GEOMETRY OF SYMMETRIC POWERS OF COMPLEX DOMAINS

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ABSTRACT

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Given a complex manifold X, its m-fold symmetric power X^m_{Sym} is the collection of all unordered m-tuples of elements of the complex manifold. Such symmetric products arise naturally in complex analysis, for example as the set of solutions of a polynomial equation. The symmetric powers inherit a complex structure, so that it is possible to define holomorphic functions and maps on them. In this work, our goal is to understand aspects of the complex geometry of the symmetric powers $(\mathbb{B}_s)^m_{\mathrm{Sym}}$, where $\mathbb{B}_s$ is the open unit ball in the complex vector space $\mathbb{C}^s$, by classifying the proper holomorphic maps from $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ to itself. These proper holomorphic self-maps are distinguished by the fact that they carry the boundary of a space to itself, and are a natural generalization of the notion of biholomorphic automorphisms.

In the case when $s = 1$, the m-th symmetric power of the unit disc in $\mathbb{C}$ is known traditionally as the m-dimensional symmetrized polydisc, and has been intensively studied in view of its importance in operator theory and control engineering. The symmetrized polydisc can be thought of as an open subset of $\mathbb{C}^m$. Edigarian and Zwonek (2005) determined the proper holomorphic self-maps of the symmetrized polydisc, and showed that these self-maps arise from proper holomorphic self-maps of the unit disc.

The main result of this thesis is a generalization of the result of Edigarian and Zwonek to the symmetric power $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ of the unit ball $\mathbb{B}_s$, where $s \geq 2$. As a first step, we show that $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ can be identified with an open subset of an affine algebraic variety in $\mathbb{C}^N$ for a large integer N depending on m and s. This is done by using a symmetric analog of the classical Segre Embedding of a product of projective varieties in a higher dimensional projective space. This gives an elementary way to endow $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ with a complex structure. Then we reduce the problem of classifying proper holomorphic self-maps of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ to a

certain lifting problem, the obstruction to solving which lies in the fundamental group of a certain open subset of $(\mathbb{B}_s)^m$, obtained by removing an analytic set from $(\mathbb{B}_s)^m$. Using a transversality argument, we show that this obstruction in fact vanishes, thus proving the result. As a consequence, we find that each proper holomorphic self-map of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ is an automorphism induced by an automorphism of $\mathbb{B}_s$.

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CHAPTER I

INTRODUCTION

I.1. The Symmetrized Polydisc

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc: $\mathbb{D} = \{z \in \mathbb{C} | \, |z| < 1\}$. The subset $\Sigma^2\mathbb{D}$ of $\mathbb{C}^2$ given by

$$\Sigma^2\mathbb{D} = \{(z+w, zw) \in \mathbb{C}^2 | z, w \in \mathbb{D}\}$$

is called the *Symmetrized Bidisc.* It can be shown that this is a *domain*, by which we mean an open connected set, which is *pseudoconvex* but has nonsmooth boundary. This domain first emerged in the study of certain problems in Electrical Engineering (see [1]). However, the symmetrized bidisc is a natural object to consider in complex analysis of several variables. Consider, for example, a quadratic equation in t with complex coefficients s and p:

$$t^2 - st + p = 0$$

such that both roots z, ω of this equation lie in the unit disc. Then clearly the coefficients (s, p) lie in $\Sigma^2\mathbb{D}$.

It is possible to generalize this construction, and define the *m-dimensional Symmetrized polydisc* $\Sigma^m\mathbb{D}$ in the following way. For $1 \leq k \leq m$, denote by σ_k the k^{th} elementary symmetric polynomial in m variables. These polynomials are defined to be $\sigma_1(z_1, \ldots z_m) = \sum_{j=1}^m z_j$, $\sigma_2(z_1, \ldots, z_m) = \sum_{j<k} z_j z_k$ and so on, with $\sigma_m(z_1, \ldots, z_m) = z_1 z_2 \ldots z_m$, i.e., the k^{th} elementary symmetric polynomial is the sum over all products of k distinct $z^1, \ldots, z^m$. Then,

$$\Sigma^m\mathbb{D} = \{(\sigma_1(z), \sigma_2(z), \ldots, \sigma_m(z)) \in \mathbb{C}^m | z = (z_1, \ldots, z_m), \text{where each } z_j \in \mathbb{D}\}.$$

As before, a point $(s_1, \ldots, s_m) \in \mathbb{C}^m$ lies in $\Sigma^m\mathbb{D}$ if and only if all the roots of the polynomial equation

$$t^m - s_1 t^{m-1} + s_2 t^{m-2} - \cdots + (-1)^m s_m = 0$$

lie in $\mathbb{D}$.

Since the appearance of [1], the domains $\Sigma^m\mathbb{D}$ have been a subject of much study from the complex analytic point of view (see e.g. [2, 3, 4, 11, 12, 20, 14, 15, 8]). Perhaps the most remarkable property of these domains is that though they are proper holomorphic images of convex domains, the theorem of Lempert on the equality of the Kobayashi and Carathedory metrics on convex domains does not extend to them (see [11]).

I.2. Holomorphic Maps and the Theorem of Edigarian and Zwonek

One of the avenues of this work has been to explore mapping properties and symmetries of the domain $\Sigma^m\mathbb{D}$. Such questions regarding holomorphic mappings are very natural and important in complex analysis. Suppose U and V are open subsets of $\mathbb{C}^n$. If we have a bijective holomorphic map $f : U \to V$ then f^{-1} is also holomorphic. In this case, we call f a *biholomorphism*. One of the famous theorems of mathematics is the Riemann Mapping Theorem which states that each simply connected proper subdomain of $\mathbb{C}$ is biholomorphic to the unit disc $\mathbb{D}$. It is also well-known how to determine the group $\mathrm{Aut}(\mathbb{D})$ of all automorphisms of $\mathbb{D}$ (i.e. biholomorphisms from $\mathbb{D}$ to itself).

For domains in $\mathbb{C}^n$ with $n \geq 2$, the study of holomorphic maps is much more complicated, and there is no higher-dimensional analog of the general theory of conformal mapping. In particular, there is no simple analog of the Riemann Mapping Theorem: not all open, simply-connected proper subsets of $\mathbb{C}^n$ are biholomorphic to the *unit ball*

$$\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}} < 1\} \tag{I.1}$$

when $n \geq 2$. The simplest counter-example is the domain $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$, where $\mathbb{D} = \mathbb{B}_1$, the unit disc. Although $\mathbb{D}^2$ is in fact homeomorphic to $\mathbb{B}_2$, there is no biholomorphism between these domains, a fact known to Poincaré. We give a proof of this result based on a more general theorem of Remmert and Stein (see [21]) in Chapter IV.

In spite of the difficulties in studying general biholomorphic mappings of several variables outlined above, it is still sometimes possible to characterize mappings between *special domains*. In this thesis, which is motivated by the results of Jarnicki, Pflug, Edigarian, and Zwonek, which we describe below, we study the structure of interesting holomorphic maps between some higher dimensional analogs of the symmetrized polydisc. We begin by summarizing the known results about holomorphic maps between symmetrized polydiscs, on which our work is based.

The first step in this direction is [20], in which Jarncki and Pflug explicitly determined the group of automorphisms of the symmetrized polydisc. We do not state this result separately, since it is a special case of the results of Edigarian and Zwonek on *proper holomorphic maps* of these domains, which we will now describe.

A *proper map* is a weakening of the notion of a homeomorphism between topological spaces. For f to be a proper map, instead of requiring that f is bijective and has a continuous inverse, we require that the inverse image of a compact set is compact. Clearly, each homeomorphism is a proper map, but not every proper map is a homeomorphism. Here, as well as in subsequent chapters, by a *map*, we mean a continuous function between topological spaces.

In complex analysis of one variable, proper holomorphic maps arise naturally as "branched coverings". A typical example of a proper holomorphic map is given by the map f from the unit disc $\mathbb{D}$ to itself given by

$$f(z) = z^d$$

where d is a positive integer. Notice that (1) the map is proper, since the inverse image of any closed subdisc $\{|z| \leq r\}$ of $\mathbb{D}$ (where $0 < r < 1$) is a closed subdisc of $\mathbb{D}$, (2) the map f takes the boundary of $\mathbb{D}$ to itself, and (3) on $\mathbb{D} \setminus \{0\}$, f is a covering map of d sheets. These are illustrations of general properties of proper holomorphic maps. See Section I.4 below for more examples and properties of proper holomorphic maps.

With these preliminaries, we can state the following result, to the generalization of which this thesis is devoted:

Theorem 1. *([14, 15]) Let* $f : \Sigma^m\mathbb{D} \to \Sigma^m\mathbb{D}$ *be a proper holomorphic map, and let* $\sigma = (\sigma_1, \ldots, \sigma_m) : \mathbb{C}^m \to \mathbb{C}^m$*. Then, there exists a proper holomorphic map* $B : \mathbb{D} \to \mathbb{D}$ *such that*

$$f(\sigma(z^1, \ldots, z^m) = \sigma(B(z^1), \ldots, B(z^m)).$$

Remark. One can easily characterize all proper holomorphic maps $\mathbb{D} \to \mathbb{D}$. See Example 1.

This theorem demonstrates that each proper holomorphic self-map of $\Sigma^m\mathbb{D}$ is induced by a proper holomorphic self-map of $\mathbb{D}$. Also note that, as a special case, this shows that every automorphism of $\Sigma^m\mathbb{D}$ is induced by an automorphism of $\mathbb{D}$, thus recapturing the main result of [20].

I.3. Proper Holomorphic Mappings of Higher Dimensional Symmetric Powers

In order to state the generalization of Theorem 1, we will first need to understand the domains $\Sigma^m\mathbb{D}$ intrinsically. To do this, given a set Z, we form what is called its *m-fold symmetric power*, denoted by Z^m_{Sym}. This is the collection of all *unordered* m-tuples of elements of Z. A typical element of Z^3_{Sym} is of the form

$$\langle z^1, z^2, z^3 \rangle$$

where $z^1, z^2, z^3 \in Z$ and the ordering of the elements is irrelevant, i.e.,

$$\langle z^1, z^2, z^3 \rangle = \langle z^2, z^1, z^3 \rangle = \langle z^3, z^1, z^2 \rangle = \langle z^3, z^2, z^1 \rangle = \ldots$$

Here and throughout this thesis, we adopt the convention of denoting elements of tuples taken from the same set by superscripts. In a very few cases when exponents are necessary, the meaning will be clear from the context.

The symmetric power Z^m_{Sym} should be contrasted with the better known *Cartesian* *power* Z^m of *ordered* m-tuples. Recall that this means for example that in Z^3, the two elements (z^1, z^2, z^3) and (w^1, w^2, w^3) are equal if and only if $z^1 = w^1, z^2 = w^2$ and $z^3 = w^3$.

Note also that if we are given a mapping of sets

$$f : Z \to W$$

it induces in a natural way a mapping

$$f^m_{\text{Sym}} : Z^m_{\text{Sym}} \to W^m_{\text{Sym}}$$

given by

$$f^m_{\text{Sym}}(\langle z^1, z^2, \ldots, z^m\rangle) = \langle f(z^1), \ldots, f(z^m)\rangle \in W^m_{\text{Sym}}.$$

This is easily seen to be well-defined. We call f^m_{Sym} the m-th symmetric power of the map f.

In order to see the relevance of this notion, consider now the map Φ from the symmetric power $\mathbb{D}^m_{\text{Sym}}$ of the unit disc to the symmetrized polydisc $\Sigma^m\mathbb{D}$ given, for $z^1, \ldots, z^m \in \mathbb{D}$ by

$$\Phi(\langle z^1, \ldots, z^m\rangle) = \big(\sigma_1(z^1, \ldots, z^m), \sigma_2(z^1, \ldots, z^m), \ldots, \sigma_m(z^1, \ldots, z^m)\big) \in \mathbb{C}^m,$$

where σ_k denotes the k-th elementary symmetric polynomial in m variables.

Proposition 1. *Φ is a well defined bijection from $\mathbb{D}^m_{\text{Sym}}$ onto $\Sigma^m\mathbb{D}$.*

Proof. Since each component of Φ is a symmetric polynomial in $z^1, \ldots, z^m$, Φ is invariant under permutation of $z^1, \ldots, z^m$, hence is well-defined. Furthermore, given an element $w \in \Sigma^m\mathbb{D}$, we have $w = (\sigma_1(z^1, \ldots, z^m), \ldots, \sigma_m(z^1, \ldots, z^m))$ for some $z^1, \ldots, z^m \in \mathbb{D}$. The unordered tuple $\langle z^1, \ldots, z^m\rangle$ are precisely the roots of the polynomial

$$p(t) = t^m - \sigma_1(z^1, \ldots, z^m)t^{m-1} + \cdots + (-1)^m\sigma_m(z^1, \ldots, z^m),$$

which are uniquely defined, up to permutation. This shows that Φ is a bijection. □

In view of the above proposition, we can identify the symmetrized polydisc with the set $\mathbb{D}^m_{\text{Sym}}$. With this understanding, Theorem 1 can be stated in the following way using the terminology introduced above.

Theorem 2. *Let $f : \mathbb{D}^m_{\text{Sym}} \to \mathbb{D}^m_{\text{Sym}}$ be a proper holomorphic map. Then there is a proper holomorphic map $g : \mathbb{D} \to \mathbb{D}$ such that $f = g^m_{\text{Sym}}$.*

We can now explain the main new results of this thesis. Consider the m-fold symmetric power $(\mathbb{B}_s)^m_{\text{Sym}}$ of the unit ball $\mathbb{B}_s \subset \mathbb{C}^s$ (see (I.1)). In the case that $s = 1$, we have $\mathbb{B}_1 = \mathbb{D}$, the unit disc, and we already saw in Proposition 1 that $\mathbb{D}^m_{\text{Sym}}$ can be identified naturally with the open subset $\Sigma^m\mathbb{D}$ of $\mathbb{C}^m$. This allows us to talk about holomorphic maps from $\mathbb{D}^m_{\text{Sym}}$ to itself.

In Chapter III we will show that a similar statement is true for the symmetric power $(\mathbb{B}_s)^m_{\text{Sym}}$ when $s \geq 2$. We construct an injective continuous map (which we call the *Segre-Whitney embedding*) which embeds $(\mathbb{B}_s)^m_{\text{Sym}}$ into a complex vector space $\mathbb{C}^N$ for some large N as a *local analytic subset*. Consequently, it makes sense to speak about holomorphic maps from $(\mathbb{B}_s)^m_{\text{Sym}}$ into $(\mathbb{B}_s)^m_{\text{Sym}}$.

We note here that the explicit construction of the Segre-Whitney map in Chapter III can be avoided if we were to use the theory of *complex analytic spaces* that, informally, can be thought of as complex manifolds with singularities, modeled after analytic sets. Using general results in this theory, one can then construct $(\mathbb{B}_s)^m_{\text{Sym}}$ directly as a quotient of the domain $(\mathbb{B}_s)^m$ in $\mathbb{C}^{sm}$ under the action of the symmetric group S_m. This is explained briefly in Section III.6. We prefer to use the simpler description of $(\mathbb{B}_s)^m_{\text{Sym}}$ as a certain subset of a Euclidean space.

Theorem 3. *Let $s \geq 2$, $m \geq 2$, and let $f : (\mathbb{B}_s)^m_{\text{Sym}} \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map. Then, there exists a holomorphic automorphism $g : \mathbb{B}_s \to \mathbb{B}_s$ such that*

$$f = g^m_{\text{Sym}},$$

that is f is the m-th symmetric power of g.

Recall that for $z^j \in \mathbb{B}_s$, $j = 1, \ldots, m$ we have

$$g^m_{\text{Sym}}(\langle z^1, \ldots, z^m \rangle) = \langle g(z^1), \ldots, g(z^m) \rangle.$$

Note the difference between Theorem 2 and Theorem 3: When $s \geq 2$, each proper holomorphic self-mapping of $(\mathbb{B}_s)^m_{\text{Sym}}$ is induced by an *automorphism* of $\mathbb{B}_s$, which gives us the following Corollary. This is because, as we explain in Section I.5, each proper holomorphic self-mapping of $\mathbb{B}_s$ is an automorphism when $s \geq 2$.

Corollary 1. *Each proper holomorphic self-map of* $(\mathbb{B}_s)^m_{\text{Sym}}$ *is an automorphism, and is of the form* g^m_{Sym}*, where* g *is an automorphism of the ball* $\mathbb{B}_s$*.*

Since it is possible to describe all automorphisms of the ball explicitly (see (I.2) below), this gives a complete and explicit answer to the question of determining all proper holomorphic self-maps of $(\mathbb{B}_s)^m_{\text{Sym}}$.

In Chapter IV, we prove Theorem 3 by first classifying all proper holomorphic maps $(\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ using a Remmert-Stein type argument, which is the content of Theorem 9. Then, we show that we can lift a map $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ to a map $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$, by considering the fundamental group of $(\mathbb{B}_s)^m$ after removing certain analytic sets. Then, using Theorem 9, we can immediately deduce the structure of $\widetilde{f}$ and hence, f. We then use our classification of proper holomorphic maps $(\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ to deduce the structure of all proper holomorphic maps $(\mathbb{B}_s)^m_{\text{Sym}} \to (\mathbb{B}_s)^m_{\text{Sym}}$.

Theorem 3 can be immediately generalized to the situation in which the unit ball $\mathbb{B}_s$ is replaced by a *strongly pseudoconvex* domain in $\mathbb{C}^s$. With a little more work, we can also generalize it to situations where $\mathbb{B}_s$ is replaced by a smoothly bounded pseudoconvex domain. For simplicity, we prefer to state the result in the form above.

I.4. Topological Properties of Proper Maps

In this section, we collect some properties of proper maps between topological spaces which will prove useful later.

Definition 1. A map from a topological space X to a topological space Y is called *proper* if the preimage of each compact set in Y is compact in X.

We note here a few elementary properties of proper maps.

Proposition 2. *Let X and Y be Hausdorff spaces and Z a subspace of Y. Let $f : X \to Z$ be a proper map. If a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ has no limit point in X, then $\{f(x_n)\}_{n=1}^{\infty}$ has no limit point in Z. Additionally, if $\overline{Z}$ is compact, then $\{f(x_n)\}_{n=1}^{\infty}$ has a subsequence converging to a point in ∂Z.*

Proof. Let $f : X \to Z$ a proper map and let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence with no limit point in X. Consider the sequence $\{f(x_n)\}_{n=1}^{\infty}$. Suppose this sequence has some limit point $z \in Z$. Then, there is a subsequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ converging to z, and the set $\{f(x_{n_k})\}_{k=1}^{\infty} \cup \{z\}$ is compact in Z. Hence, since f is proper, $f^{-1}\left(\{f(x_{n_k})\}_{k=1}^{\infty} \cup \{z\}\right)$ is compact in X. Since $\{x_{n_k}\}_{k=1}^{\infty} \subset f^{-1}\left(\{f(x_{n_k})\}_{k=1}^{\infty} \cup \{z\}\right)$, this means $\{x_{n_k}\}_{k=1}^{\infty}$ has a limit point in $f^{-1}\left(\{f(x_{n_k})\}_{k=1}^{\infty} \cup \{z\}\right)$, and hence in X. Consequently, $\{x_n\}$ has a limit point in X. Since no such limit point exists in X, this shows that $\{f(x_n)\}_{n=1}^{\infty}$ has no limit point in Z. If $\overline{Z}$ is compact, then $\{f(x_n)\}_{n=1}^{\infty}$ must have a limit point in $\overline{Z}$. Since this limit point cannot be in Z, it must be in ∂Z. □

Proposition 3. *Let X and Y be locally compact Hausdorff spaces and $f : X \to Y$ a proper map. Then f is a closed map.*

Proof. Let $E \subset X$ be closed. Let $y \in Y \setminus f(E)$. Since Y is locally compact, there is an open neighborhood U of y in Y with compact closure. Since f is proper, $f^{-1}(\overline{U})$ is compact. Since E is closed, $K = f^{-1}(\overline{U}) \cap E$ is compact. Thus, $f(K) \subset f(E)$ is compact, and hence closed in Y. Let $V = U \setminus f(K)$. Then V is an open neighborhood of y, disjoint from $f(E)$. Hence, $f(E)$ is closed. □

Proposition 4. *Let X and Y be topological spaces and $f : X \to Y$ a proper map. Let V be a subspace of Y, set $U = f^{-1}(V)$ and let g be the restriction of f to U. Then $g : U \to V$ is a proper map.*

Proof. Let $K \subset V$ be a compact set, so that K is also a compact subset of Y. By construction $g^{-1}(K) = f^{-1}(K)$, so $g^{-1}(K)$ is a compact subset of U. □

Proposition 5. *Let X, Y, and Z be topological spaces and maps $g : X \to Y$, $f : Y \to Z$ such that $f \circ g$ is a proper map. Then g is a proper map and the restriction of f to $g(X)$, $f|_{g(X)} : g(X) \to Z$, is a proper map.*

Proof. Let $K \subset Y$ be compact. Then, since f is continuous, $f(K)$ is compact, and hence, $(f \circ g)^{-1}(f(K)) = g^{-1}((f^{-1} \circ f)(K))$ is compact as well. Since $K \subset (f^{-1} \circ f)(K)$, $g^{-1}(K) \subset g^{-1}((f^{-1} \circ f)(K))$. Thus, since g is continuous, $g^{-1}(K)$ is a closed subset of a compact set, and therefore, compact. Hence, g is proper.

Now, let $K \subset Z$ be compact. Then $(f \circ g)^{-1}(K)$ is compact, and since g is continuous, so is $g((f \circ g)^{-1}(K)) = f^{-1}(K) \cap g(X) = (f|_{g(X)})^{-1}(K)$. Hence, $f|_{g(X)}$ is proper. □

Proposition 6. *Let X and Y be locally compact Hausdorff spaces and $f : X \to Y$ a proper bijective map. Then, f is a homeomorphism.*

Proof. By Proposition 3, f is a closed map. Since f is a bijection, for any open set $U \subset X$, we have $f(X \setminus U) = f(X) \setminus f(U) = Y \setminus f(U)$ is closed in Y. Hence, $f(U)$ is open in Y. Thus, f is an open map, and so f^{-1} is continuous. □

I.5. Some Examples of Proper Holomorphic Maps

Like a proper map is a generalization of a homeomorphism, a proper holomorphic map $f : M \to N$ between complex manifolds can be thought of as a generalization of a *biholomorphism*, that is, a holomorphic map with a holomorphic inverse. Since every biholomorphism is a homeomorphism, a biholomorphism is necessarily proper. However, as before, it is not always the case that a proper holomorphic map is a biholomorphism. In the following examples, we compute the proper holomorphic self-mappings of certain domains in $\mathbb{C}^n$. Let $\mathbb{D}$ denote the open unit disc: $\{z \in \mathbb{C} : |z| < 1\}$.

Example 1. Let $f : \mathbb{D} \to \mathbb{D}$ be a proper holomorphic map. Since f is proper, f must be nonconstant, and $f^{-1}(0)$ compact. Hence, $f^{-1}(0)$ is finite. Let $\{\alpha_i\}_{i=1}^n$ be the zeros of f, allowing for repetition when a zero has multiplicity. Let

$$g(z) = \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha}_i z}$$

for all $z \in \mathbb{D}$. Note that for $|z| = 1$, we have

$$\left| \frac{z - \alpha_i}{1 - \overline{\alpha}_i z} \right| = \left| \frac{z - \alpha_i}{z(\overline{z} - \overline{\alpha}_i)} \right| = 1.$$

Hence, by the maximum modulus principle, $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Now, since f and g have exactly the same zeros with the same multiplicities, $\frac{f}{g}$ has only removable singularities in $\mathbb{D}$, and therefore can be extended to a holomorphic function on $\mathbb{D}$ with no zeros in $\mathbb{D}$. Since f is proper, by Proposition 2, if $\{\zeta_n\}_{n=1}^{\infty}$ is any sequence in $\mathbb{D}$ converging to a point in $\partial\mathbb{D}$, we must have

$$\lim_{n\to\infty} |f(\zeta_n)| = 1.$$

Since the same is true for g, we have

$$\lim_{n\to\infty} \left| \frac{f(\zeta_n)}{g(\zeta_n)} \right| = 1.$$

Hence, by the maximum modulus principle, we have $\left|\frac{f}{g}\right| \leq 1$ on $\mathbb{D}$. Now, since $\frac{f}{g}$ has no zeros on $\mathbb{D}$, $\left|\frac{f}{g}\right|$ cannot have a local minimum in $\mathbb{D}$ either. Hence, $\left|\frac{f}{g}\right| \equiv 1$ on $\mathbb{D}$. Hence, we have proved that if $f : \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map, then

$$f(z) = e^{i\lambda} \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha}_i z}$$

for some $\lambda \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{D}$.

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a proper holomorphic map. Since f is entire, there is a power series expansion of f at the origin

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which converges on the whole plane. Now, consider the function

$$F(z) = f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n.$$

This gives a Laurent series expansion of $F(z)$ at the origin. Since the power series for f converges on the whole plane, the Laurent series for F converges on the punctured plane $\mathbb{C} \setminus \{0\}$. Suppose the series for F does not terminate. Then, F has an essential singularity at 0, which means that f has an essential singularity at ∞. Hence, $f(\mathbb{C} \setminus D_r)$ is dense in $\mathbb{C}$ for every $r > 0$, where D_r is the disc centered at the origin with radius r. In particular, this means that $f^{-1}(\overline{D}_r)$ is unbounded, and hence noncompact. This contradicts the properness of f. Therefore, if f is a proper holomorphic map $\mathbb{C} \to \mathbb{C}$, then the power series expansion of f must be finite, and hence, f must be a polynomial.

Example 3. In contrast with the previous two examples, the only proper holomorphic self-mappings of the unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$, $n \geq 2$, are the automorphisms of $\mathbb{B}_n$. This result, first proven by Alexander [5], the details of which can also be found in [25, page 316], shows that on some domains, the notions of biholomorphism and of proper holomorphic maps are actually equivalent.

For future reference, each automorphism of the unit ball $\mathbb{B}_n$ is of the form:

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \tag{I.2}$$

where $a \in \mathbb{B}_n$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional complex linear subspace spanned by a, $Q_a = \mathbb{1} - P_a$ is the orthogonal projection from $\mathbb{C}^n$ onto the orthogonal complement of the one dimensional complex linear subspace spanned by a, and $s_a = (1 - |a|^2)^{\frac{1}{2}}$. A proof of this fact can be found in [25, page 25].

CHAPTER II

ANALYTIC SETS AND PROPER MAPS

II.1. Analytic Subsets of Complex Manifolds

We assume the basic definitions and facts regarding real and complex manifolds, analytic sets and their mappings. Complete treatments of these topics may be found in the texts [9], [21], and [26]. However, for completeness we collect here some definitions and facts which will be needed in this work. We also provide proofs of some results which are needed in the sequel, but whose proofs cannot be found in the standard texts mentioned above.

Definition 2. The following definitions will be important in our work:

1. A *complex manifold* is a Hausdorff space M with a collection of maps $\Phi_\alpha : U_\alpha \to \mathbb{C}^n$, where each $U_\alpha \subset M$ is open, such that

   (a) $M = \bigcup_\alpha U_\alpha$,

   (b) each Φ_α is a homeomorphism $U_\alpha \to \Phi_\alpha(U_\alpha)$, and

   (c) for each pair of maps Φ_α and Φ_β, $\Phi_\alpha \circ \Phi_\beta^{-1}$ is holomorphic on $\Phi_\beta(U_\alpha \cap U_\beta)$.

   The collection of maps $\{\Phi_\alpha\}$ is called a *chart*. We call n the *dimension* of the manifold.

2. Let M and N be complex manifolds with charts $\{\Phi_\alpha\}$ and $\{\Psi_\beta\}$. A map $f : M \to N$ is a *holomorphic map* if, for every Φ_α and Ψ_β, the composition $\Psi_\beta \circ f \circ \Phi_\alpha^{-1}$ is holomorphic on $\Phi_\alpha(f^{-1}(U_\beta) \cap U_\alpha)$. This allows us to define holomorphic maps ($\mathbb{C}$-valued maps) on any open subset of M

3. Let M be a complex manifold. A set $A \subset M$ is called an *analytic subset* of M if for each point $a \in M$ there is a neighborhood U of a and functions $f_1, \ldots, f_n$ holomorphic on U such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \cdots = f_n(z) = 0\}.$$

   If A is an analytic subset of an open subset $\Omega \subset M$, we call A a *local analytic subset* of M.

4. Let A be a local analytic subset of a complex manifold M. A function $f : A \to \mathbb{C}$ is *holomorphic* if for each $a \in A$, there is an open neighborhood U of a in M and a holomorphic function $F : U \to \mathbb{C}$ such that $f \equiv F$ on $A \cap U$.

5. Let X and Y be local analytic subsets of complex manifolds M and N respectively. A map $f : X \to Y$ is a *holomorphic map* if for each $x \in X$, there is an open neighborhood U of x in M, and a holomorphic function $F : U \to N$ such that $f \equiv F$ on $X \cap U$.

6. Let A be a local analytic subset of a complex manifold M. A point $a \in A$ is *regular* if there is an open neighborhood U of a in M such that $U \cap A$ is a submanifold of M. The collection of regular points of A is denoted $\operatorname{reg} A$. The dimension of this submanifold is called the *dimension* of A at a, denoted $\dim_a A$.

7. The *dimension* of an analytic subset A of a complex manifold M is defined to be

$$\dim A = \max_{a \in \operatorname{reg} A} \{\dim_a A\}.$$

8. An analytic set A is called *reducible* if there are analytic sets A_1 and A_2 distinct from A such that $A = A_1 \cup A_2$. An analytic set that cannot be represented in this manner is called *irreducible.*

For the following results, we refer to [9] for proofs, whereas Riemann's Continuation Theorem can be found in [21].

Proposition 7. *Let A be an analytic subset of a complex manifold M. Then, we have the following:*

1. *A is closed in M.*
2. *If $A \neq M$, then $M \setminus A$ is dense in Ω.*
3. *$M \setminus A$ is connected.*

Theorem 4 (Riemann's Continuation Theorem.)**.** *Let M be a complex manifold and A an analytic subset of M such that $M \setminus A$ is dense in M. Let f be holomorphic on $M \setminus A$ and* *suppose that for any $a \in A$, there exists an open neighborhood U of a in M such that $f|_{U \setminus A}$ is bounded. Then, there is a unique function F, holomorphic on M, such that $F|_{M \setminus A} = f$.*

II.2. Proper Holomorphic Maps

The following results, which we will need, and their proofs, can be found in [9].

Proposition 8. *1. Let $A \subset \mathbb{C}^n$ be a compact local analytic set. Then A is finite.*

2. An analytic set A is irreducible if and only if the set $\operatorname{reg} A$ is connected.

3. Let A and A' be analytic subsets of a complex manifold Ω. Then $\overline{A \setminus A'}$ is also an analytic subset in Ω.

The following remarkable theorem of Remmert shows that the proper holomorphic image of an analytic set is analytic. A proof of this can be found in [9].

Theorem 5 (Remmert's Theorem)**.** *Let A be an analytic subset of a complex manifold M and let $f : A \to Y$ be a proper holomorphic map into another complex manifold Y. Then $f(A)$ is an analytic subset of Y of dimension $\dim f$.*

In the above theorem, $\dim f$ is defined to be the maximum codimension of the fibers $f^{-1}(x)$, which is equal to the rank of f at any regular point. The following theorem, which is encompassed by the famous GAGA set of analogies, is a consequence of Remmert's Theorem. A proof of this can also be found in [9].

Theorem 6 (Chow's Theorem)**.** *Every analytic subset of a complex projective space is an algebraic variety.*

Proposition 9. *Let X and Y be analytic subsets of $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$, respectively, and let $f : X \to \Omega_2$ be a proper holomorphic map such that $f(X) \subset Y$. Then,*

1. If $f(X) = Y$, then $\dim X = \dim Y$.

2. If Y is irreducible and $\dim X = \dim Y$, then $f(X) = Y$.

Proof. By Remmert's Theorem, $f(X)$ is an analytic subset of Ω_2, with $\dim f(X) = \dim f$. Since f is proper, we cannot have $\dim f < \dim X$. Otherwise, the Rank Theorem [24, page 229] would imply that for some $y \in f(X)$, $f^{-1}(y)$ is a compact analytic subset of X with positive dimension. By Proposition 8, this is impossible. Hence, we conclude that $\dim f(X) = \dim X$.

1. Suppose $f(X) = Y$. Then, evidently, $\dim X = \dim f(X) = \dim Y$.

2. Now suppose $\dim X = \dim Y$ and suppose that $f(X) \neq Y$. Then, by Proposition 8, $\overline{Y \setminus f(X)}$ is an analytic set, and since Y is closed in Ω_2, $\overline{Y \setminus f(X)}$ is contained in Y. Additionally, since $\dim f(X) = \dim Y$, $\overline{Y \setminus f(X)}$ cannot be all of Y. Now, since $Y = f(X) \cup \overline{Y \setminus f(X)}$, Y is reducible. Hence, if Y is irreducible and $\dim X = \dim Y$, then $f(X) = Y$.

□

Note that, in the case that X and Y are complex manifolds, Y is irreducible if and only if Y is connected, which immediately gives us the following corollary:

Corollary 2. *Let X and Y be complex manifolds, and let $f : X \to Y$ be a proper holomorphic map. Then,*

1. *If $f(X) = Y$, then $\dim X = \dim Y$.*
2. *If Y is connected and $\dim X = \dim Y$, then $f(X) = Y$.*

In the following definition, $T_p(M)$ denotes the *tangent space to M at the point $p \in M$*. It's important to note that the dimension of $T_p(M)$ at every point $p \in M$ is equal to the dimension of the manifold M.

Definition 3. Let $f : X \to Y$ be a smooth map of manifolds and Z a submanifold of Y. We say the map f is *transversal* to Z if $\text{Image}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$ for all $x \in f^{-1}(Z)$.

The following theorem, and its corrollary that we use to prove Proposition 10, as well as their proofs, can be found in [17].

Theorem 7 (The Transversality Theorem)**.** *Suppose that $F : X \times S \to Y$ is a smooth map of manifolds, where only X has boundary, and let Z be any boundaryless submanifold of Y. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z.*

Note that in the above theorem, ∂F denotes the restriction of F to the boundary $\partial(X \times S)$, $f_s : X \to Y$ is defined by $f_s(x) = f(x, s)$, and hence ∂f_s is the restriction of f_s to ∂X.

Corollary 3. *Suppose $f : X \to Y$ is a smooth map of manifolds, where only X has boundary, and the boundary map $\partial f : \partial X \to Y$ is transversal to Z, then there is a map $g : X \to Y$ homotopic to f with $\partial g = \partial f$, which is transversal to Z.*

The proposition below can be found in [16]. Although [16] uses a different technique, the proof we present below is based on the Transversality Theorem.

Proposition 10. *Let M be a smooth, connected real manifold without boundary, $A \subset M$ be closed submanifold, x a point in $M \setminus A$, and $i : M \setminus A \to M$ the inclusion map. Then*

1. *if the codimension of A is at least 2, the group homomorphism*

$$i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$$

is surjective, and

2. *if the codimension of A is at least 3, then the group homomorphism*

$$i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$$

is an isomorphism.

Proof. 1. Let $x_0 \in M \setminus A$ and $f : [0, 1] \to M$ be a smooth, closed path starting at x_0. Since $x_0 \notin A$, ∂f is transversal to A. By Corollary 3, there is a smooth path $g : [0, 1] \to M$ starting at x_0, which is homotopic to f and is transversal to A. By the transversality condition of Definition 3, we have $\text{Image}(dg_x) + T_{g(x)}(A) = T_{g(x)}(M)$ for all $x \in g^{-1}(A)$.

Now, since $g([0,1])$ is a one-dimensional submanifold of M, if the codimension of A is at least 2, the only way that g can satisfy the transversality condition is if $g([0,1]) \cap A = \emptyset$. Since every closed path in M starting at x_0 is homotopic to a smooth path in M starting at the same point, this proves that for every $[f] \in \pi_1(M, x_0)$, there is some $[g] \in \pi_1(M \setminus A, x_0)$ with $f \simeq g$. Hence, the map $i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$ is surjective.

2. Let $h : [0,1] \times [0,1] \to M$ be a homotopy of closed paths in $M \setminus A$ starting at x_0. Assume that h is smooth. Since the paths $h([0,1] \times \{0\})$ and $h([0,1] \times \{1\})$ are in $M \setminus A$, and $x_0 \in M \setminus A$, ∂h is transversal to A. By Corollary 3, there is a smooth map $g : [0,1] \times [0,1] \to X$ with $\partial g = \partial h$, which is homotopic to h and is transversal to A. Since $\partial g = \partial h$, g is a homotopy of the same paths that h is. By the transversality condition of Definition 3, we have $\text{Image}(dg_x) + T_{g(x)}(A) = T_{g(x)}(M)$ for all $x \in g^{-1}(A)$. Now, since $g([0,1] \times [0,1])$ is a two-dimensional submanifold of M, if the codimension of A is at least 3, the only way that g can satisfy the transversality condition is if $g([0,1] \times [0,1]) \cap A = \emptyset$. Since every closed path in $M \setminus A$ starting at x_0 is homotopic to a smooth path starting at the same point, and every homotopy of smooth paths is homotopic to a smooth homotopy, this proves that for every pair of paths f_1 and f_2 in $M \setminus A$ with $f_1 \simeq f_2$ in M, we also have $f_1 \simeq f_2$ in $M \setminus A$. Hence, the map $i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$ is injective.

□

The following proposition regarding the structure of analytic sets is called the *stratification* of analytic sets. It tells us, in particular, that each analytic set can be decomposed into a disjoint union of manifolds, and that for each integer p, $1 \leq p \leq \dim A$, the union of the manifolds with dimension at most p is an analytic set. More details can be found in [9].

We use this stratification of analytic sets to show that Proposition 10 can be generalized to the case where A is an analytic set, rather than just a closed submanifold. Let $\sqcup$ denote the disjoint union, that is, a union of disjoint sets. Then we have the following:

Proposition 11. *Let A be an analytic subset of a complex manifold M. Then, there exist disjoint local analytic subsets A_j of A such that*

$$A = \bigsqcup_{i=1}^{n} A_i,$$

where $B_k = \bigsqcup_{i=1}^{k} A_i$ is an analytic subset of M and A_k is a closed submanifold of $M \setminus B_{k-1}$ for each $1 \leq k \leq n$.

Proposition 12. *Let M be a smooth, connected complex manifold without boundary, $A \subset M$ be an analytic subset, x a point in $M \setminus A$, and $i : M \setminus A \to M$ the inclusion map. Then*

1. *if the complex codimension of A is at least 1, the group homomorphism*

$$i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$$

is surjective, and

2. *if the complex codimension of A is at least 2, then the group homomorphism*

$$i_* : \pi_1(M \setminus A, x) \to \pi_1(M, x)$$

is an isomorphism.

Proof. By Proposition 11, we can write

$$A = \bigsqcup_{i=1}^{n} A_i,$$

where $B_k = \bigsqcup_{i=1}^{k} A_i$ is an analytic subset of M, hence closed in M, and A_k is a closed submanifold of $M \setminus B_{k-1}$ for each $1 \leq k \leq n$. Note that, if A has codimension at least s in M, then A_k also has codimension at least s in $M \setminus B_{k-1}$ for each $1 \leq k \leq n$. Thus, assuming

$A \neq M$,

$$M_k = M \setminus B_k$$

is a open submanifold of M for each $1 \leq k \leq n$. Hence, since

$$M_k = M_{k-1} \setminus A_k,$$

each inclusion in the following chain

$$M \setminus A = M_n \xrightarrow{\quad i \quad} M_{n-1} \xrightarrow{\quad i \quad} \cdots \xrightarrow{\quad i \quad} M_1 \xrightarrow{\quad i \quad} M_0 = M$$

satisfies the conditions of Proposition 10. Thus, it follows that if the complex codimension of A is at least 1, $i_* : \pi_1(M_k, x) \to \pi_1(M_{k-1}, x)$ is surjective for each $1 \leq k \leq n$, and if the complex codimension of A is at least 2, $i_* : \pi_1(M_k, x) \to \pi_1(M_{k-1}, x)$ is an isomorphism for each $1 \leq k \leq n$. The conclusion follows. $\square$

CHAPTER III

SYMMETRIC POWERS

III.1. Definitions and Preliminaries

In the definitions below and throughout the paper, recall that we use superscripts to denote elements in a symmetric product. Usually the topological space will be $\mathbb{C}^s$, and we use x^i_j to denote the j^{th} component of $x^i \in \mathbb{C}^s$.

Let X be a set, and for a positive integer m, let X^m denote the m-th Cartesian power of X, which is by definition the collection of ordered tuples

$$\{(x^1, \ldots, x^m), x^j \in X, j = 1, \ldots, m\}.$$

The symmetric group S_m of bijective mappings of the set $\{1, \ldots, m\}$ acts on X^m in a natural way: for $\sigma \in S_m$, and $x = (x^1, \ldots, x^m) \in X^m$ we set

$$\sigma \cdot x = (x^{\sigma(1)}, \ldots, x^{\sigma(m)}).$$

Definition 4. The m-th **Symmetric Power** of X, denoted by X^m_{Sym} is the quotient of X^m under the action of S_m defined above, i.e, points of X^m_{Sym} are orbits of the action of S_m on X^m. We denote by

$$\pi : X^m \to X^m_{\text{Sym}}$$

the natural quotient map.

If $x = (x^1, \ldots, x^m) \in X^m$, we denote its image in X^m_{Sym} by

$$\pi(x) = \langle x^1, \ldots, x^m \rangle,$$

so that we can think of $\pi(x)$ as an *unordered m-tuple of points* of X. For elements of X^m_{Sym} we use the notation in [26]: we denote by

$$\langle x^1 : m_1, x^2 : m_2, \ldots, x^k : m_k \rangle \tag{III.1}$$

where $m_1, m_2, \ldots, m_k$ are non-negative integers (called *multiplicities*) with $\sum_{j=1}^{k} m_j = m$, the element of X^m_{Sym} in which x^j is repeated m_j times.

Also notice, that the construction of the symmetric power is *functorial*, i.e., given a mapping between sets $f : X \to Y$, there is a natural map $f^m_{\mathrm{Sym}} : X^m_{\mathrm{Sym}} \to Y^m_{\mathrm{Sym}}$, called the symmetric power of the map f, defined by

$$f^m_{\mathrm{Sym}}\left(\langle x^1, \ldots, x^m\rangle\right) = \langle f(x^1), \ldots, f(x^m)\rangle. \tag{III.2}$$

It is not difficult to see that f^m_{Sym} is well defined. Further, it is easy to verify that the construction is functorial, in the sense that $(\mathrm{id}_X)^m_{\mathrm{Sym}} = \mathrm{id}_{X^m_{\mathrm{Sym}}}$ and $(f \circ g)^m_{\mathrm{Sym}} = f^m_{\mathrm{Sym}} \circ g^m_{\mathrm{Sym}}$.

III.2. Topological Symmetric Powers

The notion of symmetric power makes sense in a category where one can define finite products of objects and quotients of objects by finite groups. For example, one can construct the n-fold symmetric power of any topological space, which will also be a topological space in a natural way.

Let X be a locally compact Hausdorff topological space and let m be a positive integer. We endow X^m_{Sym} with the quotient topology. Recall that in this topology a subset U of X^m_{Sym} is open if and only if $\pi^{-1}(U)$ is open in X^m.

We now consider two properties of the map π.

Proposition 13. *The map $\pi : X^m \to X^m_{\mathrm{Sym}}$ is a proper map.*

Proof. Let $K \subset X^m_{\mathrm{Sym}}$ be compact, and let $\bigcup_{\alpha \in A} O_\alpha$ be an open cover of $\pi^{-1}(K)$. We refine the cover as follows: For every $x \in K$, each element $\widetilde{x}$ in the fiber $\pi^{-1}(x)$ is contained in $O_{\alpha_{\widetilde{x}}}$ for some $\alpha_{\widetilde{x}} \in A$. Additionally, each $\widetilde{x}$ has an open neighborhood $U_{\widetilde{x}} \subset O_{\alpha_{\widetilde{x}}}$, and these neighborhoods can be made to be disjoint. Set

$$V_x = \bigcap_{\widetilde{x} \in \pi^{-1}(x)} \pi(U_{\widetilde{x}}).$$

Then, $\pi^{-1}(V_x)$ is a disjoint union of open neighborhoods $V_{\widetilde{x}}$, with $\widetilde{x} \in V_{\widetilde{x}} \subset U_{\widetilde{x}} \subset O_{\alpha_{\widetilde{x}}}$. Then,

$$\bigcup_{x \in K} \bigcup_{\widetilde{x} \in \pi^{-1}(x)} V_{\widetilde{x}}$$

is a refinement of the cover $\bigcup_{\alpha \in A} O_\alpha$. Now, since $x \in V_x$, and since π is a quotient map,

$$\bigcup_{x \in K} V_x$$

is an open cover of K. Since K is compact, there is a finite set $F \subset K$, where

$$\bigcup_{x \in F} V_x$$

still covers K. Now, if $y \in V_x$, we certainly have $\pi^{-1}(y) \subset \pi^{-1}(V_x)$. Hence,

$$\bigcup_{x \in F} \bigcup_{\widetilde{x} \in \pi^{-1}(x)} V_{\widetilde{x}}$$

still covers $\pi^{-1}(K)$. Since each fiber $\pi^{-1}(x)$ contains at most $m!$ elements, this is a finite cover. Now, since $\pi^{-1}(K)$ can be covered by a finite refinement of $\bigcup_{\alpha \in A} O_\alpha$, we know that $\bigcup_{\alpha \in A} O_\alpha$ must have a finite subcover. Hence, $\pi^{-1}(K)$ is compact, and so π is proper. □

Remark. In the above proposition, we only used two properties of the map π:

1. π is a quotient map of topological spaces.
2. Each element in the range of π has at most $m!$ preimages.

In general, if p is a quotient map and for some $n \in \mathbb{N}$, each element in the range of p has at most n preimages, then p is proper.

Now let $m_1, \ldots, m_k$ be a partition of m, i.e., m_j are positive integers such that $\sum_{j=1}^{k} m_j = m$. Let $V(m_1, \ldots, m_k)$ be the set of points α in X^m_{Sym} such that there are distinct $x^1, \ldots, x^k \in X$ such that

$$\alpha = \langle x^1 : m_1, \ldots, x^k : m_k \rangle,$$

where we use the notation (III.1), that is x^j is repeated m_j times. Also let

$$\widetilde{V}(m_1, \ldots, m_k) = \pi^{-1}(V(m_1, \ldots, m_k)) \subset X^m.$$

Proposition 14. *The restricted map*

$$\pi : \widetilde{V}(m_1, \ldots, m_k) \to V(m_1, \ldots, m_k)$$

is a covering map of degree

$$\frac{m!}{m_1!m_2!\cdots m_k!}.$$

Proof. Let $\alpha \in V(m_1, \ldots, m_k)$. We will show that there is an open set $U_\alpha \subset V(m_1, \ldots, m_k)$, containing α, such that $\pi^{-1}(U_\alpha)$ is a disjoint union of open subsets $\widetilde{U}^i_\alpha \subset \widetilde{V}(m_1, \ldots, m_k)$, with the restriction of π to each $\widetilde{U}^i_\alpha$ a homeomorphism onto its image $\pi(\widetilde{U}^i_\alpha)$.

We can write $\alpha = \langle x^1 : m_1, \ldots, x^k : m_k \rangle$ for some distinct $x^1, \ldots, x^k \in X$. Since X is Hausdorff, there exist disjoint open subsets $U_i \subset X$ with $x^i \in U_j$ if and only if $i = j$. Now, let

$$U_\alpha = \{\langle \alpha^1 : m_1, \ldots, \alpha^k : m_k \rangle \in V(m_1, \ldots, m_k) \mid \alpha^i \in U_i\}.$$

Note that U_α is open in in the subspace topology defined on $V(m_1, \ldots, m_k)$, since

$$U_\alpha = \pi(U_1^{m_1} \times \cdots \times U_k^{m_k}) \cap V(m_1, \ldots, m_k),$$

where $U_i^{m_i}$ denotes the m_i-fold Cartesian power of U_i. Let $\sigma(y^1, \ldots, y^m)$ denote $(y^{\sigma(1)}, \ldots, y^{\sigma(m)})$ for $\sigma \in S_m$ and $(y^1, \ldots, y^m) \in X^m$. Then, $\pi^{-1}(\langle y^1, \ldots, y^m \rangle) = \{\sigma(y^1, \ldots, y^m) \mid \sigma \in S_m\}$, and we have

$$\begin{aligned}
\pi^{-1}(U_\alpha) &= \bigcup_{\sigma \in S_m} \sigma\left((U_1^{m_1} \times \cdots \times U_k^{m_k}) \cap \widetilde{V}(m_1, \ldots, m_k)\right) \\
&= \bigcup_{\sigma \in S_m} \left(\sigma(U_1^{m_1} \times \cdots \times U_k^{m_k}) \cap \widetilde{V}(m_1, \ldots, m_k)\right) \\
&= \left(\bigcup_{\sigma \in S_m} \sigma(U_1^{m_1} \times \cdots \times U_k^{m_k})\right) \cap \widetilde{V}(m_1, \ldots, m_k).
\end{aligned}$$

Since the sets $\sigma(U_1^{m_1} \times \cdots \times U_k^{m_k})$ are open in X^m, it follows that $\pi^{-1}(U_\alpha)$ is open in $\widetilde{V}(m_a, \ldots, m_k)$ with the subspace topology. Also, since $U_i \cap U_j = \emptyset$ for $i \neq j$, the sets $\sigma(U_1^{m_1} \times \cdots \times U_k^{m_k})$ and $\tau(U_1^{m_1} \times \cdots \times U_k^{m_k})$ are either identical or disjoint for each $\sigma, \tau \in S_m$, and the number of distinct sets $\sigma(U_1^{m_1} \times \cdots \times U_k^{m_k})$ is equal to the number of distinct preimages $\pi^{-1}(\langle \alpha^1 : m_1, \ldots, \alpha^k : m_k \rangle)$, which is exactly

$$\frac{m!}{m_1! m_2! \cdots m_k!}.$$

Now, the restricted map $\pi : \sigma(U_1^{m_1} \times \cdots \times U_k^{m_k}) \to \pi(U_1^{m_1} \times \cdots \times U_k^{m_k})$ is one-to-one, and hence a homeomorphism, as π is a quotient map. Thus, by restricting π to the subspace $\sigma(U_1^{m_1} \times \cdots \times U_k^{m_k}) \cap \widetilde{V}(m_1, \ldots, m_k)$, we have $\pi : \sigma(U_1^{m_1} \times \cdots \times U_k^{m_k}) \cap \widetilde{V}(m_1, \ldots, m_k) \to U_\alpha$ is a homeomorphism as well. □

III.3. Algebraic Embedding of Symmetric Powers of Projective Spaces

In this section we describe a method of realizing the m-th symmetric power of a vector space as a subset of its m-th symmetric tensor product. In view of our applications, all our vector spaces may be assumed to be complex and finite dimensional. As usual $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ denotes the tensor product of the m vector spaces $V_1, V_2 \ldots, V_m$. For a detailed treatment of tensor products, see [10]. Then $V_1 \otimes V_2 \otimes \ldots\ldots V_m$ is a vector space of dimension $\prod_{j=1}^m \dim V_j$.

The symmetric tensor product of Hilbert spaces arise in the context of systems of identical particles in quantum mechanics. For a two particle system, with one particle in state ψ_1 and the other in state ψ_2, if these particles are *distinguishable*, the state of the system is represented by $\psi_1 \otimes \psi_2$. However, when the particles are *indistinguishable*, the convention is to *symmetrize* the state, representing it instead by $\frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1)$. This is, up to a constant, the projection of $\psi_1 \otimes \psi_2$ onto the symmetric part of the tensor algebra, and is equivalent to representing the state by $\psi_1 \odot \psi_2$.

Definition 5. The m-fold *symmetric tensor product* of a vector space V, denoted $V^{\odot m}$ is defined as $V^{\odot m} = V^{\otimes m}/\sim$, where $V^{\otimes m} = V \otimes V \otimes \cdots \otimes V$ is the m-fold tensor product of V with itself, and the equivalence relation $\sim$ is defined by:

$$v^{\sigma(1)} \otimes v^{\sigma(2)} \otimes \cdots \otimes v^{\sigma(m)} \sim v^1 \otimes v^2 \otimes \cdots \otimes v^m \tag{III.3}$$

for all $v^1, v^2, \ldots, v^m \in V$ and all $\sigma \in S_m$. We denote the equivalence class of $v^1 \otimes v^2 \otimes \cdots \otimes v^m$ under this equivalence relation (which is an element of $V^{\odot m}$) by

$$v^1 \odot v^2 \odot \cdots \odot v^m.$$

Just as the tensor product of vector spaces is itself a vector space, the symmetric tensor product $V^{\odot m}$ of a vector space is again a vector space, with a natural linear structure as a quotient vector space of $V^{\otimes m}$. Suppose V is finite-dimensional of dimension $s+1$, and let $\{e_0, \ldots, e_s\}$ be a basis of V. Let $\mu = (\mu_0, \mu_2, \ldots, \mu_s) \in \mathbb{N}^{s+1}$ be a multi-index with $|\mu| = \sum_{j=0}^{s} \mu_j = m$, and let $\mathbf{e}_\mu$ denote the element of $V^{\odot m}$ given by

$$\mathbf{e}_\mu = e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_m} \tag{III.4}$$

where exactly μ_j of the e_{i_k} are equal to e_j. For example, if $V = \mathbb{C}^3$ with basis $\{e_0, e_1, e_2\}$, the elements $\mathbf{e}_\mu \in (\mathbb{C}^3)^{\odot 2}$ are:

$$\mathbf{e}_{(2,0,0)} = e_0 \odot e_0, \quad \mathbf{e}_{(0,2,0)} = e_1 \odot e_1, \quad \mathbf{e}_{(0,0,2)} = e_2 \odot e_2,$$
$$\mathbf{e}_{(1,1,0)} = e_0 \odot e_1, \quad \mathbf{e}_{(1,0,1)} = e_0 \odot e_2, \quad \mathbf{e}_{(0,1,1)} = e_1 \odot e_2.$$

It is not difficult to see that $\{\mathbf{e}_\mu\}_{|\mu|=m}$ gives a basis for $V^{\odot m}$. Therefore the dimension of $V^{\odot m}$ is the number of solutions of $\sum_{j=0}^{s} \mu_j = m$, i.e.

$$\dim V^{\odot m} = \binom{m+s}{m}. \tag{III.5}$$

Given a vector space V, there is a natural mapping from the symmetric power, V^m_{Sym}

(which is just a set), to the symmetric tensor product $V^{\odot m}$ (which is a vector space), given by

$$\langle v^1, v^2, \ldots, v^m \rangle \mapsto v^1 \odot v^2 \odot \cdots \odot v^m. \tag{III.6}$$

This mapping is well-defined, since both the unordered tuple $\langle v^1, v^2, \ldots, v^m \rangle$ and the symmetric tensor $v^1 \odot v^2 \odot \cdots \odot v^m$ are invariant under a permutation of the variables.

Let V be a vector space, and denote by $\mathbb{P}(V)$ the *projectivization* of V, that is, the collection of all one-dimensional linear subspaces of V. In other words, we take the quotient of $V \setminus \{0\}$, subject to the equivalence relation

$$\lambda v \sim v \tag{III.7}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$, and imbue it with the quotient topology. For a vector v in a vector space V

$$[v] \in \mathbb{P}(V)$$

denotes its equivalence class in the projectivization of V. In the case that $V = \mathbb{C}^{s+1}$, $\mathbb{P}(V)$ is a complex manifold of dimension s. There is a natural map

$$\psi : (\mathbb{P}(V))^m_{\mathrm{Sym}} \to \mathbb{P}(V^{\odot m})$$

induced by the map (III.6) given by

$$\psi(\langle [v^1], [v^2], \ldots, [v^m] \rangle) = \left[v^1 \odot v^2 \odot \cdots \odot v^m \right] \in \mathbb{P}(V^{\odot m}), \tag{III.8}$$

with $v^j \in V$. We call this the *Segre-Whitney* map. A detailed treatment of symmetric powers and this mapping can be found in the appendix of [26]. This is a symmetric version of the classical *Segre embedding* of the product of two or more projective spaces as a projective variety,

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_m) \to \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_m)$$

given by

$$([v^1],[v^2],\dots,[v^m]) \mapsto [v^1 \otimes v^2 \otimes \cdots \otimes v^m].$$

More details on the Segre map and projective varieties can be found in [18].

Proposition 15. *The Segre-Whitney map* (III.8) *is well-defined and injective.*

Proof. First we show that the Segre-Whitney map is well-defined, i.e. it does not depend on the choice of the vectors v^j but only depends on the elements $[v^j]$. Let $\lambda_1,\lambda_2,\dots,\lambda_m \in \mathbb{C}\setminus\{0\}$ and let $v^1,v^1,\dots,v^m \in V$. Since both sides of (III.8) are invariant under permutation of $v^1,v^2,\dots,v^m$, we merely need to check that $\psi(\langle[\lambda_1 v^1],[\lambda_2 v^2],\dots,[\lambda_m v^m]\rangle) = \psi(\langle[v^1],[v^2],\dots,[v^m]\rangle)$:

$$\begin{aligned}\psi(\langle[\lambda_1 v^1],[\lambda_2 v^2],\dots,[\lambda_m v^m]\rangle) &= [(\lambda_1 v^1)\odot(\lambda_2 v^2)\odot\cdots\odot(\lambda_m v^m]\\ &= [\lambda_1\lambda_2\cdots\lambda_m(v^1\odot v^2\odot\cdots\odot v^m)]\\ &= [v^1\odot v^2\odot\cdots\odot v^m]\\ &= \psi(\langle[v^1],[v^2],\dots,[v^m]\rangle).\end{aligned}$$

Hence, ψ is well-defined. Now to show that ψ is injective, suppose that $[u^1\odot u^2\odot\cdots\odot u^m] = [v^1\odot v^2\odot\cdots\odot v^m]$. Then, there must be constants $\lambda_1,\lambda_2,\dots,\lambda_m \in \mathbb{C}\setminus\{0\}$ and some $\sigma\in S_m$ such that

$$u^1\otimes u^2\otimes\cdots\otimes u^m = (\lambda_1 v^{\sigma(1)})\otimes(\lambda_2 v^{\sigma(2)})\otimes\cdots\otimes(\lambda_m v^{\sigma(m)}),$$

i.e., such that $u^i = \lambda_i v^{\sigma(i)}$ for each $i\in\{1,2,\dots,m\}$. Hence, if $\psi(\langle[u^1],[u^2],\dots,[u^m]\rangle) = \psi(\langle[v^1],[v^2],\dots,[v^m]\rangle)$, then

$$\begin{aligned}\langle[u^1],[u^2],\dots,[u^m]\rangle &= \langle[\lambda_1 v^{\sigma(1)}],[\lambda_2 v^{\sigma(2)}],\dots,[\lambda_m v^{\sigma(m)}])\\ &= \langle[v^{\sigma(1)}],[v^{\sigma(2)}],\dots,[v^{\sigma(m)}]\rangle\\ &= \langle[v^1],[v^2],\dots,[v^m]\rangle.\end{aligned}$$

This proves injectivity. $\square$

When $V = \mathbb{C}^{s+1}$, with coordinates $(z_0, z_1, \ldots, z_s)$, the Segre-Whitney map is therefore a map

$$\psi : (\mathbb{CP}^s)^m_{\text{Sym}} \to \mathbb{P}\left((\mathbb{C}^{s+1})^{\odot m}\right) \cong \mathbb{CP}^{N(m,s)}, \tag{III.9}$$

where (see (III.5))

$$N(m,s) = \binom{m+s}{m} - 1 = \dim_{\mathbb{C}}\left((\mathbb{C}^{s+1})^{\odot m}\right) - 1. \tag{III.10}$$

We will now describe the map ψ as in (III.9) explicitly. To do this, we note that since $\{\mathbf{e}_\mu\}_{|\mu|=m}$ is a natural basis of $(\mathbb{C}^{s+1})^{\odot m}$, the linear coordinates of a point $w \in (\mathbb{C}^{s+1})^{\odot m}$ can be written as $(w_\mu)_{|\mu|=m}$, and the homogeneous coordinates of $[w] \in \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ as $\left[(w_\mu)_{|\mu|=m}\right]$.

Recall that a polynomial p in variables $x^1, x^2, \ldots, x^m$ is called *homogeneous* if each term of p has the same degree. If a polynomial is homogeneous, the degree of each term is the same as the degree of the polynomial.

Proposition 16. *For each $\mu \in \mathbb{N}^{s+1}$ with $|\mu| = m$, there is a homogeneous polynomial of degree m in $(s+1)m$ variables*

$$\psi_\mu : (\mathbb{C}^{s+1})^m \to \mathbb{C}$$

such that for $z^1, \ldots, z^m \in \mathbb{C}^{s+1}$ we have

$$\psi(\langle [z^1], \ldots, [z^m] \rangle) = \left[\sum_{|\mu|=m} \psi_\mu(z^1, z^2, \ldots, z^m)\mathbf{e}_\mu \right]$$

Remark: the polynomials ψ_μ are called multisymmetric.

Proof. This is a computation. Recall that $\pi : (\mathbb{P}(\mathbb{C}^{s+1}))^m \to (\mathbb{P}(\mathbb{C}^{s+1}))^m_{\text{Sym}}$ is the natural quotient map, and $\psi : (\mathbb{P}(\mathbb{C}^{s+1}))^m_{\text{Sym}} \to \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ is the Segre-Whitney map. The composition of these two maps $\psi \circ \pi$ gives a mapping $(\mathbb{P}(\mathbb{C}^{s+1}))^m \to \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$. Denote by $e_0, e_1, \ldots, e_s$ the standard basis of $\mathbb{C}^{s+1}$. Recall that for $\mu \in \mathbb{N}^{s+1}$ be a multi-index with

$|\mu| = m$ and indices $\mu_0, \mu_1, \dots, \mu_s$, $\mathbf{e}_\mu$ denotes the element of $(\mathbb{C}^{(s+1)})^{\odot m}$ given by

$$\mathbf{e}_\mu = e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_m}$$

where exactly μ_j of the e_{i_k} are equal to e_j. Let $z^1, z^2, \dots, z^m \in \mathbb{C}^{s+1}$. Then

$$\begin{aligned}(\psi \circ \pi)([z^1], [z^2], \dots, [z^m]) &= [z^1 \odot z^2 \odot \cdots \odot z^m] \\ &= [(z^1_0 e_0 + z^1_1 e_1 + \cdots + z^1_s e_s) \odot (z^2_0 e_0 + z^2_1 e_1 + \cdots + z^2_s e_s) \odot \dots \\ &\qquad \odot (z^m_0 e_0 + z^m_1 e_1 + \cdots + z^m_s e_s)] \\ &= \left[\sum_{|\mu|=m} \psi_\mu(z^1, z^2, \dots, z^m) \mathbf{e}_\mu \right], \end{aligned} \tag{III.11}$$

where the ψ_μ are the *components* of ψ. As we can see from the above equation, for each $|\mu| = m$, each ψ_μ has the form

$$\psi_\mu(z^1, z^2, \dots, z^m) = \sum z^1_{i_1} z^2_{i_2} \cdots z^m_{i_m}, \tag{III.12}$$

where the sum ranges over all sequences $i_1, \dots, i_m$ with exactly μ_j of the i_k's equal to j. These polynomials are precisely the degree m homogeneous multi-symmetric polynomials in $z^1, z^2, \dots, z^m$. □

We now show that the Segre-Whitney map can be used to embed $(\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}$ into $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ as an analytic set.

Proposition 17. *Let $\psi : (\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}} \to \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ be the Segre-Whitney map. Then, $\psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}})$ is a projective algebraic variety in the projective space $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$.*

Proof. Since $(\mathbb{P}(\mathbb{C}^{s+1}))^m$ is a product of compact spaces, it is itself compact, and since $\psi \circ \pi$ is continuous, it is automatically proper. Since from the representation (III.11), the mapping $\psi \circ \pi$ is holomorphic, by Remmert's Theorem (Theorem 5), $\psi \circ \pi$ maps the image of $(\mathbb{P}(\mathbb{C}^{s+1}))^m$ in $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ as an analytic set. Furthermore, since $(\psi \circ \pi)((\mathbb{P}(\mathbb{C}^{s+1}))^m) =$

$\psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}})$ is an analytic subset of a projective space, by Chow's Theorem (Theorem 6), it is actually a projective algebraic variety. □

III.4. Embedding of Symmetric Powers of Affine Spaces and of Balls

Let E be a subset of $\mathbb{CP}^s$, and let $i : E \to \mathbb{CP}^s$ be the inclusion map. Then we have a natural inclusion

$$i^m_{\mathrm{Sym}} : E^m_{\mathrm{Sym}} \to (\mathbb{CP}^s)^m_{\mathrm{Sym}},$$

so that we can think of E^m_{Sym} as a subset of $(\mathbb{CP}^s)^m_{\mathrm{Sym}}$. Composing with the Segre-Whitney embedding ψ of $(\mathbb{CP}^s)^m_{\mathrm{Sym}}$ in $\mathbb{CP}^{N(m,s)}$, where $N(m,s)$ is defined as in (III.10), we obtain an embedding

$$\psi \circ i^m_{\mathrm{Sym}} : E^m_{\mathrm{Sym}} \to \mathbb{CP}^{N(m,s)}$$

which we will again call the *Segre-Whitney embedding*. In this way we can think of symmetric powers as sitting in some projective space. We will denote this embedded version of the m-th symmetric power of E by $\Sigma^m E$, i.e.,

$$\Sigma^m E = \psi \circ i^m_{\mathrm{Sym}}(E^m_{\mathrm{Sym}}) \subset \mathbb{CP}^{N(m,s)}. \tag{III.13}$$

Let $e_0, \ldots, e_s$ be the natural basis of $\mathbb{C}^{s+1}$ and let $z_0, \ldots, z_s$ be the corresponding linear coordinates, which we can think of as homogeneous coordinates on the projective space $\mathbb{CP}^s$. We consider the affine piece A_0 of $\mathbb{CP}^s$ defined by

$$A_0 = \big\{[z_0 : \cdots : z_s] \in \mathbb{CP}^s \mid z_0 \neq 0\big\}.$$

We can identify A_0 with $\mathbb{C}^s$ in a natural way, by mapping a point $(z_1, \ldots, z_s) \in \mathbb{C}^s$ to the point of $A_0 \subset \mathbb{CP}^s$ given by $[1 : z_1 : \cdots : z_s]$. Recall that for each multi-index $\mu \in \mathbb{N}^{s+1}$ such that $|\mu| = m$, if we define $\mathbf{e}_\mu$ by (III.4), then the collection $\{\mathbf{e}_\mu\}_{|\mu|=m}$ forms a basis of the vector space $(\mathbb{C}^{s+1})^{\odot m}$. We denote the linear coordinates with respect to this basis by

$(z_\mu)_{|\mu|=m}$. Let $\mu_0 = (m, 0, 0, \dots) \in \mathbb{N}^{s+1}$, so the corresponding basis element of $(\mathbb{C}^{s+1})^{\odot m}$ is

$$\mathbf{e}_{\mu_0} = e_0 \odot e_0 \odot \cdots \odot e_0.$$

Let A_{μ_0} be the affine piece of $\mathbb{P}(\mathbb{C}^{s+1})^{\odot m}) = \mathbb{CP}^{N(m,s)}$ (see (III.10)) given by

$$A_{\mu_0} = \left\{ [(z_\mu)_{|\mu|=m}] \in \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})) \cong \mathbb{CP}^{N(m,s)} \mid z_{\mu_0} \neq 0 \right\}.$$

Again, we can identify A_{μ_0} with $\mathbb{C}^{N(m,s)}$ with coordinates $(z_\mu)_{\mu \neq \mu_0}$ by mapping the element $\left[\sum_{|\mu|=m} z_\mu \mathbf{e}_\mu \right] \in A_{\mu_0}$ to the point $\left(\frac{z_\mu}{z_{\mu_0}} \right)_{\mu \neq \mu_0} \in \mathbb{C}^{N(m,s)}$.

We have the following:

Proposition 18. *The mapping $\psi \circ i^m_{\mathrm{Sym}}$ maps the subset $(A_0)^m_{\mathrm{Sym}}$ of $(\mathbb{CP}^s)^m_{\mathrm{Sym}}$ into the affine piece A_{μ_0} of $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})) \cong \mathbb{CP}^{N(m,s)}$.*

Proof. Let $z \in (A_0)^m_{\mathrm{Sym}}$. Then, $z = \langle [z^1], [z^2], \dots, [z^m] \rangle$, where each $[z^i] \in A_0$, and hence $z_0^j \neq 0$ for each j. Then,

$$\begin{aligned}(\psi \circ i^m_{\mathrm{Sym}})(z) &= \psi(\langle [z^1], [z^2], \dots, [z^m] \rangle) \\ &= \left[\sum_{|\mu|=m} \psi_\mu(\langle z^1, z^2, \dots, z^m \rangle) \mathbf{e}_\mu \right],\end{aligned}$$

where the components are as in (III.12). We see that, for $\mu_0 = (m, 0, \dots, 0)$, $\psi_{\mu_0}(\langle z^1, z^2, \dots, z^m \rangle) = z_0^1 z_0^2 \cdots z_0^m \neq 0$. Hence, for $z \in (A_0)^m_{\mathrm{Sym}}$, we have $\psi \circ i^m_{\mathrm{Sym}}(z) \in A_{\mu_0}$, and so $\psi \circ i^m_{\mathrm{Sym}}((A_0)^m_{\mathrm{Sym}}) \subset A_{\mu_0}$. □

Therefore, identifying A_0 with $\mathbb{C}^s$, denoting by i the inclusion of A_0 in $\mathbb{CP}^s$ and identifying A_{μ_0} with $\mathbb{C}^{N(m,s)}$ (see (III.10)), we obtain an injective continuous map

$$\phi = \psi \circ i^m_{\mathrm{Sym}} : (\mathbb{C}^s)^m_{\mathrm{Sym}} \to \mathbb{C}^{N(m,s)},$$

which we will continue to call the Segre-Whitney embedding. Recall from (III.13) that its image is denoted by $\Sigma^m \mathbb{C}^s$.

Proposition 19. *The Segre-Whitney embedding* $\phi : (\mathbb{C}^s)^m_{\mathrm{Sym}} \to \Sigma^m\mathbb{C}^s$ *is a homeomorphism.*

Proof. Since $i : \mathbb{C}^s \to i(\mathbb{C}^s) \subset \mathbb{P}(\mathbb{C}^{s+1})$ is a homeomorphism, $i^m : (\mathbb{C}^s)^m \to i^m((\mathbb{C}^s)^m) \subset (\mathbb{P}(\mathbb{C}^{s+1}))^m$, and hence $i^m_{\mathrm{Sym}} : (\mathbb{C}^s)^m_{\mathrm{Sym}} \to i^m_{\mathrm{Sym}}((\mathbb{C}^s)^m_{\mathrm{Sym}}) \subset (\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}$ are also homeomorphisms. Since $\psi \circ \pi$ is a proper map (see proof of Proposition 17), where π is the quotient map $(\mathbb{P}(\mathbb{C}^{s+1}))^m \to (\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}$, by Proposition 5, ψ is a proper map as well. Since ψ is also injective, by Proposition 6, ψ is a homeomorphism $(\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}} \to \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}) \subset \mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$. Thus, it follows that $\phi = \psi \circ i^m_{\mathrm{Sym}}$ is a homeomorphism $(\mathbb{C}^s)^m_{\mathrm{Sym}} \to \Sigma^m\mathbb{C}^s$. □

Proposition 20. *Let* ψ *be the Segre-Whitney map,* $\phi = \psi \circ i^m_{\mathrm{Sym}}$*,* $\mu_0 = (m, 0, \ldots, 0) \in \mathbb{N}^{s+1}$*, and let* A_{μ_0} *be the affine subspace of* $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ *given by* $z_{\mu_0} \neq 0$*, i.e.,*

$$A_{\mu_0} = \left\{ \left[\sum_{|\mu|=m} z_\mu \mathbf{e}_\mu \right] : z_{\mu_0} \neq 0 \right\} = \mathbb{C}^{N(m,s)},$$

Where $N(m,s)$ *is as defined in* (III.10)*.Then,* $\Sigma^m\mathbb{C}^s = \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}) \cap A_{\mu_0}$*, and hence,* ϕ *embeds* $(\mathbb{C}^s)^m_{\mathrm{Sym}}$ *into* $\mathbb{C}^{N(m,s)}$ *as an affine algebraic variety, where* $N(m,s)$ *is as defined in* (III.10).

Proof. As we have seen above, $\Sigma^m\mathbb{C}^s \subset \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}) \cap A_{\mu_0}$. All that remains is to show that the map $\phi : (\mathbb{C}^s)^m_{\mathrm{Sym}} \to \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}) \cap A_{\mu_0}$ is surjective. Let $z = \left[\sum_{|\mu|=m} z_\mu \mathbf{e}_\mu\right] \in \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}}) \cap A_{\mu_0}$. Then, we have $z_{\mu_0} \neq 0$, and since z is an element of projective space, without loss of generality, we can assume $z_{\mu_0} = 1$. Now, since $z \in \psi((\mathbb{P}(\mathbb{C}^{s+1}))^m_{\mathrm{Sym}})$, we have $z = \psi(\langle [\omega^1], [\omega^2], \ldots, [\omega^m] \rangle)$ for some $\omega^k \in \mathbb{C}^{s+1}$, $1 \leq k \leq m$. Now, since

$$\begin{aligned} \psi(\langle [\omega^1], [\omega^2], \ldots, [\omega^m] \rangle) &= \left[\sum_{|\mu|=m} \psi_\mu(\langle \omega^1, \omega^2, \ldots, \omega^m \rangle) \mathbf{e}_\mu \right] \\ &= \left[\sum_{|\mu|=m} z_\mu \mathbf{e}_\mu \right], \end{aligned}$$

we have $\psi_{\mu_0}(\langle \omega^1, \omega^2, \ldots, \omega^m \rangle) = \omega^1_0 \omega^2_0 \cdots \omega^m_0 \neq 0$. Now, since ψ is a function of the $[\omega^k]$, we can assume that $\omega^k_0 = 1$ for all $1 \leq k \leq m$. But this means that $\langle [\omega^1], [\omega^2], \ldots, [\omega^m] \rangle \in$

$i^m_{\text{Sym}}((\mathbb{C}^s)^m_{\text{Sym}})$, and hence $z \in \Sigma^m\mathbb{C}^s$.

Since $\Sigma^m\mathbb{C}^s$ is the intersection of the projective algebraic variety $\psi(\mathbb{P}(\mathbb{C}^{s+1})^m_{\text{Sym}})$ in $\mathbb{P}((\mathbb{C}^{s+1})^{\odot m})$ with the affine plane A_{μ_0}, $\Sigma^m\mathbb{C}^s$ is an affine algebraic variety, as well as an analytic subset of A_{μ_0}. □

Proposition 21. *Let $U \subset \mathbb{C}^s$ be open. Then, $\Sigma^m U = \phi(U^m_{\text{Sym}})$ is a local analytic subset of the affine space $A_{\mu_0} = \mathbb{C}^{N(m,s)}$ (see* (III.10)*).*

Proof. Since U is open in $\mathbb{C}^s$, U^m_{Sym} is open in $(\mathbb{C}^s)^m_{\text{Sym}}$, and since $\phi : (\mathbb{C}^s)^m_{\text{Sym}} \to \Sigma^m\mathbb{C}^s$ is a homeomorphism, we know that $\Sigma^m U$ is an open subset of $\Sigma^m\mathbb{C}^s$. Now, we know that $\Sigma^m\mathbb{C}^s$ is an analytic subset of $A_{\mu_0} = \mathbb{C}^{N(m,s)}$, and since an open subset of an analytic set is a local analytic set, $\Sigma^m U$ is a local analytic subset of $\mathbb{C}^{N(m,s)}$. □

What we have shown in this section is that, for an open subset $U \subset \mathbb{C}^s$, the abstract m^{th} symmetric power of U, U^m_{Sym}, can be realized as a local analytic set $\Sigma^m U$ in $\mathbb{C}^{N(m,s)}$. For a complex manifold M and a function $f : M \to U^m_{\text{Sym}}$ or $f : U^m_{\text{Sym}} \to M$, we can then say that f is holomorphic if f is a holomorphic mapping of analytic sets as in Definition 2.

III.5. Complex Symmetric Powers

Symmetric powers of the complex plane and other domains occur naturally in mathematics, and also are important in applications. Consider a polynomial $p(z) \in \mathbb{C}[z]$. If the degree of p is m, the polynomial p has exactly m zeros in the complex plane, counting multiplicity. Denote these zeros $\alpha^1, \alpha^2, \dots, \alpha^m$. Then this collection of zeros is well-defined for every polynomial p in $\mathbb{C}[z]$, but has no natural ordering. Thus, it is natural to associate the collection of zeros of p with the unordered tuple $\langle \alpha^1, \alpha^2, \dots, \alpha^m \rangle$. This gives a natural association of the roots of a polynomial of degree m with coefficients in $\mathbb{C}$ with elements of the space $\mathbb{C}^m_{\text{Sym}}$.

We compute the Segre embedding $\Sigma^m U$ explicitly for a few simple cases with $U \subset \mathbb{C}^s$:

Example 4. In the case of $\mathbb{C}^2_{\text{Sym}}$, we have $\phi : \mathbb{C}^2_{\text{Sym}} \to \mathbb{C}^2 \subset \mathbb{P}((\mathbb{C}^2)^{\odot 2})$. Let $z^1, z^2 \in \mathbb{C}$. Then

$$\begin{aligned}\psi \circ i^2_{\text{Sym}}(\langle z^1, z^2 \rangle) &= [(e_0 + z^1 e_1) \odot (e_0 + z^2 e_1)] \\ &= [e_0 \odot e_0 + z^1 e_1 \odot e_0 + z^2 e_0 \odot e_1 + z^1 z^2 e_1 \odot e_1] \\ &= [e_{(2,0)} + (z^1 + z^2) e_{(1,1)} + z^1 z^2 e_{(0,2)}].\end{aligned}$$

It follows therefore that the Segre-Whitney embedding of $\mathbb{C}^2_{\text{Sym}}$ in $\mathbb{C}^2$ is given by

$$\phi(\langle z^1, z^2 \rangle) = (z^1 + z^2, z^1 z^2) \in \mathbb{C}^2, \tag{III.14}$$

which is traditionally known as the *symmetrization map*. Furthermore, ϕ is surjective onto $\mathbb{C}^2$. Indeed, given a point $(s, p) \in \mathbb{C}^2$, if z^1, z^2 are the roots of the equation

$$t^2 - st + p = 0,$$

then we have $\phi(\langle z^1, z^2 \rangle) = (s, p)$. It follows that we can identify

$$\Sigma^2 \mathbb{C} = \mathbb{C}^2.$$

Therefore, we have that

$$\Sigma^2 \mathbb{D} = \{(z^1 + z^2, z^1 z^2) \mid z^1, z^2 \in \mathbb{D}\},$$

so that we have recovered the *Symmetrized bidisc*.

Example 5. We now generalize the previous example from 2 to m variables. In the case of $\mathbb{C}^m_{\text{Sym}}$, we have $\phi : \mathbb{C}^m_{\text{Sym}} \to \mathbb{C}^m \subset \mathbb{P}((\mathbb{C}^2)^{\odot m})$. Let $z^1, z^2, \ldots, z^m \in \mathbb{C}$. Then

$$\begin{aligned}\psi \circ i^m_{\text{Sym}}(\langle z^1, z^2, \ldots, z^m \rangle) &= [(e_0 + z^1 e_1) \odot (e_0 + z^2 e_1) \odot \cdots \odot (e_0 + z^m e_1)] \\ &= [e_{(m,0)} + (z^1 + z^2 + \cdots + z^m) e_{(m-1,1)} + \cdots + z^1 z^2 \cdots z^m e_{(0,m)}]\end{aligned}$$

It follows that in affine coordinates, the map ϕ can be represented as

$$\phi(\langle z^1, z^2, \dots, z^m \rangle) = (\sigma_1(z^1, \dots, z^m), \dots, \sigma_m(z^1, \dots, z^m)),$$

where σ_k denotes the k-th *elementary symmetric polynomial in m variables.* Then ϕ is again a surjective map from $\mathbb{C}^m$ to $\mathbb{C}^m$, since given $s = (s_1, s_2, \dots, s_m) \in \mathbb{C}^m$, we can take $\langle z_1, \dots, z_m \rangle$ to be the unordered collection of roots of

$$t^m - s_1 t^{m-1} + s_2 t^{m-2} - \dots (-1)^m s_m = 0,$$

and it follows that $\phi(\langle z^1, z^2, \dots, z^m \rangle) = (s_1, s_2, \dots, s_m)$. Hence, we can identify

$$\Sigma^m \mathbb{C} = \mathbb{C}^m,$$

and we have

$$\Sigma^m \mathbb{D} = \{(\sigma_1(z^1, \dots, z^m), \dots, \sigma_m(z^1, \dots, z^m)) \mid z^1, \dots, z^m \in \mathbb{D}\}.$$

Hence, we have also recovered the *symmetrized polydisc.*

Example 6. In the case of $(\mathbb{C}^2)^2_{\mathrm{Sym}}$, we have $\phi : (\mathbb{C}^2)^2_{\mathrm{Sym}} \to \mathbb{C}^5 \subset \mathbb{P}((\mathbb{C}^3)^{\odot 2})$. Let $z^1, z^2 \in \mathbb{C}^2$. Then

$$\begin{aligned}
\psi \circ i^2_{\mathrm{Sym}}(\langle z^1, z^2 \rangle) &= [(e_0 + z^1_1 e_1 + z^1_2 e_2) \odot (e_0 + z^2_1 e_1 + z^2_2 e_2)] \\
&= [e_0 \odot e_0 + z^2_1 e_0 \odot e_1 + z^2_2 e_0 \odot e_2 + z^1_1 e_1 \odot e_0 + z^1_1 z^2_1 e_1 \odot e_1 + z^1_1 z^2_2 e_1 \odot e_2 + \\
&\qquad z^1_2 e_2 \odot e_0 + z^1_2 z^2_1 e_2 \odot e_1 + z^1_2 z^2_2 e_2 \odot e_2] \\
&= [e_{(2,0,0)} + (z^1_1 + z^2_1) e_{(1,1,0)} + (z^1_2 + z^2_2) e_{(1,0,1)} + z^1_1 z^2_1 e_{(0,2,0)} + z^1_2 z^2_2 e_{(0,0,2)} + \\
&\qquad (z^1_1 z^2_2 + z^1_2 z^2_1) e_{(0,1,1)}].
\end{aligned}$$

In affine coordinates, we can then represent the map ϕ as

$$\phi(\langle z^1, z^2 \rangle) = (z^1_1 + z^2_1, z^1_2 + z^2_2, z^1_1 z^2_1, z^1_2 z^2_2, z^1_1 z^2_2 + z^1_2 z^2_1).$$

Note that this map $\phi : (\mathbb{C}^2)^2_{\mathrm{Sym}} \to \mathbb{C}^5$ is *not* surjective, but rather the image $\Sigma^2\mathbb{C}^2$ is an affine algebraic hypersurface in $\mathbb{C}^5$. If we represent coordinates in $\mathbb{C}^5$ by $(x_1, x_2, x_3, x_4, x_5)$, then the map ϕ is given by $x_1 = z_1^1 + z_1^2, x_2 = z_2^1 + z_2^2, x_3 = z_1^1 z_1^2, x_4 = z_2^1 z_2^2, x_5 = z_1^1 z_2^2 + z_2^1 z_1^2$, and the image, $\Sigma^2\mathbb{C}^2$, is defined by the vanishing of the polynomial (see [22]):

$$x_2^2 x_3 - x_1 x_2 x_5 + x_1^2 x_4 + x_5^2 - 4x_3 x_4.$$

Now, for the unit ball, $\mathbb{B}_2$, we have

$$\Sigma^2\mathbb{B}_2 = \{(z_1^1 + z_1^2, z_2^1 + z_2^2, z_1^1 z_1^2, z_2^1 z_2^2, z_1^1 z_2^2 + z_2^1 z_1^2) | z^1, z^2 \in \mathbb{B}_2\},$$

which is an open subset of $\Sigma^2\mathbb{C}^2$, and hence a local analytic set in $\mathbb{C}^5$.

III.6. Symmetric Powers as Complex Spaces

Thanks to the above constructions, we have a natural identification of $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ with a local analytic set in $\mathbb{C}^{N(m,s)}$, where $N(m,s)$ is as defined in (III.10). This allows us to do complex analysis on $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ since we are now able to define holomorphic functions and maps. This construction generalizes the construction of the symmetrized polydisc for $s = 1$.

There is however, another more general and natural approach to the problem of constructing the symmetric power of a complex manifold. This is through the notion of a *Complex (Analytic) Space.* Complex spaces are a generalization of the notions of complex manifolds and analytic sets. A complex space X is essentially a complex manifold, except that the charts Ψ_α map open subsets $U_\alpha \subset X$ onto local analytic sets in $\mathbb{C}^n$, not just domains. We also require that a finite or denumerable collection of the U_α's cover X. In the same way that the charts Ψ_α give meaning to a function $f : \Omega \to X$ or $f : X \to \Omega$ being holomorphic if X is a manifold, so do they when X is a complex space. Therefore, one can do complex analysis on a complex space, since we can define holomorphic functions and maps. For details and precise definitions see [26, 9].

Thanks to a result of H. Cartan (see [6, 7]), whenever G is a finite group of biholomorphic automorphisms of the complex space X, the quotient spaces X/G has the natural structure of a complex space, such that the quotient map $p : X \to X/G$ is holomorphic. We could apply this result to the Cartesian power $X = (\mathbb{B}_s)^m$, and take $G = S_m$ to be the group of biholomorphic automorphisms of X given for $\sigma \in S_m$ by

$$\sigma(z^1, z^2, \ldots, z^m) = (z^{\sigma(1)}, z^{\sigma(2)}, \ldots, z^{\sigma(m)}).$$

Then the symmetric power $(\mathbb{B}_s)^m_{\mathrm{Sym}}$ can be identified with the quotient complex space $(\mathbb{B}_s)^m/S_m$.

One can show (but we will not do this) that the two complex spaces $\Sigma^m\mathbb{B}_s$ (which is a local analytic set in a complex Euclidean space) and $(\mathbb{B}_s)^m/S_m$ (which is a quotient space defined using Cartan's theorem stated above) are biholomorphic, so we can identify them, and think of them as different representations of the abstract symmetric power $(\mathbb{B}_s)^m_{\mathrm{Sym}}$. This is the approach we will take in the last chapter of this thesis.

CHAPTER IV

PROPER HOLOMORPHIC SELF-MAPS OF SYMMETRIC POWERS OF BALLS

In this chapter we prove the main result of this paper, Theorem 3. For completeness, we also give a simplified account of the proof of Theorem 1. Recall that $(\mathbb{B}_s)^m_{\text{Sym}}$ is the m-fold symmetric power of the unit ball in $\mathbb{C}^s$. We have shown that this can be identified with a certain local analytic set $\Sigma^m\mathbb{B}_s$, and therefore one can talk about holomorphic maps and functions on this object. In this chapter we determine the holomorphic proper self-maps of $(\mathbb{B}_s)^m_{\text{Sym}}$.

IV.1. Parametrized Analytic Sets

Definition 6. Let Ω be an open set in $\mathbb{C}^n$ and let

$$g : \Omega \to \mathbb{C}^N$$

be a holomorphic map. We call either g or $g(\Omega)$ a *parametrized analytic set*. When the dimension n of the source is 1, we call g an *analytic curve*. We say that g is non-trivial if it is not constant.

Proposition 22. *(1) There are no non-trivial parametrized analytic sets in $\partial\mathbb{B}_s$.*

(2) Let Ω be a connected open set in $\mathbb{C}^n$ and let g be a holomorphic function which takes Ω to the boundary $\partial((\mathbb{B}_s)^m)$ of $(\mathbb{B}_s)^m$. If $g = (g^1, g^2, \ldots, g^m)$, where each g^j maps Ω to $\mathbb{B}_s$, then there is at least one j for which g^j is constant with values in $\partial\mathbb{B}_s$.

Proof. (1) Let Ω be an open subset of $\mathbb{C}^n$ and $g : \Omega \to \partial\mathbb{B}_s$ be a holomorphic map. Then, we have

$$\sum_{i=1}^{s} |g_i|^2 \equiv 1$$

on Ω. Applying the operator $\frac{\partial^2}{\partial z_j \partial \overline{z_j}}$ to both sides of the above equation, we obtain

$$\frac{\partial^2}{\partial z_j \partial \overline{z_j}} \sum_{i=1}^{s} |g_i|^2 = \sum_{i=1}^{s} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} g_i \overline{g_i}$$

$$= \sum_{i=1}^{s} \frac{\partial g_i}{\partial z_j} \frac{\partial \overline{g_i}}{\partial \overline{z_j}}$$
$$= \sum_{i=1}^{s} \frac{\partial g_i}{\partial z_j} \overline{\frac{\partial g_i}{\partial z_j}}$$
$$= \sum_{i=1}^{s} \left| \frac{\partial g_i}{\partial z_j} \right|^2 \equiv 0,$$

on Ω for all $1 \leq j \leq n$. Thus, every first-order partial derivative of g is identically zero on Ω. Hence, g is constant on each connected component of Ω.

(2) The boundary $\partial(\mathbb{B}_s)^m$ is a disjoint union of $2^m - 1$ sets of the form

$$W_1 \times \cdots \times W_k,$$

where each W_j is either $\mathbb{B}_s$ or $\partial\mathbb{B}_s$, and there is at least one W_j which is $\partial\mathbb{B}_s$. Then the union of the inverse images $g^{-1}(W_1 \times \cdots \times W_k)$ is Ω, and therefore at least one of these inverse images has an interior point. Therefore, there is an open subset ω of Ω, and a j in the set of indices $\{1, \ldots, k\}$ such that $g^j(\omega) \subset \partial\mathbb{B}_s$. By part (1), g^j is a constant on ω, and by analytic continuation, since Ω is connected, g^j is constant on Ω also. □

The above proposition can be generalized: (1) holds for strongly pseudoconvex domains $U \subset \mathbb{C}^s$, and hence, (2) holds for products of strongly pseudoconvex domains $U_1 \times \ldots U_n$, $U_i \subset \mathbb{C}^{n_i}$. For a proof of the more general version of (1), see [13].

IV.2. Some Results of Remmert-Stein Theory

In this section we prove two results, using the powerful technique introduced by Remmert and Stein in [23]. The first result, which is actually found in [23], is the simplest counterexample to a Riemann Mapping Theorem for higher dimensions. In fact it is more general: it shows that there is not even a proper holomorphic map from the bidisc to the ball.

Theorem 8. *There is no proper holomorphic map $\mathbb{D}^2 \to \mathbb{B}_2$.*

Proof. Suppose to the contrary that there is a proper holomorphic map $f : \mathbb{D}^2 \to \mathbb{B}_2$. Let f_1 and f_2 denote the components of f, and let $\{\zeta_n u\}_{\nu=1}^{\infty} \subset \mathbb{D}$ be a sequence converging to some point $\zeta \in \partial\mathbb{D}$. Then, $\{f_\nu\}_{\nu=1}^{\infty}$ where $f_\nu(\tau) = f(\zeta_\nu, \tau)$ is a sequence of holomorphic maps $\mathbb{D} \to \mathbb{D}$. By Montel's Theorem, there is a subsequence $\{\zeta_{\nu_n}\}_{n=1}^{\infty}$ for which $\{f_{\nu_n}\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function g. By Proposition 2, $g : \mathbb{D} \to \partial\mathbb{B}_2$, and by Proposition 22, g, is constant. Hence, by Weierstrass' Theorem, we have

$$\frac{\partial f_i(\zeta_{\nu_n}, \tau)}{\partial \tau} \to \frac{dg_i}{d\tau} = 0$$

for $i \in \{1, 2\}$, as $n \to \infty$.

Thus, since this holds for every sequence $\{\zeta_\nu\}$ converging to a point $\zeta \in \partial\mathbb{D}$, we have

$$\frac{\partial f_i(\omega, \tau)}{\partial \tau} \to 0$$

for $i \in \{1, 2\}$, as $\omega \to \partial\mathbb{D}$. Hence, by the maximum principle,

$$\frac{\partial f_i(\omega, \tau)}{\partial \tau} \equiv 0$$

on $\mathbb{D}^2$. Hence, f is independent of τ, and thus the preimage of a point in $\mathbb{B}_2$ must be of the form $\Omega \times \mathbb{D}$, $\Omega \subset \mathbb{D}$. However, since $\mathbb{D}$ is not compact, $\Omega \times \mathbb{D}$ cannot be compact. Hence, f is not proper. □

The following result provides us with the structure of proper holomorphic maps between products of balls. While the main ideas of the proof can be found in [21, page 76], and are ultimately based on ideas of [23], since it does not appear in the literature in this form, we give details of the argument for completeness. The proof uses elementary properties of plurisubharmonic functions, which can be found in [21].

Theorem 9. *Let s, m be positive integers and let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ be a proper holomorphic map. Then, $f(\tau^1, \tau^2, \ldots, \tau^m) = (f_1(\tau^{\sigma(1)}), f_2(\tau^{\sigma(2)}), \ldots, f_m(\tau^{\sigma(m)}))$, where each f_i is a proper holomorphic self-mapping of $\mathbb{B}_s$ and σ is a permutation of $\{1, 2, \ldots, m\}$.*

Proof. Let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ be a proper holomorphic map with components $f_1, f_2, \ldots, f_m$, and let $\{\zeta_\nu\}_{\nu=1}^\infty \subset \mathbb{B}_s$ be a sequence converging to some point $\zeta \in \partial\mathbb{B}_s$. Let $(\tau^1, \tau^2, \ldots, \tau^m) \in (\mathbb{B}_s)^m$, and denote by $\tau^{(j)}$ the $(m-1)$-tuple $(\tau^1, \ldots, \tau^{j-1}, \tau^{j+1}, \ldots, \tau^m)$. Fix $j \in \{1, 2, \ldots, m\}$, and define $f_\nu^j(\tau^{(j)}) = f(\tau^1, \ldots, \tau^{j-1}, \zeta_\nu, \tau^{j+1}, \ldots, \tau^m)$. Then, $\{f_\nu^j\}_{\nu=1}^\infty$ is a sequence of holomorphic maps $(\mathbb{B}_s)^{m-1} \to (\mathbb{B}_s)^m$. By Montel's Theorem, there exists a subsequence $\{\zeta_{\nu_n}\}_{n=1}^\infty$ for which $\{f_{\nu_n}^j\}_{n=1}^\infty$ converges uniformly on compact subsets of $(\mathbb{B}_s)^{m-1}$ to a holomorphic function g. Now, by Proposition 2, $g : (\mathbb{B}_s)^{m-1} \to \partial((\mathbb{B}_s)^m)$. By Proposition 22, this implies that some component of g, say g_i is constant on $(\mathbb{B}_s)^{m-1}$. Hence, by Weierstrass' Theorem, we have

$$\frac{\partial f_i(\tau^1, \ldots, \tau^{j-1}, \zeta_{\nu_n}, \tau^{j+1}, \ldots, \tau^m)}{\partial \tau_l^k} \to \frac{\partial g_i(\tau^{(j)})}{\partial \tau_l^k} = \mathbf{0}$$

as $n \to \infty$ for all $1 \leq k \leq m$, $k \neq j$ and all $1 \leq l \leq s$.

Thus, for every sequence $\{\zeta_\nu\}$ converging to a point $\zeta \in \partial\mathbb{B}_s$, we have a subsequence $\{\zeta_{\nu_n}\}$ for which

$$\sum_{k=1;k\neq j}^{m} \sum_{l=1}^{s} \left\| \frac{\partial f_i(\tau^1, \ldots, \tau^{j-1}, \zeta_{\nu_n}, \tau^{j+1}, \ldots, \tau^m)}{\partial \tau_l^k} \right\|^2 \to 0$$

as $n \to \infty$ for some i, which may depend on the sequence $\{\zeta_\nu\}$ and the subsequence $\{\zeta_{\nu_n}\}$. However, if we consider the product over i, $1 \leq i \leq m$, we have

$$\prod_{i=1}^{s} \sum_{k=1;k\neq j}^{m} \sum_{l=1}^{s} \left\| \frac{\partial f_i(\tau^1, \ldots, \tau^{j-1}, \zeta_{\nu_n}, \tau^{j+1}, \ldots, \tau^m)}{\partial \tau_l^k} \right\|^2 \to 0 \tag{IV.1}$$

as $n \to \infty$ independently of the choice of $\zeta \in \partial\mathbb{B}_s$ and choice of sequence $\zeta_n \to \zeta$. Let

$$h_i = \sum_{k=1;k\neq j}^{m} \sum_{l=1}^{s} \left\| \frac{\partial f_i}{\partial \tau_l^k} \right\|^2. \tag{IV.2}$$

Then, not only is h_i plurisubharmonic, but $\log(h_i)$ is plurisubharmonic as well. Hence,

$$\sum_{i=1}^{m} \log(h_i) = \log\left(\prod_{i=1}^{m} h_i\right)$$

is plurisubharmonic, and so is

$$e^{\log\left(\prod_{i=1}^m h_i\right)} = \prod_{i=1}^m h_i,$$

which is precisely the expression in (IV.1). Thus, by the maximum principle, we must have $\prod_{i=1}^m h_i \equiv 0$ on $(\mathbb{B}_s)^m$. Thus, since

$$(\mathbb{B}_s)^m = \bigcup_{i=1}^m \{\tau \in (\mathbb{B}_s)^m : h_i(\tau) = 0\},$$

one of these sets must contain an interior point, and hence a neighborhood, on which $h_i \equiv 0$ for some i. By analytic continuation, $h_i \equiv 0$ on $(\mathbb{B}_s)^m$. Thus, by (IV.2), f_i depends only on τ^j. Furthermore, for each j, the choice of i is uniquely determined, and each i is distinct. For, if there were a choice of i satisfying

$$\sum_{k=1;k\neq j}^m \sum_{l=1}^s \left\| \frac{\partial f_i(\tau^1, \ldots, \tau^{j-1}, \zeta_{\nu_n}, \tau^{j+1}, \ldots, \tau^m)}{\partial \tau_l^k} \right\|^2 \equiv 0$$

for two indices $j' \neq j$, then f_i would be constant, and f would not be surjective. This is impossible, as, by Corollary 2 f must be surjective. Thus,

$$f(\tau_1, \tau_2, \ldots, \tau_m) = (f_1(\tau_{\sigma(1)}), f_2(\tau_{\sigma(2)}), \ldots, f_m(\tau_{\sigma(m)})),$$

where σ is a permutation of $\{1, 2, \ldots, m\}$. It remains to show that $f_i(\tau_{\sigma(i)})$ defines a proper mapping $\mathbb{B}_s \to \mathbb{B}_s$ for each $i \in \{1, 2, \ldots, m\}$.

Let $K \subset \mathbb{B}_s$ be compact. Then, K^m is compact, and since f is proper,

$$f^{-1}(K^m) = f^{-1}_{\sigma^{-1}(1)}(K) \times f^{-1}_{\sigma^{-1}(2)}(K) \times \cdots \times f^{-1}_{\sigma^{-1}(m)}(K) \tag{IV.3}$$

is compact. Hence, we must have $f^{-1}_{\sigma^{-1}(i)}(K)$ compact for each $i \in \{1, 2, \ldots, m\}$. Thus, each f_i is a proper holomorphic self-mapping of $\mathbb{B}_s$. □

IV.3. Results of Edigarian and Zwonek

In this section, we give a simplified proof of Theorem 1 following the original argument in [14, 15]. For simplicity, we consider only the case $m = 2$. The general case differs from this one only in algebraic complexity.

Theorem 1 for the case $m = 2$ will follow from the following result:

Theorem 10. *Let $f : \mathbb{D}^2 \to \mathbb{D}^2_{\text{Sym}}$ be a proper holomorphic map. Then, $f(\zeta, \tau) = \langle B_1(\zeta), B_2(\tau)\rangle$, where $B_1, B_2 : \mathbb{D} \to \mathbb{D}$ are proper holomorphic maps.*

Proof. Let $f : \mathbb{D}^2 \to \mathbb{D}^2_{\text{Sym}}$ be a proper holomorphic map, and let $\phi : \mathbb{D}^2_{\text{Sym}} \to \Sigma^2\mathbb{D}$ be the Segre embedding as in (III.14) defined by $\phi(\langle\alpha, \beta\rangle) = (\alpha + \beta, \alpha\beta)$. Note that $\langle\alpha, \beta\rangle$ is precisely the root system of the polynomial $p(z) = z^2 - \phi_1(\langle\alpha, \beta\rangle)z + \phi_2(\langle\alpha, \beta\rangle)$.

Let $\{\zeta_n\} \subset \mathbb{D}$ be a sequence converging to some point $\zeta_0 \in \partial\mathbb{D}$. Then, $\{f(\zeta_n, \tau)\}$ is a sequence of holomorphic maps $\mathbb{D} \to \mathbb{D}^2$. By Montel's Theorem, there exists a subsequence $\{\zeta_{n_k}\}$ for which $\{f(\zeta_{n_k}, \tau)\}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $g(\tau)$. By Proposition 2, we know $g : \mathbb{D} \to \partial(\mathbb{D}^2_{\text{Sym}})$.

Let $\Omega_1 = \partial(\mathbb{D}^2_{\text{Sym}} \cap V(1, 1))$ and $\Omega_2 = \partial(\mathbb{D}^2_{\text{Sym}} \cap V(2))$, where the notation $V(1, 1)$ and $V(2)$ is as defined in Section III.2. Now, since

$$\mathbb{D} = g^{-1}(\Omega_1) \sqcup g^{-1}(\Omega_2),$$

where $\sqcup$ represents the disjoint union, $g^{-1}(\Omega_i)$ must have an interior point for some $i \in \{1, 2\}$, and hence an open neighborhood $U \subset g^{-1}(\Omega_i)$. Since $\pi : \mathbb{D}^2 \to \mathbb{D}^2_{\text{Sym}}$ restricts to a holomorphic covering on each of $V(1, 1)$ and $V(2)$, there is some admissible neighborhood $V \subset U$, on which g lifts to a holomorphic function $\widetilde{g} : V \to \partial(\mathbb{D}^2)$ such that $\pi \circ \widetilde{g} = g$ on V. Now, by Proposition 22, one of the components of $\widetilde{g}$ is constant on V with value $C \in \partial\mathbb{D}$.

Now we consider the polynomial

$$z^2 - (\phi_1 \circ \pi \circ \widetilde{g})(\tau)z + (\phi_1 \circ \pi \circ \widetilde{g})(\tau) = z^2 - (\phi_1 \circ g)(\tau)z + (\phi_2 \circ g)(\tau),$$

where $\tau \in V$. Then, evidently, C is a root of this polynomial for each $\tau \in V$, and hence

$$C^2 - (\phi_1 \circ g)(\tau)C + (\phi_2 \circ g)(\tau) \equiv 0$$

on V. Since the mapping ϕ is a biholomorphism, we can consider f as a map $\mathbb{D}^2 \to \Sigma^2\mathbb{D}$ and g as a map $\mathbb{D} \to \partial\Sigma^2\mathbb{D}$. The equation above then takes the simpler form:

$$C^2 - g_1(\tau)C + g_2(\tau) \equiv 0 \tag{IV.4}$$

Since the left hand side of (IV.4) is holomorphic, by analytic continuation, (IV.4) holds on all of $\mathbb{D}$. Differentiating the above equation yields:

$$-\frac{dg_1}{d\tau}C + \frac{dg_2}{d\tau} \equiv 0 \tag{IV.5}$$

$$-\frac{d^2g_1}{d\tau^2}C + \frac{d^2g_2}{d\tau^2} \equiv 0. \tag{IV.6}$$

Eliminating C from (IV.5) and (IV.6) yields:

$$\frac{d^2g_2}{d\tau^2}\frac{dg_1}{d\tau} - \frac{d^2g_1}{d\tau^2}\frac{dg_2}{d\tau} \equiv 0.$$

Now, since $f(\zeta_{n_k}, \cdot)$ converges uniformly on compact subsets of $\mathbb{D}$ to the holomorphic function g, by Weierstrass' Theorem, we have

$$\frac{\partial^2 f_2(\zeta_{n_k}, \tau)}{\partial\tau^2}\frac{\partial f_1(\zeta_{n_k}, \tau)}{\partial\tau} - \frac{\partial^2 f_1(\zeta_{n_k}, \tau)}{\partial\tau^2}\frac{\partial f_2(\zeta_{n_k}, \tau)}{\partial\tau} \to \frac{d^2g_2}{d\tau^2}\frac{dg_1}{d\tau} - \frac{d^2g_1}{d\tau^2}\frac{dg_2}{d\tau} \equiv 0$$

as $k \to \infty$. As this is true for any sequence $\{\zeta_n\} \to \partial\mathbb{D}$, by the maximum principle, we have

$$\frac{\partial^2 f_2}{\partial\tau^2}\frac{\partial f_1}{\partial\tau} - \frac{\partial^2 f_1}{\partial\tau^2}\frac{\partial f_2}{\partial\tau} \equiv 0 \tag{IV.7}$$

on $\mathbb{D}^2$. Hence, defining

$$B_1 = \frac{\frac{\partial f_2}{\partial\tau}}{\frac{\partial f_1}{\partial\tau}}, \tag{IV.8}$$

by (IV.7), we have $\frac{\partial B_1}{\partial\tau} \equiv 0$ on $\mathbb{D}^2 \backslash A$, where $A = \left\{(\zeta, \tau) \in \mathbb{D}^2 \mid \frac{\partial f_1}{\partial\tau}(\zeta, \tau) = 0\right\}$ is an analytic subset of $\mathbb{D}^2$. From (IV.4) and (IV.5), we obtain

$$\left(\frac{dg_2}{d\tau}\right)^2 - g_1\frac{dg_2}{d\tau}\frac{dg_1}{d\tau} + g_2\left(\frac{dg_1}{d\tau}\right)^2 \equiv 0.$$

By Weierstrass' Theorem and the Maximum Principle, we obtain a similar equation for f:

$$\left(\frac{\partial f_2}{d\tau}\right)^2 - f_1\frac{\partial f_2}{\partial\tau}\frac{\partial f_1}{\partial\tau} + f_2\left(\frac{\partial f_1}{\partial\tau}\right)^2 \equiv 0 \tag{IV.9}$$

on $\mathbb{D}^2$. Now, using (IV.8) and (IV.9), we have

$$B_1(\zeta)^2 - f_1(\zeta,\tau)B_1(\zeta) + f_2(\zeta,\tau) \equiv 0 \tag{IV.10}$$

on $\mathbb{D}^2 \setminus A$. Note that the above equation implies $B_1(\zeta)$ is a root of the polynomial $z^2 - f_1(\zeta,\tau)z + f_2(\zeta,\tau)$. Since $f : \mathbb{D}^2 \to \Sigma^2\mathbb{D}$, this means that $B_1(\zeta) \in \mathbb{D}$. Hence, B_1 is bounded on $\mathbb{D}^2 \setminus A$, and by Riemann's Continuation Theorem (Theorem 4), extends holomorphically to $\mathbb{D}$. Thus, (IV.10) holds on all of $\mathbb{D}^2$. Selecting a sequence $\{\tau_n\} \to \tau_0 \in \partial\mathbb{D}$, and repeating the above argument, we can obtain a similar equation:

$$B_2(\tau)^2 - f_1(\zeta,\tau)B_2(\tau) + f_2(\zeta,\tau) \equiv 0. \tag{IV.11}$$

Thus, (IV.10) and (IV.11) tell us that $f(\zeta,\tau) = \phi(\langle B_1(\zeta), B_2(\tau)\rangle)$, and since f is proper, by Proposition 5, $B_1, B_2 : \mathbb{D} \to \mathbb{D}$ must be proper holomorphic maps. □

Theorem 1 for the case $m = 2$ now follows from the above result:

Proof of Theorem 1 ($m = 2$). Let $f : \mathbb{D}^2_{\mathrm{Sym}} \to \mathbb{D}^2_{\mathrm{Sym}}$. Then, $f \circ \pi$ is a proper holomorphic mapping $\mathbb{D}^2 \to \mathbb{D}^2_{\mathrm{Sym}}$, where π is the quotient map $\mathbb{D}^2 \to \mathbb{D}^2_{\mathrm{Sym}}$. By the above theorem, $f \circ \pi$ has the structure $(f \circ \pi)(\zeta,\tau) = \langle B_1(\zeta), B_2(\tau)\rangle$, where B_1 and B_2 are proper holomorphic self-mappings of $\mathbb{D}$. Since these maps must be surjective, there are $\zeta_1, \tau_1 \in \mathbb{D}$ with $B_1(\zeta_1) = B_2(\tau_1) = 0$. Since $(f \circ \pi)(\zeta_1,\tau_1) = (f \circ \pi)(\tau_1,\zeta_1)$, we evidently have $B_1(\tau_1) = B_2(\zeta_1) = 0$. Since this holds for any zeros of B_1 and B_2, these maps have the same zero set.

Now, let $\zeta \in \mathbb{D}$ such that $\mathbb{B}_1(\zeta) \neq 0$. As before, since $(f \circ \pi)(\zeta,\tau_1) = (f \circ \pi)(\tau_1,\zeta)$, we must have $B_1(\zeta) = B_2(\zeta)$. Since this holds for any $\zeta \in \mathbb{D}$, we mush have $B_1 \equiv B_2$ on $\mathbb{D}$. The conclusion follows. □

The proof of the full version of Theorem 2 can be done in entirely the same manner, but is more complex algebraically. The polynomial (IV.4) is replaced with a polynomial of degree m, and equation (IV.5) is replaced with a system of equations in the partial derivatives of f, which must be solved in order to define B_1 as in (IV.8). For details, see the original paper [15], or [8].

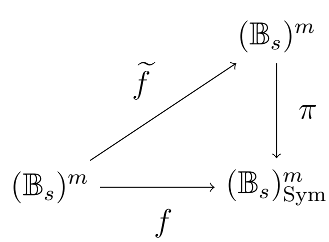
IV.4. Mappings from Cartesian to Symmetric Powers

In this section, we begin the proof of Theorem 3, and extend the results of Edigarian and Zwonek to symmetric powers of the unit ball in $\mathbb{C}^s$ for $s \geq 2$, which is the main new contribution of this thesis.

The first step in the proof of Theorem 3 is the following result, which is interesting in its own right:

Theorem 11. *Let $s \geq 2$, $m \geq 2$, and let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map. Then, there exists a proper holomorphic map $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ such that $f = \pi \circ \widetilde{f}$.*

In other words, the map f can be *lifted* to a proper holomorphic map $\widetilde{f}$ such that the following diagram commutes:



First, we will need the following lemma:

Lemma 1. *Let $A = \{(z^1, \ldots, z^m) \in (\mathbb{B}_s)^m : z^i = z^j \text{ for some } i \neq j\}$. Then $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\text{Sym}}$ of codimension s.*

Proof. Let A_{ij} be the linear subspace of $(\mathbb{C}^s)^m \cong \mathbb{C}^{sm}$ given by

$$A_{ij} = \left\{(z^1, \dots, z^m) | z^i = z^j\right\}.$$

Then A_{ij} is defined by the vanishing of s linearly independent linear functionals $z \mapsto z^i_k - z^j_k$ where $1 \le k \le s$, and consequently A_{ij} is of codimension s in $(\mathbb{C}^s)^m$. Since

$$A = \bigcup_{i<j} (A_{ij} \cap (\mathbb{B}_s)^m),$$

it now follows that A is an analytic subset of codimension s in $(\mathbb{B}^s)^m$. Recall that, by Proposition 13, the quotient map $\pi : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ is proper. Since π is also holomorphic, by Remmert's Theorem (Theorem 5), $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\text{Sym}}$. Now, by Proposition 4, $\pi|_A$ is a proper map $A \to \pi(A)$, and since π is also surjective, by Proposition 9, we must have $\dim A = \dim \pi(A)$. Since $\dim((\mathbb{B}_s)^m) = \dim((\mathbb{B}_s)^m_{\text{Sym}})$, it follows that $\pi(A)$ has the same codimension in $(\mathbb{B}_s)^m_{\text{Sym}}$ as A has in $(\mathbb{B}_s)^m$, which is s. □

Recall that, for a partition $m_1, \dots, m_k$ of m, $V(m_1, \dots, m_k)$ is the set of points α in $(\mathbb{B}_s)^m_{\text{Sym}}$ such that there are distinct $z^1, \dots, z^k \in \mathbb{B}_s$ such that

$$\alpha = \langle z^1 : m_1, \dots, z^k : m_k \rangle,$$

where this notation is as in (III.1), i.e., z^j is repeated m_j times. Also recall from Section III.2 the meaning of the notation $\widetilde{V}(m_1, \dots, m_k)$:

$$\widetilde{V}(m_1, \dots, m_k) = \pi^{-1}(V(m_1, \dots, m_k)) \subset (\mathbb{B}_s)^m.$$

Then, $(\mathbb{B}^s)^m \setminus A$ is precisely the set $\widetilde{V}(m_1, \dots, m_k)$ and $(\mathbb{B}^s)^m_{\text{Sym}} \setminus \pi(A)$ is precisely the set $V(m_1, \dots, m_k)$.

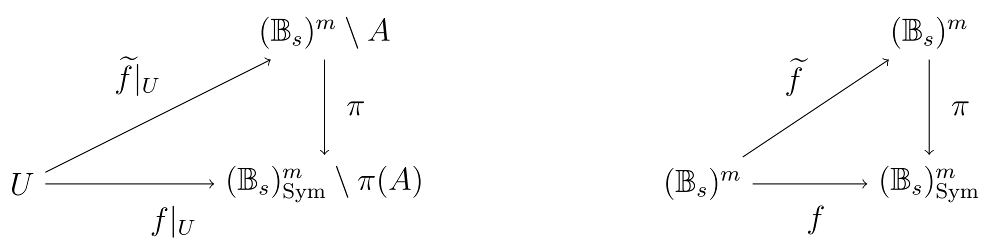
We are now ready to prove Theorem 11.

Proof of Theorem 11. Let $f : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map and let A be as defined in Lemma 1. Since $\pi(A)$ is an analytic subset in $(\mathbb{B}_s)^m_{\text{Sym}}$, $(\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A)$ is an open, connected set.

Since $(\mathbb{B}_s)^m \setminus A = \widetilde{V}(1,1,\dots,1)$ and $(\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A) = V(1,1,\dots,1)$, by Proposition 14, $\pi|_{(\mathbb{B}_s)^m \setminus A}$ is a holomorphic covering $(\mathbb{B}_s)^m \setminus A \to (\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A)$. Since a holomorphic covering is a local biholomorphism, $(\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A)$ is an sm-dimensional complex manifold. Moreover, since $\pi(A)$ is an analytic subset of $(\mathbb{B}_s)^m_{\text{Sym}}$, by Proposition 7, $(\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A)$ is connected and dense in $(\mathbb{B}_s)^m_{\text{Sym}}$. Since $\text{reg}((\mathbb{B}_s)^m_{\text{Sym}})$ is an open subset of $(\mathbb{B}_s)^m_{\text{Sym}}$ containing $(\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A)$, $\text{reg}((\mathbb{B}_s)^m_{\text{Sym}})$ must be connected, and hence by Proposition 8, $(\mathbb{B}_s)^m_{\text{Sym}}$ is an irreducible analytic set, with $\dim((\mathbb{B}_s)^m_{\text{Sym}}) = \dim((\mathbb{B}_s)^m_{\text{Sym}} \setminus \pi(A))$. Since f is a proper holomorphic map from $(\mathbb{B}_s)^m$, which is a manifold of dimension sm, to $(\mathbb{B}_s)^m_{\text{Sym}}$, which is an irreducible analytic set of dimension sm, by Theorem 9, f is surjective.

Now, by Proposition 4, $f|_{f^{-1}(\pi(A))}$ is a proper holomorphic map from $f^{-1}(\pi(A))$, which is an analytic subset of $(\mathbb{B}_s)^m$, onto $\pi(A)$, and so by Proposition 9, $\dim(f^{-1}(\pi(A)) = \dim(\pi(A))$. Since we also have $\dim((\mathbb{B}_s)^m) = \dim((\mathbb{B}_s)^m_{\text{Sym}}) = sm$, and we know from Lemma 1 that $\pi(A)$ has codimension at least s, $f^{-1}(\pi(A))$ must have codimension at least s in $(\mathbb{B}_s)^m$.

Let $U = (\mathbb{B}_s)^m \setminus f^{-1}(\pi(A))$. Since A and $f^{-1}(\pi(A))$ have complex codimension at least $s \geq 2$, by Proposition 12, both $(\mathbb{B}_s)^m \setminus A$ and U are simply-connected. Hence, $f|_U$ has a holomorphic lift $\widetilde{f}|_U : U \to (\mathbb{B}_s)^m \setminus A$, where $f|_U = \pi \circ \widetilde{f}|_U$. Since $\widetilde{f}|_U$ is bounded on U and $f^{-1}(\pi(A))$ is an analytic set, by Riemann's Continuation Theorem (Theorem 4), $\widetilde{f}|_U$ extends to a holomorphic function $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ with $f = \pi \circ \widetilde{f}$.

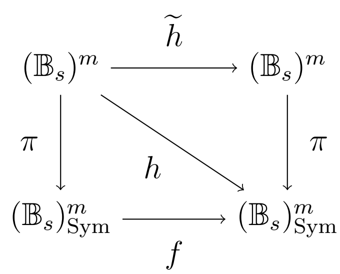


It remains to show that $\widetilde{f} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ is a proper map. However, since $f = \pi \circ \widetilde{f}$ is a proper map, it follows from Proposition 5 that $\widetilde{f}$ is proper. □

IV.5. End of the Proof of Theorem 3

In Theorem 9, we characterized all proper holomorphic self-mappings of $(\mathbb{B}_s)^m$, and in Theorem 11, we showed that every proper holomorphic mapping $(\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$ lifts to a proper holomorphic self-mapping of $(\mathbb{B}_s)^m$. We are now ready to combine these results to characterize proper holomorphic self-mappings of $(\mathbb{B}_s)^m_{\text{Sym}}$, which is the content of our main result, Theorem 3.

Proof. Let $f : (\mathbb{B}_s)^m_{\text{Sym}} \to (\mathbb{B}_s)^m_{\text{Sym}}$ be a proper holomorphic map. Then, since π is proper and holomorphic, $h = f \circ \pi$ is a proper holomorphic map $(\mathbb{B}_s)^m \to (\mathbb{B}_s)^m_{\text{Sym}}$. By Theorem 11, h lifts to a proper holomorphic map $\widetilde{h} : (\mathbb{B}_s)^m \to (\mathbb{B}_s)^m$ with $h = \pi \circ \widetilde{h}$, and by Theorem 9, we know $\widetilde{h}$ has the structure $\widetilde{h}(\tau^1, \tau^2, \ldots, \tau^m) = (\widetilde{h}_1(\tau^{\sigma(1)}), \widetilde{h}_2(\tau^{\sigma(2)}), \ldots, \widetilde{h}_m(\tau^{\sigma(m)}))$ for all $\tau^1, \ldots, \tau^m \in \mathbb{B}_s$ for some permutation σ of $\{1, 2, \ldots, m\}$, where each $\widetilde{h}_i$ is a proper holomorphic self-mapping of $(\mathbb{B}_s)^m$.



Since $f \circ \pi = \pi \circ \widetilde{h}$, and the left-hand side is invariant under the action of S_m on $(\tau^1, \tau^2, \ldots, \tau^m)$, we must have

$$\langle \widetilde{h}_1(\tau^{\sigma(1)}), \widetilde{h}_2(\tau^{\sigma(2)}), \ldots, \widetilde{h}_m(\tau^{\sigma(m)}) \rangle = \langle \widetilde{h}_1(\tau^1), \widetilde{h}_2(\tau^2), \ldots, \widetilde{h}_m(\tau^m) \rangle \qquad \text{(IV.12)}$$

for every $(\tau^1, \tau^2, \ldots, \tau^m) \in (\mathbb{B}_s)^m$ and every $\sigma \in S_m$. Let $\tau^2, \ldots, \tau^m \in \mathbb{B}_s$ be fixed. Then, both sides of equation (IV.12) are functions only of τ^1, and since equation (IV.12) must hold for all $\tau^1 \in \mathbb{B}_s$ and for all $\sigma \in S_m$, the $\widetilde{h}_i$'s must all be identical. Thus, we conclude that h has the structure

$$h(\tau^1, \tau^2, \ldots, \tau^m) = \langle g(\tau^1), g(\tau^2), \ldots, g(\tau^m) \rangle,$$

where $g = \widetilde{h}_1$ is a proper holomorphic self-mapping of $\mathbb{B}_s$. Now, since $f \circ \pi = h$, we have

$$f(\langle \tau^1, \tau^2, \ldots, \tau^m \rangle) = \langle g(\tau^1), g(\tau^2), \ldots, g(\tau^m) \rangle$$
$$= g^m_{\mathrm{Sym}}.$$

As noted in the introduction, for $s \geq 2$, the only proper holomorphic self-mappings of $\mathbb{B}_s$ are the automorphisms of $\mathbb{B}_s$. Hence, g is an automorphism of $\mathbb{B}_s$ and has the form given in (I.2).

□

REFERENCES

[1] J. Agler and N.J. Young, *A commutant lifting theorem for a domain in* $\mathbb{C}^2$ *and spectral interpolation.* J. Funct. Anal. 161 (2), 452–477 (1999)

[2] J. Agler and N.J. Young, *A Schwarz lemma for the symmetrized bidisc.* Bull. London Math. Soc. 33(2), 175–186 (2001)

[3] J. Agler and N.J. Young, *The hyperbolic geometry of the symmetrized bidisc.* J. Geom. Anal. 14(3), 375–403 (2004)

[4] J. Agler and N.J. Young, *A model theory for* Γ*-contractions.* J. Operator Theory 49(1), 45–60 (2003)

[5] H. Alexander, *Proper holomorphic mappings in* $\mathbb{C}^n$, Indiana Univ. Math. J. 26, 137–146 (1977)

[6] H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes.* In Algebraic geometry and topology. pp. 90–102. Princeton University Press, Princeton, N. J. 1957

[7] H. Cartan, *Quotients of complex analytic spaces.* Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960) pp. 1–15 Tata Institute of Fundamental Research, Bombay

[8] D. Chakrabarti and S. Gorai, *Function theory and holomorphic maps on symmetric products of planar domains.* J. Geom. Anal. 25, no. 4, 2196–2225 (2015)

[9] E. M. Chirka, *Complex Analytic Sets*, Kluwer Academic Publishers, 1989

[10] B. Cooperstein, *Advanced Linear Algebra*, Second Edition, Textbooks in Mathematics, CRC Press, 2015

[11] C. Costara, *The symmetrized bidisc and Lempert's theorem.* Bull. London Math. Soc. 36(5), 656–662 (2004)

[12] C. Costara, *The 2× 2 spectral Nevanlinna-Pick problem.* J. London Math. Soc.(2) 71(3), 684–702 (2005)

[13] J. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, Studies in Advanced Mathematics, CRC Press, 1993

[14] A. Edigarian, *Proper Holomorphic Self-Mappings of the Symmetrized Bidisc*, Annales Polonici mathematici 84, 181–184 (2004)

[15] A. Edigarian and W. Zwonek, *Geometry of the Symmetrized Polydisc*, Archiv der Mathematik 84, 364–374 (2005)

[16] C. Godbillon, *Éléments du topologie algébrique*, Hermann 1971

[17] V. Guillemin and A. Pollack, *Differential Topology*, Prentice Hall, New Jersey, 1974

[18] J. Harris, *Algebraic Geometry : A First Course*, Graduate Texts in Mathematics 133, Springer Science+Business Media, New York, 1992

[19] A. Hatcher, *Algebraic Topology*, 2001

[20] M. Jarnicki and P. Pflug, *On automorphisms of the symmetrized bidisc.* Arch. Math. (Basel) 83, no. 3, 264–266 (2004)

[21] R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Mathematics, University of Chicago Press, 1971

[22] P. Olver, *Classical Invariant Theory*, London Mathematical Society Student Texts 44, Cambridge University Press, 1999

[23] R. Remmert and K. Stein, *Eigentliche holomorphe Abbildungen*, Math. Zeit. 73, 159–189 (1960)

[24] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, 1976

[25] W. Rudin, *Function Theory in the unit ball of* $\mathbb{C}^n$, Grundlehren der mathematischen Wissenschaften 241, Springer-Verlag, 1980

[26] H. Whitney, *Complex Analytic Varieties*, Addison-Wesley Series in Mathematics, Addison-Wesley, 1972