

$$(T_\alpha)|_{\Omega \setminus Z} = \widehat{a}_\alpha(T|_{\Omega \setminus Z})e_\alpha.$$

By Proposition 3.6, the Fourier components of a holomorphic distribution are holomorphic distributions, so we have $T_\alpha \in \mathcal{O}(\Omega)$ since $T \in \mathcal{O}(\Omega)$. Therefore, since by Corollary 3.9, each holomorphic distribution is a smooth function, we know that $T_\alpha \in \mathcal{C}^\infty(\Omega)$. Therefore, the function $T_\alpha|_{\Omega \setminus Z} = \widehat{a}_\alpha(T|_{\Omega \setminus Z})e_\alpha$ admits a \mathcal{C}^∞ extension through Z . If $\widehat{a}_\alpha(T|_{\Omega \setminus Z}) \neq 0$ for some $T \in \mathcal{O}(\Omega)$, this means that e_α itself admits a \mathcal{C}^∞ extension to Ω , i.e., $\alpha \in \mathcal{S}(\Omega)$,

where $\mathcal{S}(\Omega)$ is as in (3.11). Therefore if $\alpha \notin \mathcal{S}(\Omega)$, the corresponding term in the Laurent series of $T|_{\Omega \setminus Z}$ vanishes, and the series takes the form

$$T|_{\Omega \setminus Z} = \sum_{\alpha \in \mathcal{S}(\Omega)} \hat{a}_\alpha(T|_{\Omega \setminus Z}) e_\alpha.$$

Now for each $\alpha \in \mathcal{S}(\Omega)$ define the map $a_\alpha : \mathcal{O}(\Omega) \rightarrow \mathbb{C}$ by $a_\alpha(T) = \hat{a}_\alpha(T|_{\Omega \setminus Z})$. Since both the restriction map and the coefficient functional \hat{a}_α are continuous, it follows that a_α is continuous. The extension of e_α from $\Omega \setminus Z$ to Ω is still the monomial e_α , so the Fourier series of T in $\mathcal{O}(\Omega)$ is of the form (3.48). \square

Notice that each term of (3.48) is in $\mathcal{C}^\infty(\Omega)$ and by Proposition 3.8, the series converges absolutely in $\mathcal{C}^\infty(\Omega \setminus Z)$ when its terms are restricted to $\Omega \setminus Z$, i.e., it converges uniformly along with all derivatives on those compact sets in Ω which do not intersect Z .

3.12. Extension to the relative completion. Given a Reinhardt domain $D \subset \mathbb{C}^n$, its *relative completion* is the smallest relatively complete domain containing D (see above before the statement of Theorem 3.1 for the definition of relative completeness of a domain.) Notice that the relative completion of D coincides with the unions of the sets $D^{(j)}$ of (3.10), where the union is taken over those j for which $D \cap \{z_j = 0\}$ is nonempty. The following general proposition, which encompasses classical examples of the Hartogs phenomenon, e.g. in the ‘‘Hartogs figure’’, will be needed to complete the proof of Theorem 3.1.

Proposition 3.13. *Let D be a Reinhardt domain. Then each holomorphic function on D extends holomorphically to the relative completion of D .*

Proof. We may assume that $n \geq 2$ since each Reinhardt domain in the plane is automatically relatively complete. If for each j , the intersection $D \cap \{z_j = 0\} = \emptyset$, then the domain D is annular, and its relative completion is itself so there is nothing to prove. Suppose therefore that there is $1 \leq j \leq n$ such that $D \cap \{z_j = 0\} \neq \emptyset$. We need to prove that each function in $\mathcal{O}(D)$ extends holomorphically to $D^{(j)}$.

Without loss of generality we can assume that $j = 1$. Write the coordinates of a point $z \in \mathbb{C}^n$ as $z = (z_1, \tilde{z})$ where $\tilde{z} \in \mathbb{C}^{n-1}$. Let $f \in \mathcal{O}(D)$. By Proposition 3.12 f admits a Laurent series representation

$$f = \sum_{\alpha \in \mathcal{S}(D)} a_\alpha(f) e_\alpha$$

with $\mathcal{S}(D)$ as in (3.11), and the series converges absolutely in $\mathcal{C}^\infty(D \setminus Z)$. To prove the proposition it suffices to show that the series in fact converges absolutely in $\mathcal{C}^\infty(D^{(1)})$, which by Lemma 3.7 is equivalent to the following: for each point $p \in D^{(1)}$ there is a compact neighborhood M of p in $D^{(1)}$ such that

$$\sum_{\alpha \in \mathcal{S}(D)} p_M(a_\alpha(f) e_\alpha) < \infty. \quad (3.49)$$

We claim the following: for each $p \in D^{(1)}$ there is a compact neighborhood M of p and a compact subset $K \subset D$ such that $p_M(e_\alpha) \leq p_K(e_\alpha)$ for each $\alpha \in \mathcal{S}(D)$. Assuming the claim for a moment, we see that we have

$$p_M(a_\alpha(f) e_\alpha) = |a_\alpha(f)| p_M(e_\alpha) \leq |a_\alpha(f)| p_K(e_\alpha) = p_K(a_\alpha(f) e_\alpha),$$

so that (3.49) follows since by (3.43) we do have $\sum_{\alpha \in \mathcal{S}(D)} p_K(a_\alpha(f)e_\alpha) < \infty$ for a compact subset K of D . Since each point $p \in D^{(1)}$ has such a neighborhood this completes the proof, modulo the claim above.

To establish the claim, we may assume that $p \notin D$, since otherwise there is nothing to prove. Therefore $p \in D^{(1)} \setminus D$, and consequently, there is a $z \in D$ and a $\lambda \in \mathbb{D}$ such that $p = (\lambda z_1, \tilde{z})$, where $\tilde{z} = (z_2, \dots, z_n)$. Let K be a compact neighborhood of z in D of the form $K = K_1 \times \tilde{K}$, where $\tilde{K} \subset \mathbb{C}^{n-1}$ and $K_1 = \{\zeta \in \mathbb{C} : |\zeta - z_1| \leq \epsilon\}$ is a closed disk. Let L_1 be the disk $\{\zeta \in \mathbb{C} : |\zeta| \leq |z_1| + \epsilon\}$, so that $K_1 \subset L_1$, and the point

$$\gamma = z_1 + \frac{z_1}{|z_1|} \epsilon$$

is a point of maximum modulus (i.e. maximum distance from the origin) in both sets K_1 and L_1 . Notice also that

$$|\gamma| = \left| \frac{z_1}{|z_1|} \cdot |z_1| + \frac{z_1}{|z_1|} \epsilon \right| = |z_1| + \epsilon.$$

We set $M = L_1 \times \tilde{K}$. We will show that these sets K, M satisfy the conditions of the claim.

Now let $\alpha \in \mathcal{S}(D)$. Since $D \cap \{z_1 = 0\} \neq \emptyset$, it follows that $\alpha_1 \geq 0$. Let $\tilde{\alpha} = (\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$, and set

$$B = \sup_{\tilde{w} \in \tilde{K}} \left| \tilde{w}^{\tilde{\alpha}} \right| \quad \text{where } \tilde{w} = (w_2, \dots, w_n) \in \mathbb{C}^{n-1} \text{ and } \left| \tilde{w}^{\tilde{\alpha}} \right| = |w_2^{\alpha_2}| \cdots |w_n^{\alpha_n}|,$$

so that we have

$$p_M(e_\alpha) = \sup_{w \in M} |e_\alpha(w)| = \sup_{w_1 \in L_1} |w_1|^{\alpha_1} \cdot B = (|z_1| + \epsilon)^{\alpha_1} \cdot B.$$

On the other hand

$$p_K(e_\alpha) = \sup_{w \in K} |e_\alpha(w)| = \sup_{w_1 \in K_1} |w_1|^{\alpha_1} \cdot B = (|z_1| + \epsilon)^{\alpha_1} \cdot B.$$

Consequently, in fact we have $p_M(e_\alpha) = p_K(e_\alpha)$, and the claim is proved, thus completing the proof of the result. \square

3.13. End of proof of Theorem 3.1. It only remains to put together the various pieces to note that all parts of Theorem 3.1 have been established. If Ω is annular, i.e. Ω has empty intersection with the set Z of (3.7), then Proposition 3.8 takes care of the complete proof. When Ω is allowed to be non-annular, we see from Proposition 3.12 that the Laurent series representation has only monomials which are smooth functions on Ω . Now it is not difficult to see that the smallest log-convex relatively complete Reinhardt domain $\hat{\Omega}$ containing Ω can be constructed from Ω in two steps. First, we construct the log-convex hull Ω_1 of the set $\Omega \setminus Z$. Notice that $\Omega \setminus Z$ and Ω_1 are both annular. The second step consists of constructing the relative completion of the domain Ω_1 , thus obtaining the domain $\hat{\Omega}$. Now by Proposition 3.10, the Laurent series of a holomorphic distribution on Ω converges absolutely in $\mathcal{C}^\infty(\Omega_1)$. Applying Proposition 3.13 (with $D = \Omega_1$), we see that the Laurent series actually converges absolutely in the space $\mathcal{C}^\infty(\hat{\Omega})$. The sum of this series is the required holomorphic extension of a given holomorphic distribution on Ω . The result has been completely established.

4. MISSING MONOMIALS

In Theorem 3.1, we considered the natural representation of the torus \mathbb{T}^n on the space $\mathcal{O}(\Omega)$ of holomorphic functions on a Reinhardt domain Ω . In applications, one often deals with a subspace of functions $Y \subset \mathcal{O}(\Omega)$ such that

- (1) the subspace Y is invariant under the natural representation σ , i.e., if $f \in Y$ then $f \circ R_\lambda \in Y$ for each $\lambda \in \mathbb{T}^n$, where R_λ is as in (3.6),
- (2) there is a locally convex topology on Y in which it is complete, and such that the inclusion map

$$j : Y \hookrightarrow \mathcal{O}(\Omega)$$

is continuous, and

- (3) when Y is given this topology, the representation σ restricts to a continuous representation on Y .

The *locus classicus* here is the theory of Hardy spaces on the disc. We can make the following elementary observation:

Proposition 4.1. *Let Y and σ be as above, and set*

$$\mathcal{S}(Y) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in Y\}.$$

Then the Laurent series of a function $f \in Y$ is of the form

$$f = \sum_{\alpha \in \mathcal{S}(Y)} a_\alpha(f) e_\alpha.$$

Proof. It suffices to show that if for $\alpha \in \mathbb{Z}^n$, if the monomial e_α does not belong to Y , we have that $a_\alpha(f) = 0$, where $a_\alpha(f)$ is the Laurent coefficient of f as in (3.12). By part (1) of Proposition 2.4, we see that $\pi_\alpha^\sigma(f) \in Y$. By part (2) of the same proposition, taking X to be the space $\mathcal{O}(\Omega)$, for $f \in Y$ we have that $\pi_\alpha^\sigma(j(f)) = j(\pi_\alpha^\sigma(f))$, and from the description of the Fourier components of a holomorphic function in the proof of Theorem 3.1, we see that $\pi_\alpha^\sigma(j(f)) = a_\alpha(j(f))e_\alpha = a_\alpha(f)e_\alpha$. Therefore $j(\pi_\alpha^\sigma(f)) = a_\alpha(f)e_\alpha$, which contradicts the fact that $\pi_\alpha^\sigma(f) \in Y$ unless $a_\alpha(f) = 0$. \square

This simple observation can be called the ‘‘principle of missing monomials’’, since it says that certain monomials cannot occur in the Laurent series of the function f . It can be thought to be the reason behind several phenomena associated to holomorphic functions. We consider two examples:

(1) **Bergman spaces in Reinhardt domains:** Let Ω be a Reinhardt domain in \mathbb{C}^n and let $\lambda > 0$ be a radial weight on Ω , i.e., for $z \in \Omega$ we have

$$\lambda(z_1, \dots, z_n) = \lambda(|z_1|, \dots, |z_n|).$$

The *L^p -Bergman space* $A^p(\Omega, \lambda)$ is defined to be the subspace of the weighted L^p -space $L^p(\Omega, \lambda)$ consisting of holomorphic functions, where the norm on the weighted L^p -space is given by

$$\|f\|_{L^p(\Omega, \lambda)}^p = \int_{\Omega} |f|^p \lambda dV,$$

where dV is the Lebesgue measure. It is well-known that $A^p(\Omega, \lambda)$ is a closed subspace of the Banach space $L^p(\Omega, \lambda)$ and therefore a Banach space ([DS04]). It is also easy to see (using standard facts about L^p -spaces) that the natural representation σ of \mathbb{T}^n on $L^p(\Omega, \lambda)$ is continuous for $1 \leq p < \infty$, so it follows that the representation on $A^p(\Omega, \lambda)$ is

also continuous. It now follows from Proposition 4.1 that the Laurent series expansion of a function $f \in A^p(\Omega, \lambda)$ consists only of terms with monomials e_α such that $e_\alpha \in L^p(\Omega, \lambda)$. The case $\lambda \equiv 1$ of this fact was deduced by a different argument in [CEM18].

(2) **Extension of holomorphic functions smooth up to the boundary:** Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain such that the origin (which is the center of symmetry) is on the boundary of Ω . A classic example of this is the *Hartogs triangle* $\{|z_1| < |z_2| < 1\}$ in \mathbb{C}^2 . In [Cha19], the following extension theorem was proved: there is a complete Reinhardt domain $V \subset \mathbb{C}^n$ such that $\Omega \subset V$ and each function in the space $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ of holomorphic functions on Ω smooth up to the boundary extends holomorphically to the domain V . This was noted for the Hartogs triangle in [Sib75], where V is the unit bidisk.

To deduce this from the principle of missing monomials, it suffices to consider the case when Ω is bounded. We notice that $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ is a Fréchet space in its usual topology of uniform convergence on $\overline{\Omega}$ with all partial derivatives. A generating family of seminorms for this topology is given by the norms $\{p_k\}$ where $p_k(f) = \|f\|_{C^k(\overline{\Omega})}$. The natural representation σ on $\mathcal{O}(\Omega)$ restricts to a continuous representation on $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$. Therefore, the principle of missing monomials applies and the only monomials e_α that occur in the Laurent expansion of a function $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ are such that $e_\alpha \in \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$. For such a multi-index α , write the multiindex $\alpha = \beta - \gamma$, where $\beta_j = \max(\alpha_j, 0)$ and $\gamma_j = (-\alpha_j, 0)$. Then $\beta, \gamma \in \mathbb{N}^n$, and we can apply the differential operator $(\frac{\partial}{\partial z})^\beta$ to obtain

$$\left(\frac{\partial}{\partial z}\right)^\beta e_\alpha(z) = \left(\frac{\partial}{\partial z}\right)^\beta \frac{e_\beta(z)}{e_\alpha(z)} = \frac{\beta!}{z^\gamma}.$$

Since $e_\alpha \in \mathcal{C}^\infty(\overline{\Omega})$ this means that $e_\gamma \in \mathcal{C}^\infty(\overline{\Omega})$, which is possible only if $\gamma = 0$. Therefore $\alpha \in \mathbb{N}^n$, and the Laurent series of each function $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ is a Taylor series which converges in some complete (log-convex) Reinhardt domain V , and this V must strictly contain Ω , since Ω is not complete.

5. CLASSICAL CHARACTERIZATIONS OF HOLOMORPHIC FUNCTIONS

In this section we show how one can avoid the machinery of generalized functions and weak derivatives altogether, and still use Fourier methods to prove the basic facts of function theory. We confine ourselves to one complex variable and the simple geometry of the disk.

5.1. Goursat's characterization. The starting point of a traditional account of holomorphic functions of a single variable is typically Goursat's definition ([Gou00]): a function $f : \Omega \rightarrow \mathbb{C}$ on an open subset $\Omega \subset \mathbb{C}$ is holomorphic, if it is *complex differentiable*, i.e., for each point $w \in \Omega$, the limit

$$\lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w} \tag{5.1}$$

exists. The result that a holomorphic function in this sense is infinitely many times complex differentiable and even admits a convergent power series representation near each point is rightly celebrated as one of the most elegant and surprising in all of mathematics. Unfortunately, we cannot use it as a definition, if we want to apply the theory of abstract Fourier expansions as developed in Section 2. Denoting by $G(\Omega)$ the collection of holomorphic functions in the sense of Goursat in an open set $\Omega \subset \mathbb{C}$, we notice that

the space $G(\Omega)$ does not have a nice a priori linear locally convex topology in which it is complete and such that when Ω is a disc or an annulus, the natural action of the group \mathbb{T} on the space $G(\Omega)$ is a continuous representation. Though Goursat's definition carries the weight of a century of academic tradition, we will start from an alternative definition which lends itself better to the application of the methods of Section 2. We also note that the characterization of holomorphic functions by complex-differentiability cannot be used for natural generalizations of complex analysis, e.g. quaternionic analysis, analysis on Clifford algebras etc. (see [GM91, pp. 87–93]).

5.2. Morera's definition. Let Ω be an open subset of the complex plane \mathbb{C} , and let $f : \Omega \rightarrow \mathbb{C}$ be a continuous function. In honor of [Mor02], let us say that the function f is *holomorphic in the sense of Morera* (*Morera-holomorphic* for short) if for each triangle T contained (with its interior) in Ω , we have the vanishing of the complex line integral of f around the boundary of T :

$$\int_{\partial T} f(z)dz = 0, \quad (5.2)$$

where ∂T denotes the boundary of T , oriented counterclockwise. For an open set $\Omega \subset \mathbb{C}$, let us denote by $\mathcal{O}_M(\Omega)$ the collection of Morera-holomorphic functions on Ω . It is known by Morera's theorem that a Morera-holomorphic function is Goursat-holomorphic, and one can develop function theory starting from Morera's definition (see [Hef55, MW67]). Notice that the a priori regularity of Morera-holomorphic functions (assumed to be only continuous) is even less than that assumed for Goursat-holomorphic functions (assumed also to admit the limit (5.1) at each w .)

It is immediate from the definition that $\mathcal{O}_M(\Omega)$ is a closed linear subspace of the Fréchet space $\mathcal{C}(\Omega)$ of continuous functions. "Closed" means that the limit of a sequence of Morera-holomorphic functions converging uniformly on compact subsets of Ω is itself Morera-holomorphic, a fact that was already noted in [Mor86]. The proof of this crucial fact starting from the Goursat definition must pass through a lengthy development of integral representations, so this is definitely a pedagogical advantage of Morera's definition over Goursat's.

The notion of Morera-holomorphicity is local: i.e., $f \in \mathcal{O}_M(\Omega)$ if and only if there is an open cover $\{\Omega_j\}_{j \in J}$ of Ω such that $f|_{\Omega_j} \in \mathcal{O}_M(\Omega_j)$ for each j . One half of this claim is trivial, and for the other half, for a triangle T in Ω , we can perform repeated barycentric subdivisions till the triangles so formed are each contained in some element of the open cover $\{\Omega_j\}_{j \in J}$. We therefore conclude that \mathcal{O}_M is a *sheaf* of Fréchet spaces on \mathbb{C} .

The following local description of Morera-holomorphic functions is well-known, and a proof can be found in e.g. [Rem91, pp. 186-189].

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be convex. Then the following statements about a continuous function $f \in \mathcal{C}(\Omega)$ are equivalent:*

- (A) $f \in \mathcal{O}_M(\Omega)$.
- (B) f has a holomorphic primitive, i.e., there is an F which is complex-differentiable on Ω and $F' = f$.
- (C) for each piecewise \mathcal{C}^1 closed path γ in Ω we have

$$\int_{\gamma} f(z)dz = 0. \quad (5.3)$$

Recall that assuming (A), the primitive F in (B) is constructed by fixing $z_0 \in \Omega$, and setting $F(z) = \int_{[z_0, z]} f(\zeta) d\zeta$ where $[z_0, z]$ denotes the line segment from z_0 to z . Proposition 5.1 allows us to give examples of Morera-holomorphic functions. Recall from (3.9) that for an integer n , we use the notation $e_n(z) = z^n$ for the holomorphic monomials.

Proposition 5.2. *If $n \geq 0$ then $e_n \in \mathcal{O}_M(\mathbb{C})$ and if $n < 0$ then $e_n \in \mathcal{O}_M(\mathbb{C} \setminus \{0\})$.*

Proof. First note that e_n is continuous, on all of \mathbb{C} if $n \geq 0$ and on $\mathbb{C} \setminus \{0\}$ if $n < 0$. If $g_n(z) = \frac{z^{n+1}}{n+1}$ for $n \neq -1$, we can verify from the definition (5.1) that g_n is complex-differentiable and $g'_n = e_n$ so that by part (B) of Proposition 5.1 the result follows for $n \neq -1$. For $n = -1$, we can construct for each $p \in \mathbb{C} \setminus \{0\}$, a local primitive of e_{-1} near p by setting

$$g_{-1}(z) = \ln |z| + i \arg z,$$

where \arg denotes a branch of the argument defined near the point p . A direct computation shows that $g'_{-1} = e_{-1}$ near p , so that again we see that $e_{-1} \in \mathcal{O}_M(\mathbb{C} \setminus \{0\})$. \square

5.3. Products of Morera-holomorphic functions. In the proof of Theorem 3.1, an important role is played by the fact that if U is a holomorphic distribution (in the sense of (3.5)) and f is a holomorphic function (i.e. a holomorphic distribution which is \mathcal{C}^∞ , Section 3.3 above), then the product distribution fU is also a holomorphic distribution. This is an immediate consequence of the distributional Leibniz formula (3.39). A similar result, proved in [MW67], will be needed in order to develop the properties of holomorphic functions starting from Morera's definition.

Proposition 5.3. *Let $f, g \in \mathcal{O}_M(\Omega)$, and assume that g is locally Lipschitz at each point, i.e. for each $w \in \Omega$ and each compact $K \subset \Omega$ such that $w \in K$, and there is an $M > 0$ such that for $z \in K$ we have*

$$|g(z) - g(w)| \leq M |z - w|. \quad (5.4)$$

Then the product fg also belongs to $\mathcal{O}_M(\Omega)$.

The proof is based on a version of the classical Goursat lemma ([Gou00, Pri01]). This is of course the main ingredient in the standard textbook proof of the Cauchy theorem for triangles for Goursat-holomorphic functions. Recall that two triangles are *similar* if they have the same angles.

Lemma 5.4. *Let Ω be an open subset of \mathbb{C} and let λ be a complex valued function defined on the set of triangles contained in Ω such that the following two conditions are satisfied:*

- (1) *λ is additive in the following sense: if a triangle Δ is represented as a union of smaller triangles $\Delta = \bigcup_{k=1}^n \Delta_k$ with pairwise disjoint interiors then*

$$\lambda(\Delta) = \sum_{k=1}^n \lambda(\Delta_k). \quad (5.5)$$

- (2) *For each $w \in \Omega$ and each triangle Δ_0 , we have*

$$\lim_{\substack{\Delta \downarrow w \\ \Delta \sim \Delta_0}} \frac{\lambda(\Delta)}{|\Delta|} = 0, \quad (5.6)$$

where $|\Delta|$ denotes the area of the triangle Δ , and the limit is taken along the family of triangles similar to Δ_0 and containing the point w , as these triangles shrink to the point w .

Then $\lambda \equiv 0$.

Proof. For completeness, we recall the classic argument. Let T be a triangle contained in Ω . We construct a sequence of triangles $\{T_k\}_{k=0}^\infty$ with $T_0 = T$ using the following recursive procedure. Assuming that T_k has been constructed, we divide T_k into four similar triangles with half the diameter of T_k by three line segments each parallel to a side of T_k and passing through the midpoints of the other two sides. Denote the four triangles so obtained by $\Delta_j, 1 \leq j \leq 4$. Then we have, by (5.5):

$$\lambda(T_k) = \sum_{j=1}^4 \lambda(\Delta_j).$$

Choose T_{k+1} to be one of $\Delta_j, 1 \leq j \leq 4$ such that the value of $|\lambda(T_{k+1})|$ is the largest. Then, by the triangle inequality we have $|\lambda(T_k)| \leq 4 |\lambda(T_{k+1})|$, and by induction it follows that By induction therefore we have

$$|\lambda(T)| \leq 4^k |\lambda(T_k)| = 4^k |T_k| \cdot \frac{|\lambda(T_k)|}{|T_k|} = |T| \cdot \frac{|\lambda(T_k)|}{|T_k|}, \quad (5.7)$$

where in the last step we have used the fact that T_{k+1} has one-fourth the area of T_k , so $|T_k| = 4^{-k} |T_0| = 4^{-k} |T|$. Since the diameters of the T_k go to zero, by compactness, there is a unique point w in the intersection $\bigcap_{k=0}^\infty T_k$. Since the family $\{T_k\}$ is a subfamily of all the triangles containing w , and each T_k is similar to T_0 , therefore by letting $k \rightarrow \infty$ in (5.7) and using (5.6) the result follows. \square

Proof of Proposition 5.3. For a triangle $\Delta \subset \Omega$, define

$$\lambda(\Delta) = \int_{\partial\Delta} f(z)g(z)dz.$$

To prove the result, we need to show that $\lambda \equiv 0$. Since condition (5.5) of Lemma 5.4 is obvious, we need to show (5.6) to complete the proof. Let $w \in \Omega$, let K be a compact neighborhood of w in Ω , and denote by M the Lipschitz constant corresponding to this w and this K in (5.4). Now, let Δ_0 be a triangle and let Δ be a triangle similar to Δ_0 such that $w \in \Delta \subset K$. Then observe that, by the hypothesis of Morera-holomorphicity of f and g we have

$$\lambda(\Delta) = \int_{\partial\Delta} (f(z) - f(w))(g(z) - g(w)) dz.$$

Therefore, denoting the perimeter of the triangle Δ by $|\partial\Delta|$,

$$\begin{aligned} |\lambda(\Delta)| &\leq \sup_{z \in \partial\Delta} |(f(z) - f(w))(g(z) - g(w))| |\partial\Delta| \\ &\leq M \cdot |\partial\Delta| \cdot \sup_{z \in \partial\Delta} (|f(z) - f(w)| |z - w|) \\ &\leq M \cdot |\partial\Delta| \cdot \text{diam}(\Delta) \cdot \sup_{z \in \partial\Delta} |f(z) - f(w)| \\ &= M \cdot |\Delta| \cdot \left(\frac{|\partial\Delta| \cdot \text{diam}(\Delta)}{|\Delta|} \right) \cdot \sup_{z \in \partial\Delta} |f(z) - f(w)| \end{aligned}$$

$$= M \cdot |\Delta| \cdot \left(\frac{|\partial\Delta_0| \cdot \text{diam}(\Delta_0)}{|\Delta_0|} \right) \cdot \sup_{z \in \partial\Delta} |f(z) - f(w)|,$$

where in the last step we use the fact that Δ and Δ_0 are similar, so the quantity in parantheses (which is clearly invariant under dilations) is the same. So we have

$$\frac{|\lambda(\Delta)|}{|\Delta|} \leq C \cdot \sup_{z \in \partial\Delta} |f(z) - f(w)|,$$

for a constant C independent of the triangle Δ as long as Δ is similar to Δ_0 and $w \in \Delta \subset K$. Letting Δ shrink to w , we have (5.6) and the proof is complete. \square

5.4. Fourier expansion of a Morera-holomorphic function. We will now prove the following analog of Theorem 3.1. In particular, it shows that holomorphic functions in the sense of Morera are identical to the holomorphic distributions considered in Section 3 above.

Theorem 5.1. *Let $f \in \mathcal{O}_M(\mathbb{D})$ where $\mathbb{D} = \{|z| < 1\}$ is the open unit disc. Then there is a sequence $\{a_n\}_{n=0}^\infty$ of complex numbers, such that*

$$f = \sum_{n=0}^{\infty} a_n e_n, \quad (5.8)$$

where e_n is as in (3.9), and the series on the right converges absolutely in $\mathcal{C}(\mathbb{D})$ to the function f .

Let σ be the natural representation of \mathbb{T} on $\mathcal{C}(\mathbb{D})$ given by

$$\sigma_\lambda(f)(z) = f(\lambda z), \quad \lambda \in \mathbb{T}, z \in \mathbb{D}. \quad (5.9)$$

Proposition 5.5. *The space $\mathcal{O}_M(\mathbb{D})$ is invariant under σ and the resulting representation of \mathbb{T} on $\mathcal{O}_M(\mathbb{D})$ is continuous.*

Proof. Let $\lambda \in \mathbb{T}$, let T be a triangle in \mathbb{D} . Notice that

$$\lambda^{-1}T = \{\lambda^{-1}z : z \in T\}$$

is itself a triangle, and we have

$$\int_{\partial T} (\sigma_\lambda f)(z) dz = \int_{\partial T} f(\lambda z) dz = \lambda^{-1} \int_{\partial(\lambda^{-1}T)} f(w) dw = 0.$$

It follows that $\mathcal{O}_M(\mathbb{D})$ is invariant under σ .

It suffices to show that the representation σ is continuous on $\mathcal{C}(\mathbb{D})$. For $0 < r < 1$, let p_r be the seminorm on $\mathcal{C}(\mathbb{D})$ given by

$$p_r(f) = \sup_{|z| \leq r} |f(z)|. \quad (5.10)$$

It is clear that $p_r(\sigma_\lambda(f)) = p_r(f)$ for each $0 < r < 1$, $\lambda \in \mathbb{T}$ and $f \in \mathcal{C}(\mathbb{D})$. Also $f_j \rightarrow f$ in $\mathcal{C}(\mathbb{D})$ if and only if for each r , we have $p_r(f - f_j) \rightarrow 0$, so the family $\{p_r\}$ is a σ -invariant family of seminorms that generates the topology of $\mathcal{C}(\mathbb{D})$. Further, given $f \in \mathcal{C}(\mathbb{D})$, by uniform continuity, for each $0 < r < 1$, we have

$$p_r(\sigma_\lambda(f) - f) = \sup_{|z| \leq r} |f(\lambda z) - f(z)| \rightarrow 0 \quad \text{as } \lambda \rightarrow 1,$$

so that $\lim_{\lambda \rightarrow 1} \sigma_\lambda(f) = f$ in the space $\mathcal{C}(\mathbb{D})$. Therefore both conditions of Proposition 2.3 are satisfied, and the representation σ is continuous. \square

In view of the above, the machinery of Section 2 applies. We now compute the Fourier components (2.5).

Proposition 5.6. *For $f \in \mathcal{O}_M(\mathbb{D})$, and with σ the natural representation (5.9), the Fourier components of f are of the form:*

$$\pi_n^\sigma(f) = \begin{cases} a_n e_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases}$$

where $a_n \in \mathbb{C}$ and $e_n(z) = z^n$.

The proof will use the following lemma:

Lemma 5.7. *A radial function in $\mathcal{O}_M(\mathbb{D} \setminus \{0\})$ is constant.*

Proof. Let $f \in \mathcal{O}_M(\mathbb{D} \setminus \{0\})$ be radial, and define the complex valued continuous function u on the interval $(0, 1)$ by restriction, $u(r) = f(r)$, so that we have $f(z) = u(|z|)$ by the radially of f . To prove the theorem, it suffices to show that u is a constant.

Fix $0 \leq \alpha < \beta < \pi$ and $0 < \rho < 1$. For R in the interval $(\rho, 1)$ consider the curvilinear quadrilateral defined by

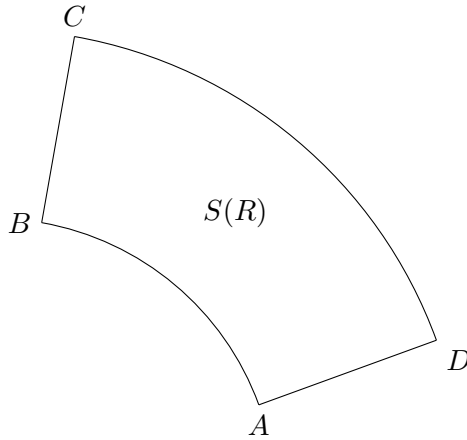
$$S(R) = \{re^{i\theta} : \rho \leq r \leq R, \alpha \leq \theta \leq \beta\}, \quad (5.11)$$

and notice that $S(R)$ lies in the upper half disc, which is convex. The region $S(R)$ is bounded by the two circular arcs

$$AB = \{\rho e^{i\theta} : \alpha \leq \theta \leq \beta\}, \quad CD = \{R e^{i\theta} : \alpha \leq \theta \leq \beta\},$$

along with the two radial line segments

$$AD = \{r e^{i\alpha} : \rho \leq r \leq R\}, \quad BC = \{r e^{i\beta} : \rho \leq r \leq R\}.$$



Orient the boundary $\partial S(R)$ counterclockwise. We can write

$$\int_{\partial S(R)} f(z) dz = \int_{AD} + \int_{DC} + \int_{CB} + \int_{BA} = 0,$$

where the vanishing of the integral follows from part (c) of Proposition 5.1 above. Parametrizing AD by $z = re^{i\alpha}$ where $\rho \leq r \leq R$, and using $dz = e^{i\alpha}dr$ we get

$$\int_{AD} f(z)dz = \int_{\rho}^R f(re^{i\alpha})e^{i\alpha}dr = e^{i\alpha} \int_{\rho}^R u(r)dr.$$

Similarly,

$$\int_{CB} f(z)dz = -e^{i\beta} \int_{\rho}^R u(r)dr.$$

Now, parametrizing DC by $z = Re^{i\theta}$, $\alpha \leq \theta \leq \beta$, we have $dz = Rie^{i\theta}d\theta$, so

$$\int_{DC} f(z)dz = \int_{\theta=\alpha}^{\beta} f(Re^{i\theta}) \cdot Rie^{i\theta}d\theta = Ru(R) \int_{\alpha}^{\beta} ie^{i\theta}d\theta = Ru(R)(e^{i\beta} - e^{i\alpha}).$$

Similarly,

$$\int_{BA} f(z)dz = \rho u(\rho)(e^{i\beta} - e^{i\alpha}).$$

Therefore, adding the four integrals we have

$$\int_{\partial S} f(z)dz = \left(\rho u(\rho) - Ru(R) + \int_{\rho}^R u(r)dr \right) (e^{i\alpha} - e^{i\beta}) = 0,$$

so that we have the relation

$$\int_{\rho}^R u(r)dr = Ru(R) - \rho u(\rho), \quad (5.12)$$

which can be written as

$$u(R) = \frac{1}{R} \left(\int_{\rho}^R u(r)dr + \rho u(\rho) \right).$$

Since u is continuous, the right hand side is a function of R which is continuously differentiable on $(\rho, 1)$. Therefore $u \in C^1(\rho, 1)$. Differentiating both sides of (5.12) with respect to R we have $u(R) = Ru'(R) + u(R)$ so that we have $u'(R) = 0$ for $R \in (\rho, 1)$. But $0 < \rho < 1$ can be chosen arbitrarily, so it follows that $u' \equiv 0$ on $(0, 1)$ and the result follows. \square

Proof of Proposition 5.6. Let $n \in \mathbb{Z}$ and for simplicity of notation, let $f_n = \pi_n^\sigma(f)$. Then by Proposition 2.4, we see that f_n lies in the Fourier mode $[\mathcal{O}_M(\mathbb{D})]_n^\sigma$ so that $f_n \in \mathcal{O}_M(\mathbb{D})$ and for $\lambda \in T, z \in \mathbb{D}$ we have $f_n(\lambda z) = \lambda^n f_n(z)$. Since clearly $f_n|_{\mathbb{D} \setminus \{0\}} \in \mathcal{O}_M(\mathbb{D} \setminus \{0\})$, it follows by Proposition 5.3 that the function h_n on $\mathbb{D} \setminus \{0\}$ defined by

$$h_n = e_{-n} \cdot f_n$$

lies in $\mathcal{O}_M(\mathbb{D} \setminus \{0\})$. Further we have for $\lambda \in T, z \in \mathbb{D}$

$$h_n(\lambda z) = e_n(\lambda z)f_n(\lambda z) = \lambda^{-n}e_{-n}(z) \cdot \lambda^n f_n(z) = h_n(z),$$

so h_n is radial, and hence by Lemma 5.7, h_n is a constant, which we call a_n . Therefore, on $\mathbb{D} \setminus \{0\}$, we have $f_n = a_n e_n$, so the product $a_n e_n$ extends for each n to a continuous function on \mathbb{D} . If $n < 0$, this is possible only if $a_n = 0$, and this completes the proof. \square

5.5. Conclusion of the proof of Theorem 5.1. We will first show that the series $\sum_{n=0}^{\infty} a_n e_n$ converges absolutely in the space $\mathcal{C}(\mathbb{D})$. It suffices to show that for each $0 < r < 1$, we have

$$\sum_{n=0}^{\infty} p_r(a_n e_n) < \infty,$$

where p_r is as in (5.10). Fix an r with $0 < r < 1$ and let ρ be such that $r < \rho < 1$. Applying the Cauchy estimate (2.9) to the seminorm p_ρ we see that for each $n \in \mathbb{Z}$:

$$p_\rho(\pi_n^\sigma f) \leq p_\rho(f)$$

Thanks to Proposition 5.6 we have for each $n \geq 0$,

$$p_r(\pi_n^\sigma f) = \sup_{|z| \leq r} |a_n z^n| = |a_n| r^n \leq \left(\frac{r}{\rho}\right)^n \cdot |a_n| \rho^n \leq \left(\frac{r}{\rho}\right)^n \cdot p_\rho(f).$$

Therefore we have

$$\sum_{n=-\infty}^{\infty} p_r(\pi_n^\sigma f) = \sum_{n=0}^{\infty} p_r(\pi_n^\sigma f) \leq \left(\sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n\right) \cdot p_\rho(f) = \frac{1}{1 - \frac{r}{\rho}} \cdot p_\rho(f) < \infty.$$

This proves that the series $\sum_{n=0}^{\infty} a_n e_n$ converges absolutely in $\mathcal{C}(\mathbb{D})$. Let g be its sum. Since the partial sums are all in $\mathcal{O}_M(\mathbb{D})$ by Proposition 5.2, we see that $g \in \mathcal{O}_M(\mathbb{D})$. However, it follows from Corollary 2.5 that the sum of the series is f , therefore $f = g$, and the proof of the theorem is complete.

5.6. Pompeiu's characterization of holomorphic functions. Goursat's definition is non-quantitative, since it is framed in terms of the *existence* of the limit (5.1), and does not provide a way to measure the degree of non-holomorphicity of a function. For example, if ϵ is small and nonzero, all it says about the functions $f(z) = z + \epsilon \bar{z}$ and $g(z) = \bar{z} + \epsilon z$ is that both are non-holomorphic. Morera's definition of a holomorphic function does not have this shortcoming, as the integral $\int_{\partial T} f(z) dz$ gives a measure of the amount of non-holomorphicity of f on the triangle T .

It is possible to normalize and localize this measure of non-holomorphicity, as was realized by Pompeiu (see [Pom12, MS98]). The quantity

$$\frac{1}{|T|} \int_{\partial T} f(z) dz \tag{5.13}$$

is a numerical measure of the ‘‘average density of non-holomorphicity’’ of a continuous function f on a triangle (or other region with piecewise smooth boundary) T , where $|T|$ denotes the area of T . To localize this, we can consider, for a point $w \in \Omega$, and a continuous function $f : \Omega \rightarrow \mathbb{C}$, the following limit (called the ‘‘areolar derivative’’ by Pompeiu) as a measure of the degree of non-holomorphicity of f at the point w :

$$\lim_{T \downarrow w} \frac{1}{|T|} \int_{\partial T} f(z) dz \tag{5.14}$$

where the limit is taken over the family of triangles containing the point w and contained in Ω , as these triangles shrink to the point w . Let us say that a function is *Pompeiu-holomorphic* if the limit (5.14) exists at each $w \in \Omega$, and is equal to zero. The following simple and well-known observations clarify the meaning of this notion:

Proposition 5.8. (a) *A continuous function $f : \Omega \rightarrow \mathbb{C}$ is Pompeiu-holomorphic if and only if it is Morera-holomorphic.*

(b) If the function $f \in \mathcal{C}^1(\Omega)$, then the limit (5.14) exists, and is equal to $2i \frac{\partial f}{\partial \bar{z}}(w)$.

Proof. For part (a), if f is Morera-holomorphic, then the quantity (5.13) vanishes for each triangle T , so the limit (5.14) vanishes. Conversely, suppose that the limit (5.14) vanishes at each point $w \in \Omega$, and let T be a triangle in Ω . In Lemma 5.4, if we take $\lambda(\Delta) = \int_{\partial\Delta} f(z)dz$, then the additivity condition (5.5) is clear, and the limit condition (5.6) holds by hypothesis. The result follows.

For part (b), using Stoke's theorem

$$\begin{aligned} \frac{1}{|T|} \int_{\partial T} f(z)dz &= \frac{1}{|T|} \int_{\partial T} f(z)dx + if(z)dy = \frac{1}{|T|} \int_T \left(i \frac{\partial f}{\partial x}(z) - \frac{\partial f}{\partial y}(z) \right) dx \wedge dy \\ &= 2i \int_T \frac{\partial f}{\partial \bar{z}} dx \wedge dy. \end{aligned}$$

Since the last integral has a continuous integrand, we may take the limit as the triangle T shrinks to the point w to obtain:

$$\lim_{T \downarrow w} \frac{1}{|T|} \int_{\partial T} f(z)dz = 2i \frac{\partial f}{\partial \bar{z}}(w).$$

□

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