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Coordinate Neighborhoods of Arcs and the Approximation of Maps into (Almost) Complex Manifolds

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1. Introduction

This paper is divided into three sections, which, though mostly independent of each other, are devoted to the study of the following question. Let $\Omega \Subset \mathbb{C}$, and let (X, J) be an almost complex manifold. Suppose that f is a continuous map from the closure $\bar{\Omega}$ to X that is J-holomorphic on Ω. Can we approximate f by maps J-holomorphic on (shrinking) neighborhoods of $\bar{\Omega}$?

In Section 2 we give some conditions under which such a map f can be approximated by J-holomorphic maps in a neighborhood of $\bar{\Omega}$. Unfortunately, this involves smoothness assumptions on the boundary $\partial\Omega$ and on f as well (see Theorem 1).

As might be expected, when the almost complex structure J is integrable we can say much more, since we have the tools of complex analysis at our disposal. In fact, Drinovec-Drnovšek and Forstnerič have proved the following: Let S and X be complex manifolds, and let $\Omega \Subset S$ be a strongly pseudoconvex Stein domain with boundary of class $\mathcal{C}^k$ $(k \geq 4)$; then every $\mathcal{C}^{k-2}$ map from $\bar{\Omega}$ to X can be approximated in the $\mathcal{C}^{k-2}$ sense by maps that are holomorphic on (shrinking neighborhoods of) $\bar{\Omega}$ (see [5, Thm. 2.1]). This is a consequence of the fact that the graph of such maps have a basis of Stein neighborhoods [5, Thm. 2.6]. Subsequently, the authors have been able to drop the smoothness assumption on the map to be approximated (see the preprints [6] and [7]). One of their main tools is the theory of sprays. In this paper we consider the case in which the source manifold S is the complex plane $\mathbb{C}$, for which we give a proof along completely different lines and, in so doing, obtain some results of interest on their own.

Section 3 is devoted to the study of arcs (injective continuous maps from the interval) in complex manifolds. We ask the following question: Under what circumstances does such an arc have a coordinate neighborhood—that is, a neighborhood biholomorphic to an open set in Euclidean space $\mathbb{C}^n$? For a real-analytic arc α embedded in a complex manifold $\mathcal{M}$ (i.e., for each $t \in [0,1]$ we have $\alpha'(t) \neq 0$), it is an old result of Royden that a coordinate neighborhood exists (see [17]). We show that embedded $\mathcal{C}^2$ arcs as well as $\mathcal{C}^1$ arcs with some additional conditions have coordinate neighborhoods (see Proposition 3.8 and Proposition 3.3). For our

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application it is important not to restrict attention to smooth arcs alone. We consider a class of nonsmooth arcs with finitely many nonsmooth points, which we call *mildly singular* arcs (see Definition 3.4). We show that such arcs have coordinate neighborhoods (Theorem 2).

As an easy consequence of the results of Section 3, in Section 4 we obtain the following result, a special case of the results of Drivonec-Drovšnek and Forstnerič (with slightly weaker hypotheses than theirs on the boundary).

Let $k \geq 0$ be an integer and let $\mathcal{M}$ be an arbitrary complex manifold. Let f be a $\mathcal{C}^k$ map from $\bar{\Omega}$ into $\mathcal{M}$, where the open set $\Omega \Subset \mathbb{C}$ is bounded by finitely many Jordan curves, which are further assumed to be $\mathcal{C}^1$ if $k \geq 1$. If f is holomorphic on Ω, then it can be approximated in the $\mathcal{C}^k$ topology by holomorphic maps from (neighborhoods of) $\bar{\Omega}$ into $\mathcal{M}$.

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2. Maps into Almost Complex Manifolds

We begin by introducing some notation. For a compact $K \subset \mathbb{R}^N$, an integer $k \geq 0$, and $0 < \theta < 1$, let $\mathcal{C}^{k,\theta}(K)$ denote the Lipschitz space of order $k + \theta$. This space is denoted by $\mathrm{Lip}(k + \theta, K)$ in [18], where it is defined as a Banach space of k-jets with kth derivatives that are Hölder continuous with exponent θ. If the set K is nice (e.g., the closure of a smooth domain), which will always be the case in our applications, then we can identify $\mathcal{C}^{k,\theta}(K)$ with the space of those k times differentiable functions on K, all of whose kth-order partial derivatives are Hölder continuous with exponent θ. The following remarkable fact is proved in [18, Thm. 4, p. 177].

LEMMA 2.1. *Given $N, k \in \mathbb{N}$ and $0 < \theta < 1$, there is a constant C with the following property. Given any compact $K \subset \mathbb{R}^N$, there is a linear extension operator $E\colon \mathcal{C}^{k,\theta}(K) \to \mathcal{C}^{k,\theta}(\mathbb{R}^N)$ such that $\|E\|_{\mathrm{op}} < C$.*

For a Riemannian manifold (X, g), we define the space $\mathcal{C}^{k,\theta}(K, X)$ of Lipschitz maps in the obvious way using local charts on X, and this space has a natural structure of a metric space. Observe, however, that the topology on $\mathcal{C}^{k,\theta}(K, X)$ does not depend on the choice of the metric g.

Let (X, J) be an almost complex manifold, where we assume that the almost complex structure J is of class $\mathcal{C}^{k,\theta}$ for some $k \geq 1$ and $0 < \theta < 1$. For compact $K \subset \mathbb{C}$, denote by $\mathcal{H}_J(K, X)$ the space of J-holomorphic maps from K to X. The map $f\colon K \to X$ belongs to $\mathcal{H}_J(K, X)$ if and only if, for some open $U_f \supset K$, the map f extends J-holomorphically to U_f. It is well known that $\mathcal{H}_J(K, X) \subset \mathcal{C}^{k+1,\theta}(K, X)$. Let $\mathcal{A}_J^{k,\theta}(K, X)$ denote the closed subspace of $\mathcal{C}_J^{k,\theta}(K, X)$ consisting of maps f that are J-holomorphic on the interior of K.

We can now state the following.

THEOREM 1. *Let (X, J) be an almost complex manifold with the structure J of class $\mathcal{C}^{k,\theta}$, where $k \geq 1$ and $0 < \theta < 1$, and let Ω be an open set in $\mathbb{C}$ with $\mathcal{C}^1$ boundary. Then, in the metric space $\mathcal{C}^{k,\theta}(\bar{\Omega}, X)$, the set $\mathcal{H}_J(\bar{\Omega}, X)$ is dense in the set $\mathcal{A}_J^{k+1,\theta}(\bar{\Omega}, X)$.*

We first prove a slightly stronger version (Proposition 2.3) for $X = \mathbb{R}^{2n}$ and then reduce the general case of an almost complex manifold X to that result.

2.1. Stronger Version of Theorem 1

LEMMA 2.2. *Let $k \geq 1$ be an integer, $\omega \Subset \mathbb{C}$ an open set, and B a $2n \times 2n$ real matrix of $\mathcal{C}^{k-1,\theta}$ functions on $\bar{\omega}$. Let L denote the differential operator given by*

$$L(h) = \frac{\partial h}{\partial \bar{z}} + Bh$$

that maps $\mathcal{C}^{k,\theta}(\bar{\omega})$ into $\mathcal{C}^{k-1,\theta}(\bar{\omega})$. Then there exists a constant C_0 such that, for any open subset $W \subset \omega$ and any $g \in \mathcal{C}^{k-1,\theta}(\bar{W})$, there is an $h \in \mathcal{C}^{k,\theta}(\bar{W})$ such that, on $\bar{W}$,

$$Lh = g \tag{1}$$

and

$$\|h\|_{\mathcal{C}^{k,\theta}(\bar{W})} \leq C_0 \|g\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}.$$

Proof. Fix $R > 0$ such that $\omega \subset \Delta_R = \{z \in \mathbb{C} : |z| < R\}$. By [16, Thm. A.2], one can solve the equation

$$\frac{\partial \tilde{h}}{\partial \bar{z}} + B\tilde{h} = \tilde{g} \tag{2}$$

on Δ_R with $\|\tilde{h}\|_{\mathcal{C}^{k,\theta}(\Delta_R)} \leq K_R \|\tilde{g}\|_{\mathcal{C}^{k-1,\theta}(\Delta_R)}$, where K_R is a constant depending only on R. (The proof in [16] assumes that $k = 1$, but it generalizes immediately.)

Let C be the absolute constant provided by Lemma 2.1 as an upper bound to the norm of linear extension operators mapping $\mathcal{C}^{k-1,\theta}$ of a compact subset of $\mathbb{R}^2$ to $\mathcal{C}^{k-1,\theta}(\mathbb{R}^2)$. Extend the data g and the coefficients B of equation (2) from $\bar{W}$ to $\tilde{g}$ and $\tilde{B}$ defined on $\mathbb{R}^2$, with $\|g\|_{\mathcal{C}^{k-1,\theta}(\mathbb{R}^2)} \leq C \|g\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}$, and similarly for B. We now solve equation (2) with estimates as mentioned previously, setting h to be the restriction of $\tilde{h}$ to $\bar{W}$. □

We now prove a version of Theorem 1 for $X = \mathbb{R}^{2n}$.

PROPOSITION 2.3. *Let $\Omega \Subset \mathbb{C}$ be an open set and let U be an open neighborhood of $\bar{\Omega}$. Let J be an almost complex structure of class $\mathcal{C}^{k,\theta}$ on $\mathbb{R}^{2n}$ with $k \geq 1$. Let β be such that $\theta < \beta < 1$. Suppose that the $\mathcal{C}^{k,\beta}$ map $f : \bar{U} \to \mathbb{R}^{2n}$ is such that*

- *$f|_\Omega$ is J-holomorphic and*
- *$J|_{f(\Omega)} = J_{\text{st}}$, the standard complex structure of $\mathbb{R}^{2n} = \mathbb{C}^n$.*

Then f can be approximated uniformly on $\bar{\Omega}$ by J-holomorphic maps.

Proof. Suppose we are given $\varepsilon_0 > 0$. We want to find a neighborhood Ω_{ε_0} of $\bar{\Omega}$ and a J-holomorphic u from $\bar{\Omega}_{\varepsilon_0}$ into R^{2n} such that $\|u - f\|_{\mathcal{C}^{k,\theta}(\bar{\Omega}_{\varepsilon_0})} < \varepsilon_0$.

We set

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + J_{\mathrm{st}}\frac{\partial u}{\partial y}\right)$$

and

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - J_{\mathrm{st}}\frac{\partial u}{\partial y}\right),$$

since J_{st} corresponds to multiplication by i in the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. We know that, provided that $J + J_{\mathrm{st}}$ is invertible, the condition that a map u from some subset of $\mathbb{C}$ into $(\mathbb{R}^{2n}, J)$ to be J-holomorphic is

$$\Phi(u) := \frac{\partial u}{\partial \bar{z}} + Q(u)\frac{\partial u}{\partial z} = 0,$$

where $Q(u)$ is a $2n \times 2n$ matrix given by $Q(u) = [J(u) + J_{\mathrm{st}}]^{-1}[J(u) - J_{\mathrm{st}}]$. Since $J = J_{\mathrm{st}}$ on the range of f, for maps u sufficiently close to f we have $J(u) \approx J_{\mathrm{st}}$ and so this equation determines the J-holomorphy of u for such maps.

We will view Φ as a map from $\mathcal{C}^{k,\theta}(\bar{U})$ to $\mathcal{C}^{k-1,\theta}(\bar{U})$. Its derivative is given by

$$\begin{aligned}\Phi'(u)h &= \frac{\partial h}{\partial \bar{z}} + Q'(u)h\frac{\partial u}{\partial z} + Q(u)\frac{\partial h}{\partial z}\\ &= \left\{\frac{\partial h}{\partial \bar{z}} + A(u)h\right\} + Q(u)\frac{\partial h}{\partial z}\\ &= L_u h + R_u h. \end{aligned} \tag{3}$$

Observe that A and Q are $2n \times 2n$ matrices with entries in $\mathcal{C}^{k-1,\theta}(\bar{U})$ and $\mathcal{C}^{k,\theta}(\bar{U})$, respectively. Since we can easily show that the assignments $u \mapsto (h \mapsto A(u)h)$ and $u \mapsto (h \mapsto Q(u)h_z)$ are continuous from $\mathcal{C}^{k,\theta}(\bar{U})$ into the Banach space of operators $BL(\mathcal{C}^{k,\theta}(\bar{U}, \mathbb{R}^{2n}), \mathcal{C}^{k-1,\theta}(\bar{U}, \mathbb{R}^{2n}))$, it follows that Φ is $\mathcal{C}^1$.

We claim the following: *There is an open $W \supset \bar{\Omega}$ such that $\Phi'(f)$ is surjective from $\mathcal{C}^{k,\theta}(\bar{W})$ to $\mathcal{C}^{k-1,\theta}(\bar{W})$.*

To prove the claim, observe that $Q(f) \in \mathcal{C}^{k,\theta}$ and so, a fortiori, $Q(f)$ is in $\mathcal{C}^k$. We can choose $W \supset \Omega$ so small that $\|Q(f)\|_{\mathcal{C}^k(\bar{W})}$ is small. Since the boundary $\partial\Omega$ of the set Ω is $\mathcal{C}^1$ by hypothesis, we can also choose the W such that the $\mathcal{C}^{k-1,\theta}$ norm is dominated by the $\mathcal{C}^k$ norm. Therefore, we can find a W such that

$$\|Q(f)\|_{\mathcal{C}^{k-1,\theta}(\bar{W})} < \frac{1}{2C_0K},$$

where C_0 is the constant in Lemma 2.2. Thus, for h in $\mathcal{C}^{k,\theta}(\bar{W})$ we have

$$\begin{aligned}\|R_f h\|_{\mathcal{C}^{k-1,\theta}(\bar{W})} &= \left\|Q(f)\frac{\partial h}{\partial \bar{z}}\right\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}\\ &\le \|Q(f)\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}\left\|\frac{\partial h}{\partial \bar{z}}\right\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}\\ &\le \|Q(f)\|_{\mathcal{C}^{k-1,\theta}(\bar{W})}\|h\|_{\mathcal{C}^{k,\theta}(\bar{W})}\\ &< \frac{1}{2C_0K}\|h\|_{\mathcal{C}^{k,\theta}(\bar{W})},\end{aligned}$$

so that $\|R_h\|_{\mathrm{op}} < 1/2C_0K$. Therefore, $\Phi'(f)$ is a small perturbation of a surjective linear map, and standard methods based on iteration show that it is surjective as a map from $\mathcal{C}^{k,\theta}(\bar{W})$ to $\mathcal{C}^{\theta}(\bar{W})$. The equation $\Phi'(f)h = g$ can then be solved with $\|h\|_{\mathcal{C}^{k,\theta}(\bar{W})} \leq 2C_0K\|g\|_{\mathcal{C}^{\theta}(\bar{W})}$.

Since $\Phi'(f)$ is surjective and Φ is $\mathcal{C}^1$, we see that there is a small ball around f that is mapped surjectively by Φ onto a ball around $\Phi(f)$. Therefore, given $\varepsilon > 0$ there is a $\delta > 0$ such that, if $g \in \mathcal{C}^{k-1,\theta}(\bar{W})$ is such that $\|g\|_{\mathcal{C}^{k-1,\theta}(\bar{W})} < \delta$, then the equation

$$\Phi(f + r) = \Phi(f) + g$$

can be solved for an $r \in \mathcal{C}^{k,\theta}(\bar{W})$ such that $\|r\|_{\mathcal{C}^{k,\theta}(\bar{W})} < \varepsilon$.

Now we fix $\varepsilon = \varepsilon_0$ (where ε_0 is as in the beginning of this proof) and denote by δ_0 the corresponding δ. Let C be a uniform bound for linear extension operators from $\mathcal{C}^{k-1,\theta}(\bar{V})$ to $\mathcal{C}^{k-1,\theta}(\mathbb{C})$ for any open subset V of $\mathbb{C}$ (see Lemma 2.1) and let $\delta_1 = \delta_0/C$. Since $f \in \mathcal{C}^{1,\beta}$ by hypothesis, it follows that $\Phi(f) \in \mathcal{C}^{\beta}$. We now use the hypothesis that $\beta > \theta$. Because Φ vanishes on Ω, in a small enough neighborhood of $\bar{\Omega}$ we have that $\|\Phi(f)\|$ is small in the $\mathcal{C}^{k-1,\theta}$ sense. Let Ω_{ε_0} be a neighborhood of $\bar{\Omega}$ such that $\|\Phi(f)\|_{\mathcal{C}^{k-1,\theta}(\Omega_{\varepsilon_0})} < \delta_1$. Denote by g the map $-\Phi(f)|_{\Omega_{\varepsilon_0}}$. Using a linear extension operator, we extend g to a function $\tilde{g}$ in $\mathcal{C}^{k-1,\theta}(\bar{W})$ such that $\|\tilde{g}\|_{\mathcal{C}^{k-1,\theta}(\bar{W})} \leq C\delta_1 = \delta_0$. Therefore, the equation $\Phi(f + r) = \Phi(f) + \tilde{g}$ can be solved for an r such that $\|r\|_{\mathcal{C}^{k,\theta}(\bar{W})} < \varepsilon_0$. If we now set $u = f + r$ then on Ω_{ε_0} we have $\Phi(u) = -g + g = 0$; that is, u is J-holomorphic. Of course,

$$\begin{aligned} \|u - f\|_{\mathcal{C}^{k,\theta}(\bar{\Omega}_{\varepsilon_0})} &\leq \|u - f\|_{\mathcal{C}^{k,\theta}(\bar{W})} \\ &\leq \|r\|_{\mathcal{C}^{k,\theta}(\bar{W})} \\ &< \varepsilon_0. \end{aligned}$$ □

2.2. *The General Case*

Now let (X, J) be an almost complex manifold with J of class $\mathcal{C}^{k,\theta}$ $(k \geq 1)$. Let $f \in \mathcal{A}_J^{k+1,\theta}(\bar{\Omega}, X)$. In order to prove Theorem 1, we need to approximate f in the $\mathcal{C}^{k,\theta}$ topology on $\bar{\Omega}$ by J-holomorphic maps.

We begin by making two observations. First, there is no loss of generality in assuming that f is an embedding. This is because we can replace X by $\mathbb{C} \times X$ and replace f by the map $F : z \mapsto (z, f(z))$, thus obtaining an approximation to F that we can subsequently project onto X. We will therefore assume to begin with that f is an embedding.

The next observation is that we can extend f as a $\mathcal{C}^{k+1,\theta}$ map to all of $\mathbb{C}$. Hence we will assume that f is defined and is an embedding on some large set $\bar{U}$ containing the set Ω compactly and that f is J-holomorphic on Ω.

Let n denote the complex dimension of the almost complex manifold X. It is easy to find $n - 1$ smooth vector fields $Y_2, Y_3, \ldots, Y_n$ on the embedded disc $f(\bar{U})$ such that, for any point z, the $\mathbb{C}$-span of the vectors $\frac{\partial f}{\partial x}(z), Y_2(f(z)), \ldots, Y_n(f(z))$ in the space $T_{f(z)}X$ with respect to the complex structure induced by $J(f(z))$ is the whole of $T_{f(z)}X$. Now consider the map from $\bar{U} \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$ into X given by

$$(z_1, \ldots, z_n) \mapsto \exp_{\sum_{j=2}^{n} z_j Y_j(f(z_1))}(f(z_1)). \tag{4}$$

There is a neighborhood of $\bar{U} \times \mathbb{C}^{n-1}$ in $\mathbb{C}^n$ that is mapped by (4) diffeomorphically onto a neighborhood $\mathcal{U}$ of $f(\bar{U})$ in X. As a result, the inverse φ of the map in (4) is a system of coordinates on $\mathcal{U}$. We note some properties of this coordinate map as follows.

- φ is of class $\mathcal{C}^{k+1,\theta}$. Consequently, the smoothness of J is preserved—that is, the induced structure $J^\sharp$ on $\mathbb{R}^{2n}$ is still $\mathcal{C}^{k,\theta}$.
- The map f is represented in these coordinates by

$$\zeta \mapsto (\zeta, \underbrace{0, \dots, 0}_{n-1}) \in \mathbb{C}^n. \tag{5}$$

- On the set $\bar{\Omega} \times \{\mathbf{0}^{n-1}\}$ we have that $J^\sharp = J_{\mathrm{st}}$, the standard almost complex structure of $\mathbb{C}^n$.

The problem is thus reduced to that considered in Section 2.1, where the approximation asserted in Theorem 1 has been shown to hold.

3. Arcs in Complex Manifolds

We will denote by $\mathcal{M}$ a complex manifold, of complex dimension n, on which we impose a Riemannian metric g. The actual choice of the metric does not affect any of our results.

An *arc* is an injective continuous map from the interval $[0,1]$. We say that a $\mathcal{C}^1$ arc α is *embedded* if $\alpha'(t) \neq 0$ for each t. For convenience of exposition we introduce the following terminology.

DEFINITION 3.1. Let α be an arc in $\mathcal{M}$, and let ϕ be a holomorphic submersion from a neighborhood of $\alpha([0,1])$ in $\mathcal{M}$ into $\mathbb{C}$. We will say that ϕ is a *good submersion* for the arc α if $\phi \circ \alpha$ is a $\mathcal{C}^1$ embedded arc in $\mathbb{C}$.

Clearly, a smooth (at least $\mathcal{C}^1$) arc that admits a good submersion is embedded. Observe, however, that the definition does not require the arc to be smooth. Indeed, the existence of good submersions will serve as a convenient substitute for being embedded when we consider nonsmooth arcs.

First we generalize Royden's result on the existence of coordinate neighborhoods of real-analytic arcs to smooth arcs. The proof of this result is based on a quantitative approximation of $\mathcal{C}^k$ arcs by real-analytic arcs (see Lemma 3.5).

PROPOSITION 3.2. *Let $k \geq 2$ and let α be an embedded $\mathcal{C}^k$ arc in $\mathcal{M}$. Then there is a family $\{\phi_j\}_{j=1}^n$ of n good submersions, associated with α, that form a coordinate system in a neighborhood of the image of α.*

In particular, smooth arcs of class at least $\mathcal{C}^2$ have coordinate neighborhoods. Also, a $\mathcal{C}^2$ arc is embedded if and only if (henceforth "iff") it has a good submersion.

We next consider $\mathcal{C}^1$ arcs in $\mathcal{M}$. Unfortunately, in this case the approximation Lemma 3.5 is not strong enough to prove the existence of coordinate neighborhoods if we simply assume that α is embedded. However, we can prove the following.

Proposition 3.3. *Let α be a $\mathcal{C}^1$ arc in $\mathcal{M}$ that admits a good submersion ϕ. Then there exist a coordinate neighborhood W of $\alpha([0,1])$ in $\mathcal{M}$ and a biholomorphic map $(\phi_1, \dots, \phi_n)$ from W onto on open subset of $\mathbb{C}^n$ such that $\phi_n = \phi|_W$.*

In other words, given a good submersion in a neighborhood of the image of an arc in $\mathcal{M}$, we can find $n-1$ other functions such that the n functions together form a system of coordinates in a neighborhood of α. Observe that, for $j = 1, \dots, n-1$, we may replace the function ϕ_j by the function $\phi_j + K\phi$ (for large enough K) and then assume that each of the coordinate functions ϕ_j is a good submersion, thus strengthening the conclusion.

We now turn to nonsmooth arcs. In view of the intended application in the next section, we introduce the following definition.

Definition 3.4. Let $k \geq 1$. Suppose $\alpha \colon [0,1] \to \mathcal{M}$ is an arc such that

- α is $\mathcal{C}^k$ outside a finite subset $P \subset [0,1]$ and
- α admits a good submersion ϕ.

We will refer to such arcs α as $\mathcal{C}^k$ *arcs with mild singularities* or as *mildly singular arcs.*

Our result concerning mildly singular arcs is as follows.

Theorem 2. *Let α be a $\mathcal{C}^3$ arc in $\mathcal{M}$ with mild singularities, and let ϕ be the associated good submersion. Then the image $\alpha([0,1])$ has a coordinate neighborhood W and a coordinate map $(\phi_1, \dots, \phi_n) \colon W \to \mathbb{C}^n$ with $\phi_n = \phi|_W$.*

3.1. Approximation of Smooth Functions by Real-Analytic Functions

The following approximation lemma is needed to prove that smooth arcs have coordinate neighborhoods.

Lemma 3.5. *Let Γ denote the image of the unit interval $[0,1]$ or the image of the unit circle S^1 under a $\mathcal{C}^k$ embedding into $\mathbb{C}$, where $k \geq 1$. Let f be a $\mathcal{C}^k$ function defined on Γ and let θ be such that $0 < \theta < 1$. Then there is a constant $C > 0$ and a $\mathcal{C}^k$ extension of f to a neighborhood of Γ such that, for small enough $\delta > 0$, there is a holomorphic map f_δ defined in the closed δ-neighborhood $\overline{B_{\mathbb{C}}(\Gamma, \delta)}$ of Γ such that,*

(i) *if α and β are nonnegative integers and if $\alpha + \beta < k$, then for $z \in \overline{B_{\mathbb{C}}(\Gamma, \delta)}$ we have*
$$\left| \left(\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} \right) (f(z) - f_\delta(z)) \right| < C\delta^{k-1/2-(\alpha+\beta)}.$$
Further,

(ii) *f_δ is bounded independently of δ in the $\mathcal{C}^{k-1,\theta}$ norm; more precisely, we have $\|f_\delta\|_{\mathcal{C}^{k-1,\theta}(\overline{B_{\mathbb{C}}(\Gamma,\delta)})} < C$.*

Proof. We will use C to denote any constant that is independent of δ.

For small $\delta > 0$, let χ_δ be a $\mathcal{C}_c^\infty$ cutoff on $\mathbb{C}$ such that $\chi_\delta \equiv 1$ in a neighborhood of $\overline{B_{\mathbb{C}}(\Gamma, \delta)}$ and vanishes off the 2δ-neighborhood $B_{\mathbb{C}}(\Gamma, 2\delta)$. We may choose the

χ_δ such that there is a constant C (which, of course, depends on k) such that, for small δ and every pair of nonnegative integers α and β where $\alpha+\beta<k$,

$$\left|\left(\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta}\right)\chi_\delta(z)\right| < \frac{C}{\delta^{\alpha+\beta}}.$$

Because Γ is totally real, we can use the Whitney extension theorem to extend f as a $\mathcal{C}^k$ function on $\mathbb{C}$ such that the $\bar\partial$-derivative $\frac{\partial f}{\partial \bar z}$ vanishes to order $k-1$ on Γ. Continuing to denote the extended function by f and denoting by $\eta(z)$ the distance from $z\in\mathbb{C}$ to the set Γ, we see that

$$\left|\left(\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta}\right)\left(\frac{\partial f}{\partial \bar z}\right)(z)\right| < C\eta(z)^{k-1-(\alpha+\beta)}.$$

Now we define (suppressing the dependence on δ in the notation)

$$\lambda_{(\alpha,\beta)}(z) = \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta}\left(\chi_\delta\cdot\frac{\partial f}{\partial \bar z}\right)(z).$$

Observe that $\lambda_{(\alpha,\beta)}$ is supported in $B_{\mathbb{C}}(\Gamma,2\delta)$ for every δ, and we have $|\lambda_{(\alpha,\beta)}| = O(\delta^{k-1-(\alpha+\beta)})$.

Let the function u_δ on $\mathbb{C}$ be defined by

$$u_\delta(z) := \frac{-1}{\pi z} * \lambda_{(0,0)}(z) = \frac{-1}{\pi z} * \left(\chi_\delta(z)\cdot\frac{\partial f}{\partial \bar z}(z)\right).$$

Then $f_\delta = f+u_\delta$ is clearly holomorphic on $\overline{B_{\mathbb{C}}(\Gamma,\delta)}$, and

$$\begin{aligned}
\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta}u_\delta(z) &= \frac{-1}{\pi z} * \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta}(\lambda_{(0,0)}(z)) = \frac{-1}{\pi z} * \lambda_{(\alpha,\beta)}(z)\\
&= -\frac{1}{\pi}\iint_{B_{\mathbb{C}}(\Gamma,2\delta)} \frac{1}{z-\zeta}\cdot\lambda_{(\alpha,\beta)}(\zeta)\,d\xi\,d\eta \quad (\zeta = \xi+\imath\eta)\\
&= -\frac{1}{\pi}\left(\iint_{B_{\mathbb{C}}(\Gamma,2\delta)\cap\{\zeta:|\zeta-z|<\sqrt{\delta}\}} + \iint_{B_{\mathbb{C}}(\Gamma,2\delta)\cap\zeta:\{|\zeta-z|\geq\sqrt{\delta}\}}\right)\\
&= -\frac{1}{\pi}(I_1+I_2).
\end{aligned}$$

We can now estimate

$$\begin{aligned}
|I_1| &\leq \frac{1}{\pi}\|\lambda_{(\alpha,\beta)}\|_{L^\infty}\iint_{\{\zeta:|\zeta-z|<\sqrt{\delta}\}}\frac{1}{|z-\zeta|}\,d\xi\,d\eta\\
&\leq \frac{1}{\pi}\cdot C\delta^{k-1-(\alpha+\beta)}\cdot 2\pi\sqrt{\delta}\\
&\leq C\delta^{k-1/2-(\alpha+\beta)}
\end{aligned}$$

and

$$
\begin{aligned}
|I_2| &\le \frac{1}{\pi}\|\lambda_{(\alpha,\beta)}\|_{L^\infty} \iint_{B_{\mathbb{C}}(\Gamma,2\delta)\cap\{\zeta:|\zeta-z|\ge\sqrt{\delta}\}} \frac{1}{|z-\zeta|}\,d\xi\,d\eta \\
&\le \frac{1}{\pi}\cdot C\delta^{k-1-(\alpha+\beta)}\cdot\frac{1}{\sqrt{\delta}}\cdot\operatorname{Area}(B_{\mathbb{C}}(\Gamma,2\delta)) \\
&\le C\delta^{k-1-(\alpha+\beta)}\cdot\frac{1}{\sqrt{\delta}}C\delta \\
&\le C\delta^{k-1/2-(\alpha+\beta)},
\end{aligned}
$$

which proves part (i) of the lemma. Moreover, it immediately follows that for $f_\delta = f + u_\delta$ we have $\|f_\delta\|_{\mathcal{C}^{k-1}} < C$ for some C independent of δ. To complete the proof, it is sufficient to recall the well-known fact that, for functions v on $\mathbb{C}$ supported in a fixed compact set E, the assignment $v \mapsto \frac{1}{z} * v$ is continuous from $\mathcal{C}(E)$ to $\mathcal{C}^{0,\theta}(E)$. □

3.2. $\mathcal{C}^k$ Embedded Arcs, $k \ge 2$

This section is devoted to the proof of Proposition 3.2. We will need the following two lemmas.

LEMMA 3.6. *For convenience, let $B = B_{\mathbb{R}^N}(0,1)$, the N-dimensional unit ball. Let $\Phi\colon \bar{B} \to \mathbb{R}^N$ be a $\mathcal{C}^1$ map such that, for some constant $C > 0$:*

- *for each tangent vector $\mathbf{v}$ we have $\|\Phi'(0)\mathbf{v}\| \ge C\|\mathbf{v}\|$; and*
- *for each $x \in \bar{B}$ we have $\|\Phi'(x) - \Phi'(0)\|_{\mathrm{op}} < C/2$.*

Then $\Phi(\bar{B}) \supset B_{\mathbb{R}^N}(\Phi(0), C/2)$.

Proof. After a translation and dilation, we can assume that $\Phi(0) = 0$ and $C = 2$. Fix $x \in \bar{B}$ and let $u(t) = \Phi(tx)$. We have

$$
\begin{aligned}
\|\Phi(x)\| &= \left\| \int_0^1 u'(t)\,dt \right\| = \left\| \int_0^1 \Phi'(tx)x\,dt \right\| \\
&= \left\| \int_0^1 \Phi'(0)x\,dt + \int_0^1 (\Phi'(tx) - \Phi'(0))\,dt \right\| \\
&\ge \|\Phi'(0)x\| - \left\| \int_0^1 (\Phi'(tx) - \Phi'(0))x\,dt \right\| \\
&\ge 2\|x\| - \|x\| \ge \|x\|.
\end{aligned}
$$

Let $\Sigma = B \cap \Phi(\bar{B})$. Then Σ is nonempty ($0 \in \Sigma$) and is closed in the relative topology of B. Because Φ is expanding, $B \cap \Phi(\bar{B}) = B \cap \Phi(B)$. And since Φ is a local diffeomorphism, it follows that Σ is open in B as well, which implies by connectedness that $\Sigma = B$—that is, $B \subset \Phi(\bar{B})$, which is the required conclusion. □

LEMMA 3.7. *Let $\mathcal{S}$ be a sufficiently smooth compact totally real submanifold of $\mathcal{M}$. Then there is an $\eta > 0$ such that* (i) *the η-neighborhood $B_{\mathcal{M}}(\mathcal{S},\eta)$ of $\mathcal{S}$*

is a Stein open subset of $\mathcal{M}$ *and* (ii) *any continuous function on* $\mathcal{S}$ *can be uniformly approximated by the restrictions to* $\mathcal{S}$ *of functions holomorphic on this* η*-neighborhood.* ("Sufficiently smooth" in this context means $\mathcal{C}^s$, where s is an integer ≥ 2 and greater than $\frac{1}{2}\dim_{\mathbb{R}} \mathcal{S} + 1$.)

For the case where $\mathcal{S}$ is $\mathcal{C}^\infty$, this result is due to Nirenberg and Wells (see [12, Thm. 6.1, Cor. 6.2]). Because the submanifold $\mathcal{S}$ is of class at least $\mathcal{C}^2$, the square of the distance to $\mathcal{S}$ is strictly plurisubharmonic in a neighborhood; it thus follows that $B_{\mathcal{M}}(\mathcal{S}, \eta)$ is Stein. After embedding it in some $\mathbb{C}^N$ and using a retraction onto the embedded submanifold, this reduces to [1, Thm. 17.1]. (It is known that the smoothness assumed in this result is not the best possible.) In our application, we need only prove the case where $\mathcal{S}$ is diffeomorphic to the circle.

Now we turn to the proof of Proposition 3.2. We will, in fact, prove the following proposition.

PROPOSITION 3.8. *Let* $k \geq 2$ *and let* $\alpha\colon S^1 \to \mathcal{M}$ *be a* $\mathcal{C}^k$ *embedding of the circle. Then the image* $\alpha(S^1)$ *has a coordinate neighborhood* W *in* $\mathcal{M}$ *such that there is a coordinate map* $(\phi_1, \dots, \phi_n)\colon W \to \mathbb{C}^n$, *where each* $\phi_j \circ \alpha$ *is a* $\mathcal{C}^k$ *embedding of* S^1 *into* $\mathbb{C}$.

Indeed, any embedding of the interval can be extended to an embedding of the circle, so Proposition 3.8 immediately implies Proposition 3.2.

Proof of Proposition 3.8. It is sufficient to consider the case of $k = 2$. Denote by A_δ the δ-neighborhood $B_{\mathbb{C}^n}(S^1 \times 0_{\mathbb{C}^{n-1}})$ of the circle $S^1 \times 0_{\mathbb{C}^{n-1}}$ in $\mathbb{C}^n$. For small $\delta > 0$ we will construct a biholomorphic map Φ_δ from A_δ onto an open subset of $\mathcal{M}$ such that the image of Φ_δ will contain the embedded circle $\alpha(S^1)$. Consequently, Φ_δ^{-1} is a coordinate map in a neighborhood of $\alpha(S^1)$.

For $\eta > 0$, let $\mathcal{X}_\eta = B_{\mathcal{M}}(\alpha(S^1), \eta)$. Also, for a vector field V on a manifold and a point p on the manifold, let $\exp_V p$ be the point to which p flows in unit time along the field V—that is, $X(1)$, where $X(0) = p$ and $X'(t) = V(X(t))$. The map $\exp_V p$ depends holomorphically on the vector field V and the point p.

We define a map from $A_\delta \subset \mathbb{C}^n$ to $\mathcal{X}_\eta \subset \mathcal{M}$ by setting

$$\Phi_\delta(z_1, \dots, z_n) = \exp_{\sum_{j=2}^n z_j f_j} \alpha_\delta(z_1),$$

where the number $\eta > 0$, the vector fields $\{f_j\}_{j=2}^n$ on the open submanifold $\mathcal{X}_\eta$, and the map $\alpha_\delta\colon B_{\mathbb{C}}(S^1, \delta) \to \mathcal{X}_\eta$ are as follows.

(A) The holomorphic vector fields $\{f_j\}_{j=2}^n$ are such that, for each $z \in S^1 \subset \mathbb{C}$, the set of vectors $\{\alpha'(z), f_2(\alpha(z)), \dots, f_n(\alpha(z))\}$ spans the tangent space $T_\alpha(z)\mathcal{M}$ over $\mathbb{C}$.

To see that such f_j exist, we note that $\mathcal{X}_\eta$ is diffeomorphic to an open solid torus in $\mathbb{C}^n$ and is Stein for small η. Therefore, by an application of the Oka principle, the tangent bundle $T\mathcal{X}_\eta$ is trivial. Also, thanks to Lemma 3.7, any continuous function on the one-dimensional totally real submanifold $\alpha(S^1)$ may be approximated by holomorphic functions in some neighborhood. Therefore, the existence of the f_j follows after (i) approximating smooth vector fields $\{g_j\}_{j=2}^n$ on $\alpha(S^1)$

such that the set $\{\alpha'(z), g_2(\alpha(z)), \dots, g_n(\alpha(z))\}$ spans $T_{\alpha(z)}\mathcal{M}$ and (ii) shrinking η to ensure that the holomorphic approximants f_j are defined on $\mathcal{X}_\eta$.

(B) Now we specify the map α_δ. This will be a holomorphic map defined on $B_{\mathbb{C}}(S^1, \delta)$ and taking values in $\mathcal{X}_\eta \subset \mathcal{M}$ such that, for some $0 < \theta < 1$:

- on S^1 we have $\operatorname{dist}(\alpha_\delta, \alpha) < C\delta^{3/2}$ as well as $\operatorname{dist}(\nabla\alpha_\delta, \nabla\alpha) < C\delta^{1/2}$;
- there is a constant C (independent of δ) such that, on $B_{\mathbb{C}}(S^1, \delta)$, we have $\|\alpha_\delta\|_{\mathcal{C}^{1,\theta}} < C$.

To construct α_δ we note that, since $\mathcal{X}_\eta$ is Stein, there is an embedding $j\colon \mathcal{X}_\eta \to \mathbb{C}^M$ for large M as well as a holomorphic retraction of a neighborhood of $j(\mathcal{X}_\eta)$ onto $\mathcal{X}_\eta$. Fix θ, where $0 < \theta < 1$. Since α is of class $\mathcal{C}^2$, we can use Lemma 3.5 to find a holomorphic approximation α_δ, defined in a δ-neighborhood of the circle $B_{\mathbb{C}}(S^1, \delta)$ of S^1 in $\mathbb{C}$, taking values in $\mathcal{X}_\eta$ such that the two conditions listed above are satisfied.

For small δ, the set $\{\alpha_\delta'(z), f_2(\alpha_\delta(z)), \dots, f_n(\alpha_\delta(z))\}$ spans $T_{\alpha_\delta(z)}\mathcal{M}$ for $z \in B_{\mathbb{C}}(S^1, \delta)$. Moreover, for small δ, the map α_δ is an embedding. Therefore, for small enough δ, the map Φ_δ is well-defined and is a biholomorphism from A_δ into $\mathcal{X}_\eta$. Because the $\mathcal{C}^{1,\theta}$ norm of α_δ on $B_{\mathbb{C}}(S^1, \delta)$ is bounded independently of δ, we conclude that Φ_δ must be bounded in the $\mathcal{C}^{1,\theta}$ norm on A_δ. Recall that the tangent bundle of $T\mathcal{X}_\eta$ is holomorphically trivial, and fix a trivialization. Then $\Phi_\delta'\colon A_\delta \to \operatorname{Mat}_{n\times n}(\mathbb{C})$ is a $\mathcal{C}^\theta$ map. Hence, for a constant C_1 independent of δ and for any $Z, W \in A_\delta$, we have $\|\Phi_\delta'(W) - \Phi_\delta'(Z)\|_{\mathrm{op}} \le C_1\|W - Z\|^\theta$.

In particular, if Z lies on the circle $S^1 \times 0_{\mathbb{C}^{n-1}}$ and if W is in the ball $B_{\mathbb{C}^n}(Z, \delta) \subset A_\delta$, then

$$\|\Phi_\delta'(W) - \Phi_\delta'(Z)\|_{\mathrm{op}} \le C_1\delta^\theta. \tag{6}$$

We claim that there is a constant C_2 independent of δ such that, if $\delta > 0$ is small, then for every $Z \in A_\delta$ and every tangent vector v we have

$$\|\Phi_\delta'(Z)v\| \ge C_2\|v\|. \tag{7}$$

To see this, let $\tilde\alpha$ be an extension of α to a neighborhood of S^1 in $\mathbb{C}$ such that $\|\nabla\tilde\alpha - \nabla\alpha_\delta\| = O(\delta^{1/2})$ (see Lemma 3.5(i); the map $\tilde\alpha$ is $\mathcal{C}^2$, and $\bar\partial\tilde\alpha$ vanishes along Γ). We define a map $\tilde\Phi$ by setting

$$\tilde\Phi(z_1, \dots, z_n) = \exp_{\sum_{j=2}^n z_j f_j} \tilde\alpha(z_1).$$

Then $\tilde\Phi$ is a diffeomorphism from a neighborhood A of $S^1 \times 0_{\mathbb{C}^{n-1}}$ in $\mathbb{C}^n$ into $\mathcal{M}$ and satisfies $\|\Phi_\delta' - \tilde\Phi'\| = O(\delta^{1/2})$. Clearly, there is a constant $C > 0$ such that, for any $Z \in A$ and every tangent vector v, we have $\|\tilde\Phi'(Z)(v)\| \ge C\|v\|$. The existence of the constant C_2 of estimate (7) now follows immediately.

Applying Lemma 3.6 to the inequalities shows that there exist a $\delta_0 > 0$ and a constant K independent of δ such that, for $\delta < \delta_0$ and $Z \in S^1 \times 0_{\mathbb{C}^{n-1}}$, we have $\Phi_\delta(B_{\mathbb{C}^n}(Z, \delta)) \supset B_{\mathcal{M}}(\Phi_\delta(Z), K\delta)$. For a point of the form $Z = (z, 0, \dots) \in S^1 \times 0_{\mathbb{C}^{n-1}}$ we have $\Phi_\delta(Z) = \alpha_\delta(z)$, so it follows that $\Phi_\delta(A_\delta) \supset B_{\mathcal{M}}(\alpha_\delta(S^1), K\delta)$. On the other hand, for $z \in S^1$,

$$\operatorname{dist}_{\mathcal{M}}(\alpha(z), \Phi_\delta(z,0,\ldots,0)) \le \operatorname{dist}_{\mathcal{M}}(\alpha(z), \alpha_\delta(z)) = O(\delta^{3/2}).$$

Hence for small δ we have $\alpha(S^1) \subset \Phi_\delta(A_\delta)$, which shows that $\Phi_\delta^{-1}\colon \Phi_\delta(A_\delta) \to \mathbb{C}^n$ is a coordinate map (biholomorphism onto an open subset of $\mathbb{C}^n$) defined in the neighborhood $\Phi_\delta(A_\delta)$ of $\alpha(S^1)$ in $\mathcal{M}$.

Note that as $\delta \to 0$, the maps $\Phi_\delta^{-1} \circ \alpha_\delta \to j$ on S^1 in the $\mathcal{C}^1$ sense, where j denotes the embedding of S^1 in $\mathbb{C}^n$ as $S^1 \times 0^{n-1}$. Since $\alpha_\delta \to \alpha$ in $\mathcal{C}^1$ it follows that, for small δ, the first coordinate of $\Phi_\delta^{-1} \circ \alpha$ is an embedding of $[0,1]$ into $\mathbb{C}$. Now writing Φ_δ^{-1} in coordinates as $(\phi_1, \ldots, \phi_n)$, we see that $\phi_1 \circ \alpha$ is an embedded arc in $\mathbb{C}$ (which is obviously of class $\mathcal{C}^2$). We now consider the coordinate system $(\phi_1, \phi_2 + K\phi_1, \ldots, \phi_n + K\phi_1)$ in which, for $j > 1$, ϕ_j is replaced by $\phi_j + K\phi_1$. For large K, every coordinate of this map is a $\mathcal{C}^1$ embedding when restricted to α. □

3.3. Coordinate Neighborhoods of $\mathcal{C}^1$ Arcs

We now prove Proposition 3.3. The first step is to show that the image $\alpha([0,1])$ has a *Stein* neighborhood. We begin with the following elementary observations.

OBSERVATION 3.9. *Let $\gamma\colon [0,1] \to \mathcal{M}$ be an arc. Then $\gamma([0,1])$ has a neighborhood W such that there is a strictly plurisubharmonic function ρ defined on W.*

Note that no regularity assumption apart from injectivity has been made on γ.

Proof of Observation 3.9. There is, of course, a strictly plurisubharmonic function in a neighborhood of $\gamma(0)$. Suppose that, for some $0 < p < q < 1$, the segment $\gamma([p,q])$ is in a coordinate chart of $\mathcal{M}$ and that ρ is a strictly plurisubharmonic function in a neighborhood of $\gamma([0,p])$. By an induction on a cover of $\gamma([0,1])$ by coordinate charts, it is sufficient to construct a strictly plurisubharmonic function in a neighborhood of $\gamma([0,q])$. Subtracting a constant yields $\rho(\gamma(p)) = 0$, and there is a coordinate map Z on a neighborhood of $\gamma([p,q])$ such that $Z(p) = 0 \in \mathbb{C}^n$. Fix an r with $p < r < q$ so that ρ is defined on $[p,r]$. Now we can define the function $\tilde\rho$ as follows:

$$\tilde\rho = \begin{cases} \rho & \text{near } \gamma([0,p]), \\ \max(\rho, K\|Z\|^2 - 1) & \text{near } \gamma([p,r]) \text{ with } K \text{ large (see below)}, \\ K\|Z\|^2 - 1 & \text{near } \gamma([r,q]). \end{cases}$$

In this definition we take K so large that $K\|Z(\gamma(r))\|^2 - 1 > \rho(\gamma(r))$. Then $\tilde\rho$ is ρ near $\gamma(p)$ and is $K\|Z\|^2 - 1$ near the other endpoint $\gamma(r)$ of $\gamma([p,r])$ and then continues to the next chart. □

The following is a well-known general fact regarding polynomially convex sets.

OBSERVATION 3.10. *Let X be a compact polynomially convex subset of $\mathbb{C}^N$, and let $\lambda \ge 0$ be a continuous function on $\mathbb{C}^N$ such that $\lambda = 0$ exactly on X. Then, given any neighborhood W of X, there is a continuous plurisubharmonic function $\rho \ge 0$ such that $\rho = 0$ exactly on X and $\rho < \lambda$ on W.*

Proof. For each $p \in \mathbb{C}^n \setminus X$ there exists a continuous plurisubharmonic function $\rho_p \geq 0$ defined on $\mathbb{C}^n$ such that $\rho_p(p) > 0$ and ρ vanishes in a neighborhood of X. There is a sequence x_j in $\mathbb{C}^n \setminus X$ such that, for all $z \in \mathbb{C}^n \setminus X$, there exists an integer j such that $\rho_{x_j}(z) > 0$. We set $\rho = \sum \varepsilon_j \rho_{x_j}$, with $\varepsilon_j > 0$ small enough to ensure that $\varepsilon_j \rho_{x_j} \leq 2^{-j}\lambda$ on $W \cup B_{\mathbb{C}^n}(0, j)$. □

We next prove the following lemma.

LEMMA 3.11. *Let $\gamma : [0,1] \to \mathcal{M}$ be a $\mathcal{C}^1$ embedded arc in a complex manifold $\mathcal{M}$. Then there is a $\delta > 0$ with the following properties.*

(a) *Let $0 \leq t_0 < t_1 \leq 1$ be such that $|t_0 - t_1| < \delta$, and let W_0 and W_1 be given neighborhoods of $\gamma(t_0)$ and $\gamma(t_1)$ in $\mathcal{M}$. Then there is a plurisubharmonic function $\rho \geq 0$ defined in a neighborhood of $\gamma([t_0,t_1])$ in $\mathcal{M}$ such that $\rho^{-1}(0) = V_0 \cup \gamma([t_0,t_1]) \cup V_1$, where $V_j \subset W_j$ are compact neighborhoods of $\gamma(t_j)$ for $j = 0,1$.*

(b) *Let t_1 be such that $0 < t_1 < \delta$ (resp., $0 < 1 - t_1 < \delta$), and let W_1 be a neighborhood of $\gamma(t_1)$ in $\mathcal{M}$. Then there is a plurisubharmonic function $\rho \geq 0$ defined in a neighborhood of $\gamma([0,t_1])$ (resp., of $\gamma([t_1,1])$) such that $\rho^{-1}(0) = V_1 \cup \gamma([0,t_1])$ (resp., $\rho^{-1}(0) = \gamma([t_1,1]) \cup V_1$.)*

Proof. Using compactness and local coordinates, it clearly suffices to prove the result for $\mathcal{M} = \mathbb{C}^n$. We prove only part (a), since the proof of part (b) involves only minor changes.

Let $t_0 \in [0,1]$. After a linear change of coordinates, we can assume that for each $j = 1,\dots,n$, the components of the tangent vector $\gamma_j'(t_0)$ are nonzero. We let δ_{t_0} be so small that, on $[t_0 - \delta_{t_0}, t_0 + \delta_{t_0}]$, the component functions γ_j are $\mathcal{C}^1$ embeddings into $\mathbb{C}$.

Now suppose we are given neighborhoods W_0 and W_1 of $\gamma(t_0)$ and $\gamma(t_1)$ in $\mathcal{M}$. Choose $r > 0$ so small that, for $j = 1,\dots,n$, the closed discs $\overline{B_{\mathbb{C}}(\gamma_j(t_0), r)}$ and $\overline{B_{\mathbb{C}}(\gamma_j(t_1), r)}$ are contained in the sets $\pi_j(W_0)$ and $\pi_j(W_1)$ respectively, where $\pi_j : \mathbb{C}^n \to \mathbb{C}$ is the jth coordinate function. For small r, the subset

$$K_j = \overline{B_{\mathbb{C}}(\gamma_j(t_0), r)} \cup \gamma_j([t_0,t_1]) \cup \overline{B_{\mathbb{C}}(\gamma_j(t_1), r)}$$

of $\mathbb{C}$ is polynomially convex, so there must be a plurisubharmonic $\rho_j \geq 0$ on $\mathbb{C}$ that vanishes exactly on K_j. Let $\rho := \sum_{j=1}^n \rho_j \circ \pi_j$. Then clearly $\rho^{-1}(0)$ is the union of the subarc $\gamma([t_0,t_1])$ with two closed polydiscs of polyradius r centered at the endpoints $\gamma(t_0)$ and $\gamma(t_1)$, which are contained (respectively) in W_0 and W_1. Choosing δ uniformly for all t_0 by compactness, conclusion (a) follows. □

We can now prove Proposition 3.3.

It is clear that α is an embedded arc. We consider two partitions of the interval $[0,1]$,

$$0 < t_1 < \cdots < t_{N-1} < 1 \quad \text{and} \quad 0 < t_1' < \cdots < t_{N-1}' < 1,$$

such that $t_j \neq t'_j$ for each j. We set $t_0 = t'_0 = 0$ and $t_N = t'_N = 1$. Choose the partitions in such a way that $|t_j - t_{j+1}| < \delta$ and $|t'_j - t'_{j+1}| < \delta$ for $j = 0, 1, \ldots, N-1$, where δ is as in the conclusion of Lemma 3.11. Suppose that, for $j = 1, \ldots, N-1$, the open neighborhoods W_j and W'_j of $\gamma(t_j)$ and $\gamma(t_{j+1})$ are such that $W_j \cap W'_j = \emptyset$.

We now apply Lemma 3.11. For $j = 1, \ldots, N-1$, let ρ_j be a nonnegative plurisubharmonic function in a neighborhood of $\alpha([t_j, t_{j+1}])$ that vanishes exactly on $\alpha([t_j, t_{j+1}]) \cup V_j \cup V_{j+1}$, where $V_j \subset W_j$ and $V_{j+1} \subset W_{j+1}$ contain the points $\alpha(t_j)$ and $\alpha(t_{j+1})$, respectively. Let ρ_0 and ρ_N be nonnegative and plurisubharmonic on neighborhoods of $\alpha([0, t_1])$ and $\alpha([t_N, 1])$, respectively, such that $\rho_0^{-1}(0) = \alpha([0, t_1]) \cup V_1$ and $\rho_N^{-1}(0) = \alpha([t_{N-1}, 1]) \cup V_{N-1}$, where $\alpha(t_1) \in V_1 \subset W_1$ and $\alpha(t_{N-1}) \in V_{N-1} \subset W_{N-1}$. Then there is a function $\rho \geq 0$ in a neighborhood of $\alpha([0,1])$ that is equal to ρ_j in a neighborhood of $\alpha([t_j, t_{j+1}])$. Because ρ is locally the maximum of plurisubharmonic functions, ρ is itself plurisubharmonic and vanishes exactly on the arc $\alpha([0,1])$ and on small neighborhoods of $\alpha(t_j)$ contained in W_j ($j = 1, \ldots, N-1$).

In the same way we obtain a plurisubharmonic $\rho' \geq 0$ that vanishes exactly on the arc $\alpha([0,1])$ and on small neighborhoods of $\alpha(t'_j)$ contained in W'_j, $j = 1, \ldots, N-1$. Then $\tilde{\rho} = \rho + \rho'$ is a plurisubharmonic function in a neighborhood of $\alpha([0,1])$ that vanishes exactly on $\alpha([0,1])$. Let $\varepsilon > 0$ be small and let ψ be a strictly plurisubharmonic function in a neighborhood of $\alpha([0,1])$. Then the open set $\Omega = \{\tilde{\rho} < \varepsilon\}$ supports the strictly plurisubharmonic exhaustion function $(\varepsilon - \tilde{\rho})^{-1} + \psi$ and is consequently Stein. If ε is small, then the submersion ϕ is defined on Ω.

There is an embedding $j \colon \Omega \hookrightarrow \mathbb{C}^N$ for large enough N. Let $\tilde{j} \colon \Omega \hookrightarrow \mathbb{C}^{N+1}$ be the map $\tilde{j}(z) := (j(z), \phi(z))$, where ϕ is the good submersion associated with the arc α (whose existence is assumed in the hypothesis). Then $\tilde{j}$ is again an embedding. Let $\mathcal{X} := \tilde{j}(\Omega)$. Then the following statements hold.

- $\mathcal{X}$ is a complex submanifold of $\mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$.
- $z_{N+1} \colon \mathcal{X} \to \mathbb{C}$ is a submersion.
- Let $\tilde{\alpha} = \tilde{j} \circ \alpha$; then $\tilde{\alpha}$ is a $\mathcal{C}^1$ embedded arc in $\mathcal{X} \subset \mathbb{C}^{N+1}$ such that the last coordinate $\alpha_{N+1} \colon [0,1] \to \mathbb{C}$ is a $\mathcal{C}^1$ embedding.
- $\phi \circ \alpha \colon [0,1] \to \mathbb{C}$ is a $\mathcal{C}^1$ embedding.

Set $\Gamma = \phi(\alpha([0,1]))$ and let $\psi \colon \Gamma \to [0,1]$ be the inverse $\psi = (\phi \circ \alpha)^{-1}$. We let

$$\begin{aligned}\tilde{\beta}(z) &:= (\beta(z), z)\\ &:= (j \circ \alpha \circ \psi(z), z).\end{aligned}$$

To prove our result, it is sufficient to show that $\tilde{\beta}(\Gamma)$ has a neighborhood W in $\mathcal{X}$ such that (i) W is biholomorphic to an open subset of $\mathbb{C}^n$ and (ii) there is a biholomorphism $w = (w_1, \ldots, w_n)$ from W into $\mathbb{C}^n$ such that $w_n = z_{N+1}|_{\mathcal{X}}$ (where $(z_1, \ldots, z_N)$ are the coordinates of $\mathbb{C}^{N+1}$ in which $\mathcal{X}$ is embedded). We will construct the map w by first defining it on a neighborhood of $\tilde{\beta}(\Gamma)$ in $\mathbb{C}^{N+1}$; its restriction to $\tilde{X}$ will then provide us with the required biholomorphic map.

To do this, let $\{g_i\}_{i=1}^{n-1}$ be smooth maps from Γ into $\mathbb{C}^N$ such that, for each $z \in \Gamma$, the maps (along with $\beta'(z)$) span the tangent space $T_{\beta(z)}(j(\mathcal{X})) \subset \mathbb{C}^N$. If the $\mathbb{C}^{N+1}$-valued maps $\{f_i\}_{i=1}^{n-1}$ are formed from g_i by taking the last coordinate to be 0, then the $f_i(z)$ along with the vector $\tilde{\beta}'(z) = (\beta'(z), 1)$ span the tangent space $T_{\tilde{\beta}(z)}(\mathcal{X})$.

Let A be a $\mathrm{Mat}_{n\times N}(\mathbb{C})$-valued smooth map on Γ such that $A(z)g_i(z) = e_i$ for each $i = 1, \dots, n-1$. We can approximate A uniformly on Γ by a holomorphic matrix-valued map B defined in a neighborhood of Γ in $\mathbb{C}$. Consider the map

$$\Lambda(z_1, \dots, z_{N+1}) := \left(B(z_{N+1}) \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, z_{N+1} \right),$$

which is defined in a neighborhood of the arc $\tilde{\beta}(\Gamma)$ in $\mathbb{C}^{N+1}$. Its derivative is given by the matrix

$$\Lambda'(z_1, \dots, z_{N+1}) = \begin{pmatrix} B(z_{N+1}) & B'(z_{N+1}) \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \\ 0 & 1 \end{pmatrix}.$$

By construction, if the approximation B is close enough then this map is surjective from $T_{\tilde{\beta}(z)}\mathcal{X} \subset \mathbb{C}^{N+1}$ to $\mathbb{C}^n$ at each point of $\tilde{\beta}(\Gamma)$. Moreover, it is clearly continuous. Hence Λ maps a neighborhood of the arc $\tilde{\beta}$ in $\mathcal{X}$ to $\mathbb{C}^n$ biholomorphically, and its last coordinate is z_{N+1}. This completes the proof of Lemma 3.11. $\square$

Remark. There is another approach to the construction of the Stein neighborhood of the arc in the first part of this proof, an approach that consists of extending the map α to a $\mathcal{C}^1$ embedding of the circle S^1 in such a way that $\alpha(S^1)$ is contained in the domain of ϕ and that $\phi \circ \alpha$ is still a $\mathcal{C}^1$ arc in $\mathbb{C}$. We can then apply [9, Cor. 2] to obtain a $\mathcal{C}^2$ plurisubharmonic function in a neighborhood of $\alpha(S^1)$ that vanishes precisely on $\alpha(S^1)$. The rest of the proof is completed as before.

3.4. Mildly Singular Arcs, Step 1: Stein Neighborhoods

The remainder of Section 3 is devoted to a proof of Theorem 2, which proceeds in several steps. In this step we establish the existence of certain Stein neighborhoods Ω_δ of the arc $\alpha([0,1])$, which allows us to solve $\bar{\partial}$ equations in these neighborhoods. In the next step (Section 3.5) we establish a result regarding the gluing together of immersions defined in neighborhoods of compact sets K_1 and K_2 in a manifold to a single immersion defined in a neighborhood of their union. In the last step (Section 3.6) we use these two results to obtain a proof of Theorem 2.

The main result of this section is as follows.

LEMMA 3.12. *Let $\alpha : [0,1] \to \mathcal{M}$ be a $\mathcal{C}^2$ arc with mild singularities. Let $P \subset [0,1]$ be the set of points where α is not smooth, and let ϕ be the good submersion associated with α. Let $\mathcal{U}$ be a fixed neighborhood of $\alpha(P)$ in $\mathcal{M}$. For $\delta > 0$ sufficiently small, there is a neighborhood Ω_δ of $\alpha([0,1])$ in $\mathcal{M}$ such that*:

- *Ω_δ is Stein*;
- *Ω_δ contains the δ-neighborhood of the arc $\alpha([0,1])$*; *and*
- *away from the nonsmooth points $\alpha(P)$ of α, the set Ω_δ coincides with the δ-neighborhood of the arc—more precisely, $\Omega_\delta \subset \mathcal{U} \cup B_{\mathcal{M}}(\alpha([0,1]),\delta)$.*

We will assume (without loss of generality) that the points 0 and 1 are in P. We will need the following lemma, which gives a simple condition for the union of two polynomially convex sets to be polynomially convex (see [19, Lemma 29.21(a), p. 386] for a proof). Here $\hat{X}$ will denote the polynomial hull of a compact set $X \subset \mathbb{C}^N$.

LEMMA 3.13. *Let X_1 and X_2 be compact polynomially convex sets in $\mathbb{C}^n$, and let p be a polynomial such that $\widehat{p(X_1)} \cap \widehat{p(X_2)} \subset \{0\}$. If $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex, then $X_1 \cup X_2$ is polynomially convex.*

We will also require the following.

OBSERVATION 3.14. *Let $\gamma : [0,1] \to \mathcal{M}$ be an arc, and let $0 < s < t < 1$. Suppose that* (a) *in a neighborhood of $\gamma([0,t])$ is defined a strictly plurisubharmonic function $\nu \geq 0$ that vanishes precisely on the arc γ and* (b) *in a neighborhood of $\gamma([s,1])$ is defined a continuous plurisubharmonic $\mu \geq 0$ that also vanishes precisely on γ. Further suppose that, where both μ and ν are defined, $\mu < \nu$. Then there is a continuous plurisubharmonic λ in a neighborhood of $\gamma([0,1])$ such that λ coincides with μ near $\gamma([0,s])$, coincides with ν near $\gamma([t,1])$, and is bounded above by ν near $\gamma([s,t])$.*

Proof. Let $\psi \leq 0$ be a function of small $\mathcal{C}^2$ norm such that:

- $\nu + \psi$ is still plurisubharmonic; and
- ψ is 0 except in a small neighborhood of $\gamma(t)$, where it is negative.

We set

$$\lambda = \begin{cases} \nu & \text{near } \gamma([1,s]), \\ \max(\mu, \nu+\psi) & \text{near } \gamma([s,t]), \\ \mu & \text{near } \gamma([t,1]). \end{cases}$$

This will be plurisubharmonic provided the definition makes sense. Now, near $\gamma(s)$ we have $\lambda = \nu$ because $\mu < \nu$, so λ is continuous in a neighborhood of $\gamma([1,t])$. Near $\gamma(t)$ we have $\nu + \psi < 0$; as a result, $\lambda = \mu$ there and so λ defines a continuous function in a neighborhood of $\alpha([0,1])$. □

Next we prove Proposition 3.12. Let $p \in P$. In a neighborhood of $q = \alpha(p)$ we can find a system of coordinates $(z_1,\ldots,z_n)$ such that the last coordinate z_n is equal to ϕ. Now consider a polydisc W of the type

$$W = \{(z_1, \dots, z_n) : |z_j| < R \text{ for } j = 1, \dots, n-1;\ |z_n| < r\},$$

where r and R are so chosen that:

- r is much smaller than R;
- $W \Subset V_q$, where V_q is a polydisc centered at q such that $V_q \Subset \mathcal{U}$; and
- the arc α enters and exits $\bar{W}$ exactly once transversally through the part of the boundary ∂W given by

$$\{(z_1, \dots, z_n) : |z_j| < R \text{ for } j = 1, \dots, n-1;\ |z_n| = r\}.$$

That such r and R exist follows easily from the fact that $\phi \circ \alpha = z_n \circ \alpha$ is smooth.

Consider the compact set $K := \bar{W} \cup (\alpha([0,1]) \cap \overline{V_q})$. This is the union of the polydisc $\bar{W}$ with two "whiskers" (the two components of $\alpha([0,1]) \cap (\overline{V_q} \setminus W)$). The projection of K on the last coordinate is of the form $\overline{B(0,r)} \cup \phi(\alpha(I))$ for a subinterval I of $[0,1]$. Given our choice of r and R, this set is the disc $\overline{B(0,r)}$ attached with two arcs, each at exactly one point. Two applications of Lemma 3.13 (with $p = z_n$) shows that K is polynomially convex. By applying Observation 3.10 to the polynomially convex K and the continuous function $\operatorname{dist}(\cdot, K)^2$, we obtain a continuous plurisubharmonic function $\mu \geq 0$ in a neighborhood of $\alpha(P)$ such that $\mu(z) \leq \operatorname{dist}_{\mathcal{M}}(z, \alpha([0,1])^2$. Moreover, $\mu = 0$ exactly on a disjoint union of "soup can with whiskers" neighborhoods of the points of $\alpha(P)$. We take ν to be the square of the distance to $\alpha([0,1])$. Then, applying Observation 3.14 to the plurisubharmonic μ and strictly plurisubharmonic ν (twice for each point of $\alpha(P)$), we can obtain a plurisubharmonic λ such that λ vanishes on the arc and such that, away from $\alpha(P)$, $\lambda = \nu$ (the square of the distance).

We now define $\Omega_\delta = \{Q \in \mathcal{M} : \lambda(Q) < \delta^2\}$. It is easily verified that Ω_δ has the two geometric properties required; that is, it contains the δ-neighborhood of $\alpha([0,1])$ and is actually the δ-neighborhood away from $\alpha(P)$. To see that it is Stein we note that, for small δ, the set Ω_δ has the strictly plurisubharmonic exhaustion $(\delta^2 - \lambda)^{-1} + \rho$, where ρ is a strictly plurisubharmonic function in a neighborhood of $\alpha([0,1])$ (see Observation 3.9).

3.5. Mildly Singular Arcs, Step 2: Gluing of Immersions

We now prove the induction step used to glue locally defined coordinate maps and so obtain a coordinate map in a neighborhood of a mildly singular arc.

Let K_1 and K_2 be compact subsets of $\mathcal{M}$. Suppose we are given immersions Φ and Ψ from neighborhoods of K_1 and K_2 (respectively) into $\mathbb{C}^n$. The main question considered in this step is whether there is an immersion from a neighborhood of the union $K = K_1 \cup K_2$ into $\mathbb{C}^n$. In the application, $K_1 \cup K_2$ will be a mildly singular arc without any singular points in $K_1 \cap K_2$.

3.5.1. Hypotheses on the Sets K_1, K_2 and the Maps Φ, Ψ

Of course, concluding that the immersions can be glued requires additional hypotheses, which should correspond to the intended application. The ones that we shall use are described as follows.

1. *Intersection is a smooth arc.* The basic hypothesis is that the intersection $K_1 \cap K_2$ should be a smooth arc. More precisely, there is a $\mathcal{C}^3$ arc $\alpha\colon [0,1] \to \mathcal{M}$ such that $K_1 \cap K_2$ is its image.

2. *Already glued in one coordinate ("Special").* We assume that the two immersions have already been glued in one coordinate. More precisely: Suppose that Φ and Ψ are the immersions from neighborhoods of K_1 and K_2 into $\mathbb{C}^n$; then we assume that the last coordinates Φ_n and Ψ_n are equal in a neighborhood of the arc $K_1 \cap K_2$.

We denote by ϕ the map from a neighborhood of K into $\mathbb{C}$ that is equal to Φ_n near K_1 and to Ψ_n near K_2. In order to simplify the exposition, we shall denote as *special* a map whose last coordinate is ϕ. Hence, the hypothesis is that Φ and Ψ are special. We will insist that, while modifying Φ and Ψ so that they become glued, the last coordinate is always ϕ (i.e., that they remain special).

3. *Good submersion.* We will assume that the map ϕ just described is a good submersion associated with the arc α. Furthermore, the sets $\phi(K_2 \setminus K_1)$ and $\phi(K_1 \setminus K_2)$ are disjoint.

4. *Ghost of (3.12).* This hypothesis will be required in the last step of the proof. We assume that, for small $\delta > 0$ and for a given relatively compact neighborhood $\mathcal{U}$ of $(K_1 \setminus K_2) \cup (K_2 \setminus K_1)$, there is a Stein open neighborhood Ω_δ of K that has the following properties.

- Ω_δ contains the δ-neighborhood of K.
- Near the arc $K_1 \cap K_2$, the set Ω_δ is in fact the δ-neighborhood. More precisely, $\Omega_\delta \subset \mathcal{U} \cup B_{\mathcal{M}}(K_1 \cap K_2, \delta)$. Consequently, there is a fixed compact H independent of δ such that each $\Omega_\delta \subset H$.

With these hypotheses we state the following proposition, whose proof will be given in Sections 3.5.2 and 3.5.4.

PROPOSITION 3.15. *There is a special immersion Ξ from a neighborhood of K into $\mathbb{C}^n$.*

3.5.2. First Step in Proof of Proposition 3.15: Approximate Gluing of Special Immersions

We fix the map Φ and, for each small $\delta > 0$, we modify the special immersion Ψ to a new special immersion Ψ_δ such that, near the arc $K_1 \cap K_2$, the difference $\Psi_\delta - \Phi$ is small. Once this "approximate solution" is obtained, in Section 3.5.4 we solve a standard Cousin problem to modify both Φ and Ψ_δ so that they now match on the intersection, and the result is an immersion.

We state the goal of this section as a proposition.

PROPOSITION 3.16. *After possibly shrinking the sets K_1 and K_2 (in such a way that their union $K_1 \cup K_2$ is always K), we can find a constant $C > 0$ such that, for each $\delta > 0$ small, there is a special immersion Ψ_δ into $\mathbb{C}^n$ defined in a neighborhood of K_2 that contains the δ-neighborhood of $K_1 \cap K_2$ and such that $\|(\Psi'_\delta)^{-1}\|_{\mathrm{op}} < C$ and, on $B_{\mathcal{M}}(K_1 \cap K_2, \delta)$, we have $\|\Phi - \Psi_\delta\| = O(\delta^2)$.*

For convenience, we divide the proof into several steps.

Step 1. In this step we shrink K_1 and K_2 and modify Ψ to $\tilde{\Psi}$ in such a way that the derivatives of the immersions Φ and $\tilde{\Psi}$ match at one point. This leads to a new transition function $\tilde{\chi}$ with nicer properties.

By hypothesis 3 of Section 3.5.1, $\phi \circ \alpha$ is a smooth embedded arc in $\mathbb{C}$ (where ϕ is the common last coordinate of Φ and Ψ). It follows that there is a neighborhood W of $K_1 \cap K_2 = \alpha([0,1])$ on which both Ψ and Φ are injective and hence are biholomorphisms onto the image of W.

After translating in $\mathbb{C}^n$, we can assume that $\Phi(\alpha(\frac{1}{2})) = 0$ and $\Psi(\alpha(\frac{1}{2})) = 0$. The transition map $\chi := \Phi \circ \Psi^{-1}$ from the open set $\Psi(W) \subset \mathbb{C}^n$ onto $\Phi(W) \subset \mathbb{C}^n$ is biholomorphic. Since each of Φ and Ψ is special (i.e., each has last coordinate equal to ϕ), it follows that χ has the form $\chi(Z,w) = (\xi(Z,w), w)$ with $w \in \mathbb{C}$ and $Z, \xi(Z,w) \in \mathbb{C}^{n-1}$. Moreover, $0 \in W \cap \chi(W)$; in fact, $\chi(0) = 0$.

Set $A = \chi'(0) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, and define $\tilde{\Psi} := A \circ \Psi$. Then $\tilde{\Psi}$ is again a special immersion from a neighborhood of K_2 into $\mathbb{C}^n$, and its restriction to W is a biholomorphic map onto the image. We can also define a new transition function $\tilde{\chi} := \Phi \circ \tilde{\Psi}^{-1} = \chi \circ A^{-1}$, which is a biholomorphism from $\tilde{\Psi}(W) \subset \mathbb{C}^n$ onto $\Phi(W) \subset \mathbb{C}^n$. This new $\tilde{\chi}$ has the same form as χ:

$$\tilde{\chi}(Z,w) = (\tilde{\xi}(Z,w), w), \tag{8}$$

where again $w \in \mathbb{C}$ and both Z and $\tilde{\xi}(Z,w)$ are in $\mathbb{C}^{n-1}$. The additional feature (not present before) is that $\tilde{\chi}'(0) = \mathbb{I}$.

The derivative of $\tilde{\chi}$ is given by

$$\tilde{\chi}' = \begin{pmatrix} \tilde{\xi}_Z & \tilde{\xi}_w \\ 0 & 1 \end{pmatrix},$$

where $\tilde{\xi}_Z \in \mathrm{GL}_{n-1}(\mathbb{C})$ and $\tilde{\xi}_w$ is a vector of $n-1$ components (here subscripts denote differentiation).

Because $\tilde{\chi}'(0) = \mathbb{I}$, we can shrink the compact sets K_1 and K_2 (while not changing their union K), and the neighborhood W of $K_1 \cap K_2$, so that $\tilde{\chi}' \approx \mathbb{I}$ on $\tilde{\Psi}(W)$ in the sense that there is a holomorphic $v\colon \tilde{\Psi}(W) \to \mathrm{Mat}_{(n-1)\times(n-1)}(\mathbb{C})$ such that, on $\tilde{\Psi}(W)$,

$$\tilde{\xi}_Z = \exp \circ v. \tag{9}$$

Step 2. In this step we obtain, for $\delta > 0$ small, an approximation of the map $\tilde{\xi}$ of equation (8) by a map $\hat{\xi}_\delta$ that is affine in every coordinate except the last one and such that $\|\hat{\xi}_\delta - \tilde{\xi}\| = O(\delta^2)$ near the arc α.

Let $\lambda = \tilde{\Psi} \circ \alpha$. Then $\lambda\colon [0,1] \to \tilde{\Psi}(W) \subset \mathbb{C}^n$ is a $\mathcal{C}^3$ arc. The last coordinate λ_n is the embedded $\mathcal{C}^3$ arc $\phi \circ \alpha$; denote its image $\lambda_n([0,1])$ by Γ. Since λ_n is an embedding, we can define a $\mathcal{C}^3$ map $\gamma\colon \Gamma \to \mathbb{C}^{n-1}$ by setting

$$\gamma(\lambda_n(t)) := (\lambda_1(t), \ldots, \lambda_{n-1}(t)).$$

Now apply Lemma 3.5 to γ. Hence, for $\delta > 0$ small, we can find a holomorphic $\gamma_\delta\colon B_{\mathbb{C}}(\Gamma,\delta) \to \mathbb{C}^{n-1}$ from the δ-neighborhood $B_{\mathbb{C}}(\Gamma,\delta)$ of $\Gamma \in \mathbb{C}$ into $\mathbb{C}^{n-1}$

such that γ_δ is bounded in the $\mathcal{C}^2$ norm on $B_{\mathbb{C}}(\Gamma,\delta)$ and, on Γ, we have $\|\gamma_\delta - \gamma\| = O(\delta^{5/2})$. For small $\delta > 0$ we now define a $\mathbb{C}^{n-1}$-valued holomorphic map $\hat{\xi}_\delta$ on the open set $\mathbb{C}^{n-1} \times B_{\mathbb{C}}(\Gamma,\delta) \subset \mathbb{C}^n$ by putting

$$\hat{\xi}_\delta(Z,w) := \tilde{\xi}(\gamma_\delta(w),w) + \tilde{\xi}_Z(\gamma_\delta(w),w)(Z-\gamma_\delta(w)),$$

where $\hat{\xi}_\delta$ is the first-order Taylor polynomial of the $\mathbb{C}^{n-1}$-valued map $\tilde{\xi}(\cdot,w)$ of $n-1$ variables around the point $\gamma_\delta(w) \in \mathbb{C}^{n-1}$.

We can rewrite $\hat{\xi}_\delta$ as

$$\hat{\xi}_\delta(Z,w) = f_\delta(w) + \exp g_\delta(w) Z, \tag{10}$$

where f_δ, g_δ are holomorphic maps defined on $B_{\mathbb{C}}(\Gamma,\delta)$. The $\mathrm{Mat}_{(n-1)\times(n-1)}(\mathbb{C})$-valued map g_δ is given by

$$g_\delta(w) := v(\gamma_\delta(w),w), \tag{11}$$

where v is as in equation (9) (i.e., $\tilde{\xi}_Z = \exp \circ v$). The $\mathbb{C}^{n-1}$-valued map f_δ is defined by

$$f_\delta(w) := \xi(\gamma_\delta(w),w) - (\exp g_\delta(w))\gamma_\delta(w). \tag{12}$$

We now observe the following two facts, which will be of use later.

1. *f_δ and g_δ are bounded in $\mathcal{C}^2$.* On $B_{\mathbb{C}}(\Gamma,\delta)$ the map γ_δ is bounded in the $\mathcal{C}^2$ norm independently of δ, so the same will be true of the functions f_δ and g_δ. In other words, there is a constant $C > 0$ independent of δ and such that, for $j = 0,1,2$, we have $\|f_\delta^{(j)}\| < C$ and $\|g_\delta^{(j)}\| < C$.
2. *$\hat{\xi}_\delta - \tilde{\xi}$ is small.* More precisely, suppose that the point $\mathbf{Z} = (Z,w)$ is in the δ-neighborhood $B_{\mathbb{C}^n}(\lambda([0,1]),\delta)$ of the arc $\lambda([0,1]) = \Psi(K_1 \cap K_2)$ in $\mathbb{C}^n$. Then
$$\|\hat{\xi}_\delta(\mathbf{Z}) - \tilde{\xi}(\mathbf{Z})\| = O(\delta^2). \tag{13}$$

To see this, note that $(Z,w) \in B_{\mathbb{C}^n}(\lambda([0,1]),\delta)$ means that there is a $t \in \Gamma$ such that $\|Z - \gamma(t)\| < \delta$ and $|w - t| < \delta$. Therefore, using the properties of γ_δ, we have

$$\begin{aligned}\|Z-\gamma_\delta(w)\| &\le \|Z-\gamma(t)\| + \|\gamma(t)-\gamma_\delta(t)\| + \|\gamma_\delta'\|_{\sup}|t-w| \\ &\le \delta + O(\delta^{5/2}) + C\delta = O(\delta).\end{aligned}$$

Now applying Taylor's theorem to the first-order Taylor polynomial $\hat{\xi}_\delta(\cdot,w)$ of $\tilde{\xi}(\cdot,w)$ around the point $\gamma_\delta(w)$, we see that

$$\|\hat{\xi}_\delta(Z,w) - \tilde{\xi}(Z,w)\| = O(\|Z-\gamma_\delta(w)\|^2) = O(\delta^2).$$

Step 3. We now construct an approximation χ_δ of $\tilde{\chi}$. At this point we must use a lemma regarding the approximation of functions of one variable. In order not to interrupt the flow of the proof we state it here but postpone its proof to Section 3.5.3.

LEMMA 3.17. *Let B_1, B_2, B_3 be compact subsets of $\mathbb{C}$ such that $B_1 \cap B_3 = \emptyset$ and $B_1 \cap B_2$ is a single point, which we call z_0. Let $0 < \theta < 1$ and let p be a positive* *integer. Then, if L is a closed subset of B_1 such that $L \cap B_2 = \emptyset$ (i.e., if $z_0 \notin L$), then there is a constant C with the following property. For $\delta > 0$ small, if f is a holomorphic function in the closed δ-neighborhood $\overline{B_{\mathbb{C}}(B_1 \cup B_2, \delta)}$ of $B_1 \cup B_2$ such that*

$$\|f\|_{\mathcal{C}^{1,\theta}(\overline{B_{\mathbb{C}}(B_1 \cup B_2, \delta)})} \leq 1,$$

then there exists a holomorphic f_δ defined in the δ-neighborhood of $B := B_1 \cup B_2 \cup B_3$ such that

$$\|f_\delta\|_{\mathcal{C}^1(B_{\mathbb{C}}(B,\delta))} \leq C$$

and on $B_{\mathbb{C}}(L, \delta)$ we have

$$|f - f_\delta| < C\delta^p.$$

So that we may apply Lemma 3.17 to our situation, we let $p = 2$, $B_1 = \lambda_n([0, \frac{3}{4}])$, $L = \lambda_n([0, \frac{1}{2}]) \subset B_1$, and $B_2 = \lambda_n([\frac{3}{4}, 1])$. Then, as required, we have that $B_1 \cap B_2$ is a single point $z_0 = \lambda_n(\frac{3}{4})$ and $z_0 \notin L$. Also, we have $B_1 \cup B_2 = \lambda_n([0, 1]) = \Gamma$. For B_3 we take a relatively compact neighborhood of $\phi(K_2 \setminus K_1)$ such that $B_1 \cap B_3 = \emptyset$. Then $B = B_1 \cup B_2 \cup B_3 \supset \phi(K_2)$.

Now the holomorphic functions f_δ and g_δ defined in (12) and (11) are holomorphic in the closed δ-neighborhood of $\Gamma = B_1 \cup B_2$. Moreover, thanks to the $\mathcal{C}^2$ boundedness of the maps, for any θ with $0 < \theta < 1$ we actually have $\|f_\delta\|_{\mathcal{C}^{1,\theta}(\overline{B_{\mathbb{C}}(B_1 \cup B_2, \delta)})}$ and $\|g_\delta\|_{\mathcal{C}^{1,\theta}(\overline{B_{\mathbb{C}}(B_1 \cup B_2, \delta)})}$ bounded independently of δ.

Therefore, an application of Lemma 3.17 yields holomorphic maps F_δ and G_δ, defined on the δ-neighborhood A_δ of $B = \Gamma \cup B_3$, such that $\{F_\delta\}$ and $\{G_\delta\}$ are uniformly bounded in the $\mathcal{C}^2$ norm independent of δ and, on the set $B_{\mathbb{C}}(\lambda_n([0, \frac{1}{2}]), \delta)$, we have $\|F_\delta - f_\delta\| = O(\delta^2)$ and $\|G_\delta - g_\delta\| = O(\delta^2)$.

We shrink the sets K_1 and K_2 again so that $K_1 \cap K_2 = \alpha([0, \frac{1}{2}])$. Now define the map ξ_δ from $\mathbb{C}^{n-1} \times A_\delta$ into $\mathbb{C}^{n-1}$ by

$$\xi_\delta(Z, w) := F_\delta(w) + (\exp G_\delta(w))Z, \tag{14}$$

and let

$$\chi_\delta(Z, w) := (\xi_\delta(Z, w), w). \tag{15}$$

Observe the following properties of the maps χ_δ and ξ_δ.

- $\xi_\delta(Z, w)$ (and consequently $\chi_\delta(Z, w)$) is defined on the set $\mathbb{C}^{n-1} \times A_\delta$, which contains a neighborhood of $\tilde{\Psi}(K_2)$ in $\mathbb{C}^n$.
- χ_δ is a biholomorphic automorphism of the set $\mathbb{C}^{n-1} \times A_\delta$, and there is a constant $C > 0$ independent of δ such that

$$\|(\chi_\delta^{-1})'\|_{\text{op}} < C. \tag{16}$$

The derivative of χ_δ is of the form

$$\chi'_\delta(Z, w) = \begin{pmatrix} \exp G_\delta(w) & * \\ 0 & 1 \end{pmatrix},$$

which shows that χ'_δ is a local biholomorphism. Now suppose that $Z' \in \mathbb{C}^{n-1}$ and $w' \in A_\delta$. Then we can solve the equation $\chi_\delta(Z, w) = (Z', w')$ explicitly to obtain the representations $Z = \exp(-G_\delta(w'))(Z' - F_\delta(w'))$ and $w = w'$. Hence χ_δ is a biholomorphism, and its inverse is given by

$$\chi_\delta^{-1}(Z, w) = (\exp(-G_\delta(w))(Z - F_\delta(w)), w).$$

By construction, each of F_δ and G_δ is bounded in the $\mathcal{C}^1$ norm independently of δ. The bound (16) follows immediately.

- In the δ neighborhood $B_{\mathbb{C}}(\tilde{\Psi}(K_1 \cap K_2), \delta)$ of $\tilde{\Psi}(K_1 \cap K_2)$, we have

$$\|\tilde{\chi} - \chi_\delta\| = O(\delta^2).$$

This follows immediately from the fact that the coefficients F_δ and $\exp G_\delta$ of ξ_δ are $O(\delta^2)$ perturbations of the coefficients f_δ and $\exp g_\delta$ of $\hat{\xi}_\delta$.

Step 4 (End of the proof of Proposition 3.16). We now define the promised map

$$\Psi_\delta := \chi_\delta \circ \tilde{\Psi}.$$

We make three remarks as follows.

1. By construction, χ_δ is a biholomorphic map from the set $\mathbb{C}^{n-1} \times A_\delta$ into itself. Recall that $A_\delta = B_{\mathbb{C}}(\phi(K_1 \cap K_2), \delta) \cup L$, where L is a fixed compact neighborhood of $\phi(K_2 \setminus K_1) \subset \mathbb{C}$ such that $L \cap \phi(K_1 \cap K_2)$ is a single point. It easily follows that Ψ_δ is defined on a set of the form $B_{\mathcal{M}}(K_1 \cap K_2, \delta) \cup \mathcal{U}$, where $\mathcal{U}$ is a fixed (independent of δ) neighborhood of $K_2 \setminus K_1$.

2. Ψ_δ^{-1} and $\tilde{\Psi}^{-1}$ exist locally and satisfy the equation $\Psi_\delta^{-1} = \tilde{\Psi}^{-1} \circ \chi_\delta^{-1}$. Using the chain rule and the fact (already proved) that $\|(\chi_\delta^{-1})'\|_{\text{op}} \leq C$ for C independent of δ, we see that there is a constant C' independent of δ such that $\|(\Psi_\delta^{-1})'\|_{\text{op}} < C'$. Another application of the chain rule yields $\|(\Psi_\delta')^{-1}\|_{\text{op}} < C'$.

3. On the δ-neighborhood $B_{\mathcal{M}}(K_1 \cap K_2, \delta)$ of $K_1 \cap K_2$, we have $\|\Phi - \Psi_\delta\| = O(\delta^2)$.

This completes the proof of Proposition 3.16, except we still need to prove Lemma 3.17. □

3.5.3. Proof of Lemma 3.17

We let C stand for any constant not depending on δ. The proof will be in two similar steps that each involve the solution of a $\bar{\partial}$ problem in one variable.

Step 1. The aim of the first step is to construct a holomorphic function g_δ on the δ-neighborhood of $B = B_1 \cup B_2 \cup B_3$ with the following properties.

- The $\mathcal{C}^1$ norm of g_δ is bounded independently of δ; that is, $\|g_\delta\|_{\mathcal{C}^1(B_{\mathbb{C}}(B,\delta))} \leq C$, where C is independent of δ.
- By hypothesis, the intersection $B_1 \cap B_2$ is a single point z_0. Denote henceforth the disc $B_{\mathbb{C}}(z_0, \delta)$ by N_δ and the holomorphic function $f - g_\delta$ (defined on a neighborhood of $B_1 \cup B_2$) by k_δ. Then $\|k_\delta\|_{\mathcal{C}^{1,\theta}(N_\delta)} = O(\delta^{p+2})$.

To construct g_δ, let ψ be a $\mathcal{C}_c^\infty$ cutoff on $\mathbb{C}$ that is 1 on a neighborhood of B_1 (and hence in a neighborhood of the point z_0) and 0 on a neighborhood of B_3. Now set

$$\lambda_\delta(z) := \frac{1}{(z-z_0)^{p+4}} \frac{\partial \psi}{\partial \bar{z}} f(z).$$

Observe that λ_δ is defined on the closed δ-neighborhood $\overline{B_{\mathbb{C}}(B_1 \cup B_2)}$ of $B_1 \cup B_2$ (and this justifies the subscript δ on λ). After extending by 0 at points where $\psi = 0$, we can assume that λ_δ is defined and smooth on $B_{\mathbb{C}}(B, \delta)$. Moreover, since f is bounded by 1 in the $\mathcal{C}^{1,\theta}$ norm, it follows that there is a constant C (independent of δ) such that $\|\lambda_\delta\|_{\mathcal{C}^{1,\theta}(\overline{B_{\mathbb{C}}(B,\delta)})} < C$. By Lemma 2.1, there is a compactly supported extension $\tilde{\lambda}_\delta$ of λ_δ to $\mathbb{C}$ such that $\|\tilde{\lambda}_\delta\|_{\mathcal{C}^{1,\theta}(\mathbb{C})} \leq C$ (with C independent of δ).

We now define $g_\delta := \psi f + (z-z_0)^{p+4}(-\frac{1}{\pi z} * \tilde{\lambda}_\delta)$, where ψf is assumed to be 0 when $\psi = 0$. Then:

- g_δ is holomorphic in the δ-neighborhood of B;
- since $\tilde{\lambda}_\delta$ is bounded in the $\mathcal{C}^{1,\theta}$ norm, it follows that $\|g_\delta\|_{\mathcal{C}^{1,\theta}(B_{\mathbb{C}}(B,\delta))} \leq C$, where C is independent of δ; and
- for small enough t we have $k_\delta(z_0 + t) = t^{p+4}(-\frac{1}{\pi z} * \lambda)$.

The first factor t^{p+4} is $O(\delta^{p+2})$ in the $\mathcal{C}^2$ norm on the disc N_δ. The second term is bounded on this disc in the $\mathcal{C}^{1,\theta}$ norm. Therefore, we easily have

$$\|k_\delta\|_{\mathcal{C}^{1,\theta}(N_\delta)} = O(\delta^{p+2}) \tag{17}$$

and so the function g_δ and $k_\delta = f - g_\delta$ satisfy the conditions stated in the beginning of this step.

Step 2. Now we write $f = k_\delta + g_\delta$. Observe that g_δ is already defined in the δ-neighborhood of $B = B_1 \cup B_2 \cup B_3$. To approximate f we will approximate $k_\delta = f - g_\delta$ (which is holomorphic on the δ-neighborhood of $B_1 \cup B_2$) by a holomorphic function h_δ defined in the δ-neighborhood of B. Set $f_\delta = h_\delta + g_\delta$. Then, for f_δ to satisfy the conclusion of Lemma 3.17, it is sufficient that:

- $\|h_\delta\|_{\mathcal{C}^1(B_{\mathbb{C}}(B,\delta))}$ be bounded independently of δ; and
- on the set $B_{\mathbb{C}}(L, \delta)$ (where, as in the hypothesis, L is a fixed subset of B_1 not containing the point z_0), $|h_\delta - k_\delta| = O(\delta^p)$.

To construct h_δ we proceed as follows. For $\delta > 0$ small, there exists a $\mathcal{C}^\infty$ function α_δ defined on the closed δ-neighborhood of B such that:

- $0 \leq \alpha_\delta \leq 1$, with $\alpha_\delta \equiv 1$ on the neighborhood of the set L and $\alpha_\delta \equiv 0$ on a fixed neighborhood of B_3; and
- $\nabla \alpha_\delta$ is supported in $N_\delta = B_{\mathbb{C}}(z_0, \delta)$ and, moreover, α_δ satisfies

$$\|\alpha_\delta\|_{\mathcal{C}^2(N_\delta)} = O\left(\frac{1}{\delta^2}\right). \tag{18}$$

Now define a smooth function μ_δ on $\overline{B_{\mathbb{C}}(B, \delta)}$ by setting $\mu_\delta := \partial\alpha_\delta / \partial\bar{z} \cdot k_\delta$ and then extending by 0 outside N_δ. By equations (18) and (17), it follows that $\|\mu_\delta\|_{\mathcal{C}^{1,\theta}(B_{\mathbb{C}}(B,\delta))} = O(\delta^p)$. An application of Lemma 2.1 leads to the construction of a compactly supported extension $\tilde{\mu}_\delta$ of μ_δ to the whole of $\mathbb{C}$ such that

$$\|\tilde{\mu}_\delta\|_{\mathcal{C}^{1,\theta}(\mathbb{C})} = O(\delta^p).$$

We can assume that the supports of the $\tilde{\mu}_\delta$ lie in a fixed compact of $\mathbb{C}$ (independently of δ).

Define a holomorphic h_δ on the δ-neighborhood $B_{\mathbb{C}}(B,\delta)$ by setting

$$h_\delta := \alpha_\delta \cdot k_\delta + \left(-\frac{1}{\pi z} * \tilde{\mu}_\delta\right),$$

where $\alpha_\delta \cdot k_\delta$ is understood to be 0 if $\alpha_\delta = 0$ (even if k_δ is not defined). It is clear that $\|h_\delta\|_{\mathcal{C}^1(B_{\mathbb{C}}(B,\delta))} \leq C$. Now consider the δ-neighborhood $B_{\mathbb{C}}(L,\delta)$ of the set L. For small δ, we have $\alpha_\delta \equiv 1$ on this set. Then, on $B(L,\delta)$,

$$|h_\delta - k_\delta| = \left|\frac{1}{\pi z} * \tilde{\mu}_\delta\right| = O(\delta^p).$$

This concludes the proof of Lemma 3.17.

3.5.4. Last Step in Proof of Proposition 3.15: A Cousin Problem on a Neighborhood of $K_1 \cup K_2$

LEMMA 3.18. *For $\delta > 0$ small, there exist holomorphic maps H_1^δ and H_2^δ from neighborhoods of K_1 and K_2 (respectively) into $\mathbb{C}^n$ such that*

- *$\Phi + H_1^\delta = \Psi_\delta + H_2^\delta$ in a neighborhood of $K_1 \cap K_2$;*
- *$\|(H_j^\delta)'\|_{\text{op}} = O(\delta^{1/2})$ for $j = 1,2$; and*
- *the last coordinates of H_1^δ and H_2^δ are both 0.*

We note that this completes the proof of Proposition 3.15. Consider the map Ξ defined in a neighborhood of $K = K_1 \cup K_2$ by

$$\Xi = \begin{cases} \Phi + H_1^\delta & \text{near } K_1, \\ \Psi_\delta + H_2^\delta & \text{near } K_2. \end{cases}$$

This is a well-defined special holomorphic map. To show that it is an immersion, we need only show that the derivative $\Xi'(Z)$ is an isomorphism of vector spaces for Z near K. Near K_2 we have $\Xi' = \Psi_\delta'(\mathbb{I} + (\Psi_\delta')^{-1} \circ (H_2^\delta)')$; but $\|(\Psi_\delta')^{-1}\|_{\text{op}} \leq C$ (with C independent of δ) and $\|(H_2^\delta)'\|_{\text{op}} = O(\delta^{1/2})$, so for small δ the linear operator Ξ' is an isomorphism. The same conclusion holds in a neighborhood of K_1 for $\Xi = \Phi + H_1^\delta$.

The proof of Lemma 3.18 will require the following two lemmas.

LEMMA 3.19. *Let $\mathcal{M}$ be Stein and let $\Omega \Subset \mathcal{M}$ be open. Then there is a constant C with the following property. Let $U \subset \Omega$, where U is a Stein open subset of $\mathcal{M}$, and let g be a smooth $\bar\partial$-closed $(0,1)$-form on U that is in $L^2_{(0,1)}(U)$. Then there is a smooth $u\colon \omega \to \mathbb{C}$ such that $\bar\partial u = g$ and u satisfies the estimate*

$$\|u\|_{L^2(U)} \leq C\|g\|_{L^2_{(0,1)}(U)}. \tag{19}$$

The point of the lemma is that the constant C depends not on the open set U but only on the relatively compact Ω. This is well known in the case when $\mathcal{M} = \mathbb{C}^n$ (see e.g. [10]). The general case may be reduced to the Euclidean case by embedding $\mathcal{M}$ in some $\mathbb{C}^N$ (for details see [3]).

LEMMA 3.20. *Let $\Omega \Subset \mathcal{M}$ be a smoothly bounded domain. Then there is a constant C such that, for any smooth function $w\colon \Omega \to \mathbb{C}$ and any compact $K \subset \Omega$,*

$$\sup_K |w| \le C\left\{\frac{1}{\operatorname{dist}(K,\Omega^c)^n}\|w\|_{L^2(\Omega)} + \operatorname{dist}(K,\Omega^c)\|\bar\partial w\|_{L^\infty(\Omega)}\right\}. \qquad (20)$$

Proof. After using local coordinates and scaling, we can see that it is sufficient to establish the following inequality for smooth functions w defined on the closed unit ball B in $\mathbb{C}^n$:

$$|w(0)| \le K\left\{\|w\|_{L^2(B)} + \sup_B\left(\max_j\left|\frac{\partial w}{\partial \bar z_j}\right|\right)\right\}.$$

This is proved in [1, Lemma 16.7, p. 130]. □

Proof of Proposition 3.18. For notational convenience we suppress δ whenever possible, writing $\Phi = (\hat\Phi,\phi)$, $\Psi_\delta = (\hat\Psi_\delta,\phi)$, and $H_j^\delta = (h_j,0)$, where $\hat\Phi, \hat\Psi_\delta, h_j$ are all $\mathbb{C}^{n-1}$-valued. Then the result is equivalent to solving the $\mathbb{C}^{n-1}$-valued additive Cousin problem

$$h_1 - h_2 = \hat\Psi_\delta - \hat\Phi := R_\delta$$

with bounds on h_1 and h_2. We do this using the well-known standard method. It is clear that $(K_1 \setminus K_2) \cap (K_2 \setminus K_1) = \emptyset$, so we can find a smooth cutoff μ in a neighborhood of $K_1 \cup K_2$ such that μ is 1 in a neighborhood of $K_1 \setminus K_2$ and 0 in a neighborhood of $K_2 \setminus K_1$. We obtain a smooth solution to the Cousin problem given by

$$\begin{aligned} \tilde h_1 &= \mu R_\delta && \text{extended by 0 where } \mu = 0,\\ \tilde h_2 &= (\mu - 1)R_\delta && \text{extended by 0 where } \mu = 1. \end{aligned}$$

Observe that $\|\tilde h_j\| = O(\delta^2)$ for $j = 1,2$.

We want a "correction" u, defined in a neighborhood of $K_1 \cup K_2$, such that $h_j = \tilde h_j + u$ will be holomorphic; that is, we want to solve the equations $\bar\partial(\tilde h_j + u) = 0$. Both of these equations are equivalent to the $\bar\partial$ equation in a neighborhood of $K_1 \cup K_2$ given by

$$\bar\partial u = g, \qquad (21)$$

where g is a $\mathbb{C}^{n-1}$-valued $(0,1)$ form (to be defined shortly) on a Stein neighborhood Ω_δ of $K_1 \cup K_2$ of the type whose existence was assumed in the hypotheses—that is, Ω_δ contains a δ-neighborhood of $K_1 \cup K_2$ and *is* the δ-neighborhood near $K_1 \cap K_2$. The smooth form g is defined by $g := -R_\delta\bar\partial\mu$ on the δ-neighborhood of $K_1 \cap K_2$, and it is extended by 0 to Ω_δ. We thus have

$$\begin{aligned} \|g\|_{L^2_{(0,1)}(\Omega_\delta)} &\le \|g\|_{L^\infty}(\operatorname{Vol}(\operatorname{support} g))^{1/2}\\ &= O(\delta^2)(O(\delta^{2n-1}))^{1/2} = O(\delta^{n+3/2}). \end{aligned}$$

Lemma 3.19 allows us to obtain a function u on Ω_δ such that, for some C independent of δ,

$$\|u\|_{L^2(\Omega_\delta)} \le C\|g\|_{L^2_{(0,1)}(\Omega_\delta)} \le C\delta^{n+3/2}.$$

For convenience, use K_δ to denote the δ-neighborhood of $K = K_1 \cup K_2$. Then $K_\delta \subset \Omega_\delta$ by hypothesis, and it follows that $\operatorname{dist}(K_{\delta/2}, \Omega_\delta^c) \geq \delta/2$. We now apply inequality (20) to u, concluding that

$$\begin{aligned} \|u\|_{L^\infty(K_{\delta/2})} &\leq C\left\{\frac{1}{\operatorname{dist}(K_{\delta/2}, \Omega_\delta^c)^n}\|u\|_{L^2(\Omega_\delta)} + \operatorname{dist}(K_{\delta/2}, \Omega_\delta^c)\|\bar{\partial}u\|_{L^\infty(\Omega_\delta)}\right\} \\ &\leq C\left\{\frac{1}{\delta^n}\cdot\delta^{n+3/2} + \delta\cdot\delta^2\right\} \leq C\delta^{3/2}. \end{aligned}$$

Define $h_j = \tilde{h}_j + u$; then $\|h_j\| \leq C\delta^2 + C\delta^{3/2}$. Therefore, on $K_{\delta/2}$ we have $\|h_j\| = O(\delta^{3/2})$. The required bound on h_j' (and hence on $(H_j^\delta)'$) follows after applying the Cauchy estimate. □

3.6. Mildly Singular Arcs, Step 3: End of Proof of Theorem 2

Let $\alpha\colon [0,1] \to \mathcal{M}$ be a mildly singular arc, and let ϕ be the associated good submersion into $\mathbb{C}$. We begin by covering the compact set $\alpha([0,1])$ by a finite cover of open sets $\{U_i\}_{i=1}^N$ such that

- on each U_i is defined a coordinate map whose last coordinate is ϕ, and
- the parts of the arc α in the intersections $U_i \cap U_{i+1}$ are all smooth.

A simple induction argument applied to this cover shows that it is sufficient to consider the case when $\alpha([0,1])$ is covered by charts U and V such that there exist coordinates $\Phi\colon U \to \mathbb{C}^n$ and $\Psi\colon V \to \mathbb{C}^n$, each having last coordinate ϕ. On the intersection, α is $\mathcal{C}^3$ and $\phi \circ \alpha\colon [0,1] \to \mathbb{C}$ is a $\mathcal{C}^3$ arc. By Theorem 3.12, for small $\delta > 0$ we have Stein neighborhoods Ω_δ exactly of the type required. We thus obtain an immersion Ξ, from a neighborhood of $\alpha([0,1])$ into $\mathbb{C}^n$, whose last coordinate is ϕ. But $\phi \circ \alpha$ is injective and so Ξ is also injective near $\alpha([0,1])$; that is, there exists a neighborhood of the arc on which Ξ is a biholomorphism. □

4. Approximation of Maps into Complex Manifolds

We will continue to denote by $\mathcal{M}$ a complex manifold of complex dimension n that has been endowed with a Riemannian metric (as before, the actual choice of the metric will be irrelevant). In analogy with the notation of Section 2, we introduce the following conventions. For a compact K in $\mathbb{C}$, we let $\mathcal{H}(K, \mathcal{M})$ denote the space of holomorphic maps from K to $\mathcal{M}$. A map $f\colon K \to \mathcal{M}$ is in $\mathcal{H}(K, \mathcal{M})$ iff there is an open set U_f in $\mathbb{C}$ with $K \subset U_f$ and a holomorphic $F\colon U_f \to \mathcal{M}$ such that F restricts to f on K. By $\mathcal{A}^k(K, \mathcal{M})$ we denote the closed subspace of $\mathcal{C}^k(K, \mathcal{M})$ consisting of those maps that are holomorphic in the topological interior $\operatorname{int} K$ of the compact set K. The space $\mathcal{A}^k(K, \mathcal{M})$ will always be considered to have the topology inherited from $\mathcal{C}^k(K, \mathcal{M})$. When $\mathcal{M}$ is the complex plane $\mathbb{C}$, we will abbreviate $\mathcal{H}(K, \mathbb{C})$ and $\mathcal{A}^k(K, \mathbb{C})$ by $\mathcal{H}(K)$ and $\mathcal{A}^k(K)$, respectively.

By a *Jordan domain* Ω in the plane we mean a domain whose boundary $\partial\Omega$ consists of finitely many Jordan curves (homeomorphic images of circles in the plane). A Jordan domain is said to be *circular* if each component of $\partial\Omega$ is a circle in the plane. A $\mathcal{C}^1$ domain is a Jordan domain in which each component of $\partial\Omega$ is a $\mathcal{C}^1$ embedded image of a circle.

We now state the approximation results in our new notation.

THEOREM 3. *Let $\Omega \Subset \mathbb{C}$ be a Jordan domain. Then $\mathcal{H}(\bar{\Omega}, \mathcal{M})$ is dense in $\mathcal{A}^0(\bar{\Omega}, \mathcal{M})$.*

For an analogous result for $\mathcal{C}^k$ maps with $k \geq 1$, we must assume more regularity on the boundary.

THEOREM 4. *Let $\Omega \Subset \mathbb{C}$ be a $\mathcal{C}^1$ domain; that is, Ω is bounded by finitely many $\mathcal{C}^1$ Jordan curves. If $k \geq 1$, then the space $\mathcal{H}(\bar{\Omega}, \mathcal{M})$ is dense in $\mathcal{A}^k(\bar{\Omega}, \mathcal{M})$.*

Before proceeding to prove Theorems 3 and 4, we will show that the boundary regularity required in Theorem 3 can be reduced, so that it is sufficient to consider the case when Ω is a circular domain. In other words, it is sufficient to prove the following.

THEOREM 3′. *For a circular domain W, the subspace $\mathcal{H}(\overline{W}, \mathcal{M})$ is dense in $\mathcal{A}^0(\overline{W}, \mathcal{M})$.*

We will require the following two facts from the theory of conformal mapping:

(i) (Köbe) Let Ω be a Jordan domain. Then there is a circular domain that is conformally equivalent to Ω.

(ii) (Carathéodory) Let Ω_1 and Ω_2 be finitely connected Jordan domains, and let $f : \Omega_1 \to \Omega_2$ be a biholomorphism. Then f extends to a homeomorphism from Ω_1 onto Ω_2.

(See e.g. [20, Thm. IX.35] and [20, Thm. IX.2], respectively; in [20] our (ii) is stated for simply connected domains, but the proof readily extends to the multiply connected case.)

LEMMA 4.1. *Theorem 3′ and Theorem 3 are equivalent.*

Proof. It is clear that Theorem 3 implies Theorem 3′, since a circular domain is also a Jordan domain. For the converse we proceed as follows. Let Ω be a Jordan domain. By our facts (i) and (ii) from the theory of conformal mapping, it follows that there exists a circular domain W such that there is a homeomorphism $\chi : \bar{\Omega} \to \overline{W}$ that maps Ω conformally onto W. Let $f \in \mathcal{A}^0(\bar{\Omega}, \mathcal{M})$. Then $f \circ \chi^{-1}$ is in $\mathcal{A}^0(\overline{W}, \mathcal{M})$, so by hypothesis we can approximate it uniformly by functions $g \in \mathcal{H}(\overline{W}, \mathcal{M})$. Since $\chi \in \mathcal{A}^0(\bar{\Omega}, \mathbb{C})$, it follows (by a version of Mergelyan's theorem; see [8, Thm. 12.2.7]) that χ can be approximated uniformly on $\bar{\Omega}$ by functions $\tilde{\chi} \in \mathcal{H}(\bar{\Omega}, \mathbb{C})$. Therefore, $g \circ \tilde{\chi}$ is a holomorphic map defined in a neighborhood of $\bar{\Omega}$ that approximates f uniformly. This establishes the lemma. □

4.1. Approximation on Good Pairs

This section is devoted to the development of some tools that will be used (along with the results of Section 3) in Section 4.2. We begin with a few definitions.

DEFINITION 4.2. We say that a pair (K_1, K_2) of compact subsets of $\mathbb{C}$ is a *good pair* if the following hold:

- K_1 and K_2 are "well-glued" together in the sense that
$$\overline{K_1 \setminus K_2} \cap \overline{K_2 \setminus K_1} = \emptyset.$$
- $K_1 \cap K_2$ has finitely many connected components, each of which is star shaped.

We now state the basic approximation result that will be used in the proof of Theorems 3 and 4.

THEOREM 5. *Let (K_1, K_2) be a good pair of compact sets, and let V be a compact subset of $\mathbb{C}$ disjoint from K_1 such that the following statement holds: For a fixed $k \geq 0$, given any $g \in \mathcal{A}^k(K_2, \mathbb{C})$ and an $\eta > 0$, there is a $g_\eta \in \mathcal{A}^k(K_2 \cup V, \mathbb{C})$ such that $\|g - g_\eta\|_{\mathcal{C}^k(K_2)} < \eta$.*

Let $f \in \mathcal{A}^k(K_1 \cup K_2, \mathcal{M})$ be such that each of the sets $f(K_j)$, $j = 1, 2$, is contained in a coordinate neighborhood of $\mathcal{M}$. Then, given $\varepsilon > 0$, there is an $f_\varepsilon \in \mathcal{A}^k(K_1 \cup K_2, \mathcal{M})$ such that $\operatorname{dist}_{\mathcal{C}^k(K_1 \cup K_2, \mathcal{M})}(f, f_\varepsilon) < \varepsilon$, and f_ε extends as a holomorphic map to a neighborhood of $(K_2 \cap V)$.

Of course, this is of interest only in the case when $K_2 \cap V \neq \emptyset$. We split the proof into several steps.

OBSERVATION 4.3 (Additive Cousin problem $\mathcal{C}^k$ to the boundary). *Let (K_1, K_2) be a good pair. For each $k \geq 0$, there exist bounded linear maps*
$$T_j \colon \mathcal{A}^k(K_1 \cap K_2, \mathbb{C}) \to \mathcal{A}^k(K_j, \mathbb{C})$$
such that, for any function f in $\mathcal{A}^k(K_1 \cap K_2, \mathbb{C})$, on $K_1 \cap K_2$ we have
$$T_1 f + T_2 f = f. \tag{22}$$

Proof. We reduce the problem to a $\bar{\partial}$ equation in the standard way. Let χ be a smooth cutoff that is 1 near $\overline{K_1 \setminus K_2}$ and 0 near $\overline{K_2 \setminus K_1}$. Let $\lambda := f \cdot \frac{\partial \chi}{\partial \bar{z}}$, so that $\lambda \in \mathcal{A}^k(K_1 \cap K_2, \mathbb{C})$. We can now define (assuming $(1 - \chi) \cdot f = 0$ where $\chi = 1$, even if f is not defined)
$$(T_1 f)(z) = (1 - \chi(z)) \cdot f(z) + \frac{1}{2\pi i} \int_{K_1 \cap K_2} \frac{\lambda(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta$$
and (assuming $\chi \cdot f = 0$ where $\chi = 0$, even if f is not defined)
$$(T_2 f)(z) = \chi(z) \cdot f(z) - \frac{1}{2\pi i} \int_{K_1 \cap K_2} \frac{\lambda(\zeta)}{\zeta - z} \, d\bar{\zeta} \wedge d\zeta. \qquad \square$$

We use Observation 4.3 to prove a version of the Cartan lemma on factoring matrices that is similar to one found in [4, pp. 47–48].

LEMMA 4.4. *Let (K_1, K_2) be a good pair and let $g \in \mathcal{A}^k(K_1 \cap K_2, \mathrm{GL}_n(\mathbb{C}))$, where $k \geq 0$. Then, for $j = 1, 2$, there exist $g_j \in \mathcal{A}^k(K_j, \mathrm{GL}_n(\mathbb{C}))$ such that $g = g_2 \cdot g_1$ on $K_1 \cap K_2$.*

Proof. We use $\mathfrak{G}_j$ to denote the group $\mathcal{A}^k(K_j, \mathrm{GL}_n(\mathbb{C}))$, and we use $\mathfrak{G}$ to denote the group $\mathcal{A}^k(K_1 \cap K_2, \mathrm{GL}_n(\mathbb{C}))$. Let $\mu \colon \mathfrak{G}_1 \times \mathfrak{G}_2 \to \mathfrak{G}$ be the map $\mu(g_1, g_2) = g_2|_{K_1 \cap K_2} \cdot g_1|_{K_1 \cap K_2}$. The derivative of μ at the point $(1_{\mathfrak{G}_1}, 1_{\mathfrak{G}_2})$ is given by the linear map from the Banach space $\mathcal{A}^k(K_1, \mathrm{Mat}_{n\times n}(\mathbb{C})) \oplus \mathcal{A}^k(K_2, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ into the Banach space $\mathcal{A}^k(K_1 \cap K_2, \mathrm{Mat}_{n\times n}(\mathbb{C}))$, which in turn is given by $(h_1, h_2) \mapsto h_1|_{K_1\cap K_2} + h_2|_{K_1\cap K_2}$. By Observation 4.3, this mapping is surjective. Consequently, there is a neighborhood U of the identity in $\mathfrak{G}$ such that, for any $g \in U$, there exist g_j in $\mathfrak{G}_j$ such that $g = g_2 g_1$ on $K_1 \cap K_2$. This proves the assertion when g is in the neighborhood U. We may assume without loss of generality that the exponential map is a surjective diffeomorphism onto U from a neighborhood V of 0 in $\mathcal{A}^k(K_1 \cap K_2, \mathrm{Mat}_{n\times n}(\mathbb{C}))$.

For the general case we observe that, since each component of $K_1 \cap K_2$ is contractible, it follows that the group $\mathfrak{G}$ is connected and hence $\mathfrak{G}$ is generated by the neighborhood U of the identity. We may therefore write $g = \prod_{i=1}^N \exp(h_i)$, where $h_i \in V$. Now the set $\mathbb{C} \setminus K_1 \cap K_2$ is connected, so it is possible to approximate each h_j by an entire matrix-valued $\tilde{h}_j$ on $K_1 \cap K_2$. Let $\tilde{g} = \prod \tilde{h}_j$ and $\hat{g} = \tilde{g}^{-1} \cdot g$. If the approximation of h_j by $\tilde{h}_j$ is close enough then $\hat{g} \in U$, and consequently it is possible to write $\hat{g} = a_2 a_1$, where $a_j \in \mathfrak{G}_j$. We can take $g_1 = a_1$ and $g_2 = \tilde{g} a_2$ to complete the proof. □

The following solution of a nonlinear Cousin problem is due to Rosay ([14], also see comments in [15]).

Lemma 4.5. *Let ω be an open subset of $\mathbb{C}^n$ and let $F\colon \omega \to \mathbb{C}^n$ be a holomorphic immersion. Let (K_1, K_2) denote a good pair of compact subsets of $\mathbb{C}$ and, for some $k \geq 0$, let $u_1 \in \mathcal{A}^k(K_1, \mathbb{C}^n)$ be such that $u_1(K_1 \cap K_2) \subset \omega$. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $u_2 \in \mathcal{A}^k(K_2, \mathbb{C}^n)$ is such that $\|u_2 - F(u_1)\| < \delta$, then for $j = 1, 2$ there exist $v_j \in \mathcal{A}^k(K_j, \mathbb{C}^n)$ such that $\|v_j\| < \varepsilon$ and $u_2 + v_2 = F(u_1 + v_1)$.*

It is important to note that the map u_1 is *fixed*. In [15], a version is proved in which this restriction is removed. This requires a version of Cartan's lemma for *bounded* matrices (see [2]), a result that is valid if K_1, K_2, and $K_1 \cup K_2$ are simply connected. Unfortunately, such a result could not be proved for the more general K_1, K_2 considered here. Our proof will use the following well-known fact from the theory of Banach spaces, which can be proved using a standard iteration argument (see [11, pp. 397–398]).

Lemma 4.6. *Let $\mathcal{E}$ and $\mathcal{F}$ be Banach spaces, and let $\Phi\colon B_{\mathcal{E}}(p, r) \to \mathcal{F}$ be a $\mathcal{C}^1$ map. Suppose there is a constant $C > 0$ such that:*

- *for each $h \in B_{\mathcal{E}}(p, r)$, the linear operator $\Phi'(h)\colon \mathcal{E} \to \mathcal{F}$ is surjective and the equation $\Phi'(h)u = g$ can be solved for $u \in \mathcal{E}$ for all $g \in \mathcal{F}$ with the estimate $\|u\|_{\mathcal{E}} \leq C\|g\|_{\mathcal{F}}$; and*
- *for any h_1 and h_2 in $B_{\mathcal{E}}(p, r)$, we have $\|\Phi'(h_1) - \Phi'(h_2)\| \leq 1/2C$.*

Then

$$\Phi(B_{\mathcal{E}}(p, r)) \supset B_{\mathcal{F}}\left(\Phi(p), \frac{r}{2C}\right).$$

Proof of Lemma 4.5. Denote by $\mathcal{E}$ the Banach space $\mathcal{A}^k(K_1, \mathbb{C}^n) \oplus \mathcal{A}^k(K_2, \mathbb{C}^n)$, which we endow with the norm $\|\cdot\|_{\mathcal{E}} := \max(\|\cdot\|_{\mathcal{A}^k(K_1,\mathbb{C}^n)}, \|\cdot\|_{\mathcal{A}^k(K_2,\mathbb{C}^n)})$. Also, let the open subset $\mathcal{U}$ of $\mathcal{E}$ be given by $\{(w_1, w_2) : w_1(K_1 \cap K_2) \subset \omega\}$. Then $(u_1, w_2) \in \mathcal{U}$ for any $w_2 \in \mathcal{A}^k(K_2, \mathbb{C}^n)$. Let $\mathcal{F}$ be the Banach space $\mathcal{A}^k(K_1 \cap K_2, \mathbb{C}^n)$, and consider the map $\Phi\colon \mathcal{U} \to \mathcal{F}$ given by $\Phi(w_1, w_2) := w_2|_{K_1 \cap K_2} - F \circ (w_1|_{K_1 \cap K_2})$. A computation shows that $\Phi'(w_1, w_2)$ is the linear map from $\mathcal{E}$ to $\mathcal{F}$ given by $(v_1, v_2) \mapsto v_2|_{K_1 \cap K_2} - F'(w_1|_{K_1 \cap K_2})(v_1|_{K_1 \cap K_2})$. Observe that w_2 plays no role whatsoever in this expression; therefore, $\Phi'(w_1, w_2) \in BL(\mathcal{E}, \mathcal{F})$ is in fact a smooth function of w_1 alone, and we will henceforth denote it by $\Phi'(w_1, *)$.

We construct a right inverse to $\Phi'(u_1, *)$. Let $\gamma = F'(u_1)|_{K_1 \cap K_2}$. Then $\gamma \in \mathcal{A}^k(K_1 \cap K_2, \mathrm{GL}_n(\mathbb{C}))$, and by Lemma 4.4 we may write $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_j \in \mathcal{A}^k(K_j, \mathrm{GL}_n(\mathbb{C}))$. (We henceforth suppress the restriction signs.) For $g \in \mathcal{F}$, let $S(g) = (-\gamma_1^{-1} T_1(\gamma_2^{-1} g), \gamma_2 T_2(\gamma_2^{-1} g))$, where the T_j are as in equation (22). Then S is a bounded linear operator from $\mathcal{F}$ to $\mathcal{E}$, and a computation shows that $\Phi'(u_1, *) \circ S(g)$ is the identity map on $\mathcal{F}$. Choose $\theta > 0$ so small that if $w_1 \in \mathcal{F}$ is such that $\|w_1 - u_1\| < \theta$ then (a) the equation $\Phi'(w_1, *)u = g$ can be solved with the estimate $\|u\| \leq 2\|S\|\|g\|$ and (b) $\|\Phi'(w_1, *) - \Phi'(u_1, *)\|_{\mathrm{op}} < 1/8\|S\|$. (Here (a) and (b) follow from continuity and the fact that small perturbations of a surjective linear operator are still surjective.) Consequently, if $\varepsilon < \theta$ and $u_2 \in \mathcal{A}^k(K_2, \mathbb{C}^n)$ then, for the ball $B_{\mathcal{E}}((u_1, u_2), \varepsilon)$, the hypotheses of Lemma 4.6 are verified with $C = 2\|S\|$. Thus we have

$$\begin{aligned}\Phi(B_{\mathcal{E}}((u_1, u_2), \varepsilon)) &\supset B_{\mathcal{F}}\left(\Phi(u_1, u_2), \frac{\varepsilon}{2C}\right) \\ &= B_{\mathcal{F}}\left(u_2 - F(u_1), \frac{\varepsilon}{2C}\right).\end{aligned}$$

Therefore, if $\|u_2 - F(u_1)\| < \varepsilon/4C$ then $0 \in \Phi(B_{\mathcal{E}}((u_1, u_2), \varepsilon))$. This is exactly the conclusion required. □

Proof of Theorem 5. For notational clarity, we omit the restriction signs on maps. For $j = 1, 2$ let the coordinate neighborhoods V_j of $\mathcal{M}$ be such that $f(K_j) \subset V_j$. We begin by fixing biholomorphic maps $F_j\colon V_j \to F_J(V_j) \subset \mathbb{C}^n$ and setting $F = F_2 \circ F_1^{-1}$. Then F is a biholomorphism from the open set $\omega = F_1(V_1 \cap V_2)$ onto the open set $F_2(V_2 \cap V_1)$. Moreover, a pair of maps w_1 and w_2 from (respectively) K_1 and K_2 "glue together" to form a map from $K_1 \cup K_2$ (i.e., there is a map $h\colon K \to \mathcal{M}$ such that $w_j = F_j \circ h$) *only* if $w_2 = F(w_1)$.

Let $u_1 = F_1 \circ f \in \mathcal{A}^k(K_1, \mathbb{C}^n)$. Since $V \cap K_1 = \emptyset$ by hypothesis, the pair of compact sets $(K_1, K_2 \cup V)$ is good. Fix $\varepsilon_0 > 0$, and let $\delta_0 > 0$ be the δ corresponding to $\varepsilon = \varepsilon_0$ in Lemma 4.5 for the good pair $(K_1, K_2 \cup V)$ and F, ω, u_1 as defined previously. Let $u_2 \in \mathcal{A}^k(K_2 \cup V, \mathbb{C}^n)$ be a $\mathcal{C}^k$ approximation to $F_2 \circ (f|_{K_2})$ such that $u_2(K_2) \subset V_2$, and let $\|u_2 - F(u_1)\| < \delta_0$. (Such a u_2 exists by hypothesis.)

Then, by Proposition 4.5, there is a $v_1 \in \mathcal{A}^k(K_1, \mathbb{C}^n)$ and a $v_2 \in \mathcal{A}^k(K_2 \cup V\mathbb{C}^n)$ such that $\|v_j\| < \varepsilon_0$ and $u_2 + v_2 = F(u_1 + v_1)$. Hence the maps $u_1 + v_1$ and $u_2 + v_2$ glue together to form a map f_{ε_0} given by

$$f_{\varepsilon_0} := \begin{cases} F_1^{-1}(u_1 + v_1) & \text{on } K_1, \\ F_2^{-1}(u_2 + v_2) & \text{on } K_2 \text{ and near } K_2 \cap V. \end{cases}$$

Clearly, f_{ε_0} is in $\mathcal{A}^k(K_1 \cup K_2, \mathcal{M})$ and extends to a holomorphic map near $K_2 \cap V$. Moreover, $\operatorname{dist}_{\mathcal{C}^k}(f_{\varepsilon_0}, f) = O(\varepsilon_0)$. The result follows. $\square$

4.2. Proof of Theorems 3′ and 4

Let $k \geq 0$ be an integer, and let the domain Ω be circular if $k = 0$ or be $\mathcal{C}^1$ if $k \geq 1$. Fix $f \in \mathcal{A}^k(\bar{\Omega}, \mathcal{M})$. We want to approximate f in the $\mathcal{C}^k$ sense on $\bar{\Omega}$.

The basic idea of this proof is to slice the $\bar{\Omega}$ by a system of parallel lines. If the slices are narrow enough, we will show that (thanks to the results of Section 3) the graph of f over each slice is contained in a coordinate neighborhood of $\mathcal{M}$. We will further show that the slicing can be done in a way that the unions of alternate slices form a good pair. Consequently, we can use the results of Section 4.1 to prove the approximation results. We break the proof up into a sequence of lemmas.

LEMMA 4.7. *Denote by F the map in $\mathcal{A}^k(\bar{\Omega}, \mathbb{C} \times \mathcal{M})$ given by $F(z) = (z, f(z))$. For real ξ, let L_ξ be the vertical straight line $\{z : \Re z = \xi\}$. Then*:

- *$F(L_\xi \cap \bar{\Omega})$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$; and*
- *there is a nowhere dense $E \subset \mathbb{R}$ such that if $\xi \notin E$ then the line L_ξ meets $\partial\Omega$ transversely. If $k = 0$, the set E can even be taken to be finite.*

Proof. We first prove that $F(L_\xi \cap \bar{\Omega})$ has a coordinate neighborhood. Each connected component of $L_\xi \cap \bar{\Omega}$ is a point or a compact interval. Now F is injective and so, if we show that for each such component I the set $F(I)$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$, then it will follow that $F(L_\xi \cap \bar{\Omega})$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$. This is trivial if the component I is a point. Therefore, let I be a compact interval. We consider three cases as follows.

Case 1: $k = 0$. In this case Ω is a circular domain and hence, for each ξ, the set $L_\xi \cap \partial\Omega$ (and, a fortiori, $I \cap \partial\Omega$) is finite. Consider the arc $F|_I$ in $\mathbb{C} \times \mathcal{M}$. This arc is real analytic off the finite set of points $I \cap \partial\Omega$, and the projection $\phi\colon \mathbb{C} \times \mathcal{M} \to \mathbb{C}$ has the property that $\phi \circ (F|_I)$ is the inclusion map $I \hookrightarrow \mathbb{C}$. Hence $F|_I$ is a real-analytic mildly singular arc in the sense of Definition 3.4 and so, by Theorem 2, $F(I)$ has a coordinate neighborhood.

Case 2: $k = 1$. In this case, the arc $F|_I$ is $\mathcal{C}^1$. As in Case 1, let ϕ be the projection $\phi\colon \mathbb{C} \times \mathcal{M} \to \mathbb{C}$ with the property that $\phi \circ (F|_I)$ is the inclusion map $I \hookrightarrow \mathbb{C}$. Therefore, Proposition 3.3 applies and so $F(I)$ has a coordinate neighborhood.

Case 3: $k \geq 2$. In this case, $F|_I$ is a $\mathcal{C}^k$ embedded arc with $k \geq 2$ and hence, by Corollary 3.2, it has a coordinate neighborhood.

We now turn to the second conclusion. In the case $k = 0$, the domain Ω is circular and so $\partial\Omega$ is a disjoint union of circles. Note that L_ξ is *not* transverse to $\partial\Omega$ iff it is tangent to some component circle of $\partial\Omega$. So we can take E to be the finite set of all ξ such that L_ξ is tangent to $\partial\Omega$.

In the case $k \geq 1$, let Γ be a connected component of $\partial\Omega$. We can parameterize Γ by a $\mathcal{C}^1$ map $\gamma = \gamma_1 + i\gamma_2\colon S^1 \to \Gamma \subset \mathbb{C}$. Here the line L_ξ is not transverse to Γ iff ξ is a critical value of $\gamma_2\colon S^1 \to \mathbb{R}$. By Sard's theorem, the set E_Γ of critical values of γ_2 is of measure 0. Of course, E is closed. We let $E = \bigcup E_\Gamma$, a union over the finitely many components Γ of $\partial\Omega$. Then E is nowhere dense. $\square$

We make the following simple observation, whose proof is clear.

OBSERVATION 4.8. *Let u and v be real-valued $\mathcal{C}^1$ functions defined on a neighborhood of 0 in $\mathbb{R}$ such that, for each x, we have $u(x) < 0 < v(x)$. Then there is an $\eta > 0$ such that, for $0 < \theta \le \eta$, the vertical strip*

$$S := \{(x, y) \in \mathbb{R}^2 : x \in [-\theta, \theta],\ u(x) \le y \le v(x)\}$$

is star shaped with respect to the origin.

We will now decompose $\bar{\Omega}$ into a good pair (K_1, K_2).

LEMMA 4.9. *Let F be as in Lemma 4.7. There is a good pair (K_1, K_2) such that $K_1 \cup K_2 = \bar{\Omega}$, and each $F(K_j)$ has a coordinate neighborhood $\mathcal{V}_j$ in $\mathbb{C} \times \mathcal{M}$.*

Proof. By Lemma 4.7, for each vertical line L the set $F(L \cap \bar{\Omega})$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$. By compactness we can find finitely many points

$$x_0 < x_1 < \cdots < x_N$$

such that, for $j = 0, \dots, N-1$, the set $\{F(z) : x_j \le \Re z < x_{j+1}, z \in \bar{\Omega}\}$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$ and each component of $\{z \in \mathbb{C} : x_j \le \Re z < x_{j+1}, z \in \bar{\Omega}\}$ is simply connected. We impose the following condition on the points x_j: for each j, the vertical line $\Re z = x_j$ meets $\partial\Omega$ transversely at each point of intersection. This can be easily done because we have already shown that the set E of all ξ such that $\Re z = \xi$ is *not* transverse to $\partial\Omega$ is closed and of measure 0.

Therefore, $\partial\Omega$ is a union of (an even number of) graphs of $\mathcal{C}^1$ functions in a neighborhood of each of the vertical lines $\Re z = x_j$. By Observation 4.8 there is an η such that, if $\theta \le \eta$, then each component of the intersection of $\bar{\Omega}$ with a vertical strip of width θ about the line $\Re z = x_j$ is star shaped. Define the compact subsets K_1 and K_2 of $\mathbb{C}$ given by

$$K_1 := \{z \in \bar{\Omega} : x_{2j-1} - \theta \le \Re z \le x_{2j} + \theta,\ j = 1, 2, \dots\}$$

and

$$K_2 := \{z \in \bar{\Omega} : x_{2j} \le \Re z \le x_{2j+1},\ j = 1, 2, \dots\}.$$

In other words, K_2 (resp., K_1) consists of the slices of $\bar{\Omega}$ over the odd-numbered intervals in the partition $x_0 < x_1 < \cdots < x_N$ (resp., the slices over the even-numbered ones slightly enlarged). The sets $\overline{K_1 \setminus K_2}$ and $\overline{K_2 \setminus K_1}$ are disjoint, and each $F(K_j)$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$, which will be our $\mathcal{V}_j$. $\square$

We can now use Theorem 5 to prove the following approximation result.

PROPOSITION 4.10. *There is a point $p \in \partial\Omega$ with the following property. Given any $\varepsilon > 0$, there is a neighborhood U_ε of p in $\mathbb{C}$ and a map $g \in \mathcal{A}^k(\bar{\Omega}, \mathcal{M})$ such that* $\operatorname{dist}_{\mathcal{C}^k(\bar{\Omega}, \mathcal{M})}(f, g) < \varepsilon$ *and g extends as a holomorphic map to U_ε.*

Proof. The proof is an application of Theorem 5. As before, let $F(z) = (z, f(z))$. In the notation of that theorem we choose the following data.

- The good pair (K_1, K_2) will be the one in the conclusion of Lemma 4.9, so that $K_1 \cup K_2 = \Omega$, and $F(K_1)$ and $F(K_2)$ each has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$.
- Let $p \in (K_2 \setminus K_1) \cap \partial\Omega$, and let V be a closed disc around p such that $V \cap K_1 = \emptyset$. It is clear (e.g., by an easily established $\mathcal{C}^k$ version of Mergelyan's theorem) that any function in $\mathcal{A}^k(K_2)$ can be approximated in the $\mathcal{C}^k$ sense by entire functions and hence, a fortiori, by functions in $\mathcal{A}^k(K_2 \cup V)$.
- We will let the target manifold be $\mathbb{C} \times \mathcal{M}$ (denoted by $\mathcal{M}$ in the statement of Theorem 5) and let the map to be approximated be F. As remarked previously, $F(K_j)$ has a coordinate neighborhood in $\mathbb{C} \times \mathcal{M}$.

Therefore, by Theorem 5 we obtain a $\mathcal{C}^k$ approximation G to F on $K_1 \cup K_2 = \bar{\Omega}$, where G extends holomorphically to some neighborhood U_ε of p, and we have $\mathrm{dist}_{\mathcal{C}^k(\bar{\Omega}, \mathcal{M})}(F, G) < \varepsilon$. Then let $g = \pi_{\mathcal{M}} \circ G$, where $\pi_{\mathcal{M}} \colon \mathbb{C} \times \mathcal{M} \to \mathcal{M}$ is the projection onto the second component. □

At this point, the approximation has been achieved in a neighborhood of one point p in the boundary. We could repeat this process, thus obtaining a proof of Theorems 3′ and 4. This would require a strengthened version of Proposition 4.10 in which (i) we can choose the point p arbitrarily and (ii) the diameter of the set $\partial\Omega \cap U_\varepsilon$ is independent of p. This is the route followed in [3].

However, in this paper we complete the proofs using a technique found in [5], which in the setting of our problem allows us to avoid entirely the method just outlined of successive bumpings. We explain this technique of [5] in the following lemma.

LEMMA 4.11. *Let D be a $\mathcal{C}^1$ domain in the plane, and let V be an open set in $\mathbb{C}$ such that $\partial D \cap V \neq \emptyset$. Then there is a one-parameter family $\psi_t \in \mathcal{H}(\bar{D})$ such that* (i) *for small $t \geq 0$ we have $\psi_t(\bar{D}) \subset D \cup V$ and* (ii) *as $t \to 0$, $\psi_t \to \psi_0$ in the $\mathcal{C}^\infty$ sense, where ψ_0 is the identity map $\psi_0(z) = z$.*

Proof. The case of connected ∂D is trivial, so we assume that ∂D has at least two components. For a Jordan curve $C \subset \mathbb{C}$, denote by $\beta(C)$ the bounded component of $\mathbb{C} \setminus C$. Let $\{C_k\}_{k=1}^{M+1}$ be the components of ∂D, where $V \cap C_{M+1} \neq \emptyset$. There is a component C_j of ∂D with the following properties: for $k \neq j$, we have $C_k \subset \beta(C_j)$ and $\beta(C_k) \cap D = \emptyset$. Call C_j the *outer boundary* of D.

We may assume without loss of generality that the outer boundary of D is C_{M+1}. If this is not already the case then, in a neighborhood of $\bar{D}$, make the change of coordinates $z \mapsto \rho^2/(z - z_0)$, where $z_0 \in \beta(C_{M+1})$ and $\rho > 0$ is so small that $B_{\mathbb{C}}(z_0, \rho) \Subset \beta(C_{M+1})$.

Let $V_0 \Subset V$ be such that $V_0 \cap C_{M+1} \neq \emptyset$, and for $1 \leq k \leq M$ we have $V_0 \cap C_k = \emptyset$. Let $\Gamma = \partial D \setminus V_0$. Then $\mathbb{C} \setminus \Gamma$ has $M + 1$ components. Exactly one of these is unbounded, and this component contains the set $C_{M+1} \setminus V_0$. Let P be a set of M points such that, for $j = 1, \ldots, M$, each bounded component $\beta(C_j)$ of $\mathbb{C} \setminus \Gamma$ contains exactly one point of P. Of course, $P \cap D = \emptyset$.

Let N be the inward-directed unit normal vector field on ∂D. If we dentify $T\mathbb{C}$ with $\mathbb{C}$, then the restriction $N|_\Gamma$ is a continuous function on the set Γ. Since Γ has no interior and since $\mathbb{C} \setminus \Gamma$ has finitely many components, it follows that N can be uniformly approximated on Γ by rational functions holomorphic on $\mathbb{C} \setminus P$. We thus obtain a holomorphic vector field X on $\bar{D}$ such that $X|_\Gamma$ is directed inward—that is, toward D. Denote by ψ_t the holomorphic flow generated by X. Clearly, on any compact set, ψ_t approaches the identity in all $\mathcal{C}^k$ norms as $t \to 0$. For small $t \geq 0$ we have $\psi_t(\Gamma) \subset D$, and by continuity $\psi_t(\partial D \cap \overline{V_0}) \subset V$. Therefore, $\psi_t(\partial D) \subset D \cup V$ and so $\psi_t(\bar{D}) \subset D \cup V$. □

We can now conclude the proofs of Theorem 3′ and Theorem 4. Suppose that $\varepsilon > 0$ is given and we want to find an $h \in \mathcal{A}^k(\bar{\Omega}, \mathcal{M})$ such that $\operatorname{dist}_{\mathcal{C}^k(\bar{\Omega},\mathcal{M})}(f, h) < \varepsilon$. Using Lemma 4.10, we can construct an approximation g that extends holomorphically to a neighborhood U_ε of a point p on the boundary, and we have $\operatorname{dist}_{\mathcal{C}^k(\bar{\Omega},\mathcal{M})}(f, g) < \varepsilon/2$. In Lemma 4.11 let $V = U_\varepsilon$ and $D = \Omega$. Let ψ_t be the family of biholomorphisms in the conclusion of Lemma 4.11. Then $g_t = g \circ \psi_t$ is in $\mathcal{H}(\bar{\Omega}, \mathcal{M})$ for small t, and as $t \to 0$ we have $g_t \to g$ in the $\mathcal{C}^k$ sense on $\bar{\Omega}$. Therefore, taking $t > 0$ small enough, we obtain $h = g_t$ such that $\operatorname{dist}_{\mathcal{C}^k(\bar{\Omega},\mathcal{M})}(h, g) < \varepsilon/2$. Theorem 3′ (and hence Theorem 3) and Theorem 4 are thus proved.

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