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Sets of Approximation and Interpolation in $\mathbb{C}$ for Manifold-Valued Maps

Debraj Chakrabarti

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Abstract We give examples of non-smooth sets in the complex plane with the property that every holomorphic map continuous to the boundary from these sets into any complex manifold may be uniformly approximated by maps holomorphic in some neighborhood of the set (Mergelyan-type approximation for manifold-valued maps.) Similar results are proved for sections of complex-valued holomorphic submersions from complex manifolds.

Keywords Mergelyan-type Approximation · Manifold-valued maps

Mathematics Subject Classification (2000) 32Q99 · 32H02 · 30E10

1 Introduction

1.1 Statement of Results

This article is devoted to the study of approximation and interpolation of maps from a compact $K \subset \mathbb{C}$ into a complex manifold $\mathcal{M}$, and in particular to giving examples of sets K for which certain types of approximation and interpolation are possible. For brevity, we introduce two properties A_2 and A_3 that compact subsets of the plane may possess. We first define property A_2.

For a compact $K \subset \mathbb{C}$, let K° be the interior of K. We will let $\mathcal{A}(K, \mathcal{M})$ denote the continuous maps from K into the complex manifold $\mathcal{M}$ which are holomorphic on K°. Let $\mathcal{O}(K, \mathcal{M}) \subset \mathcal{A}(K, \mathcal{M})$ denote the subspace of those maps f which extend to a holomorphic map from some neighborhood U_f of K to $\mathcal{M}$. We endow $\mathcal{M}$ with any metric. This makes $\mathcal{A}(K, \mathcal{M})$ and $\mathcal{O}(K, \mathcal{M})$ into metric spaces with the

D. Chakrabarti (✉)
Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada
e-mail: dchakra@uwo.ca

uniform metric on maps. We will say that a compact $K \subset \mathbb{C}$ has the property A_2, if, *for every complex manifold* $\mathcal{M}$*, every finite set* $\mathcal{P} \subset K$*, and every* $\epsilon > 0$*, we can approximate any* $f \in \mathcal{A}(K, \mathcal{M})$ *by a map* $f_\epsilon \in \mathcal{O}(K, \mathcal{M})$ *such that* $\mathrm{dist}_{\mathcal{M}}(f, f_\epsilon) < \epsilon$*, and for each* $p \in \mathcal{P}$*, we have* $f_\epsilon(p) = f(p)$. That is, f_ϵ is an uniform approximation to f which interpolates the values of f on $\mathcal{P}$.

We now define property A_3. Let $\phi : \mathcal{M} \to \mathbb{C}$ be a holomorphic submersion such that $\phi(\mathcal{M}) \supset K$. Let $\mathcal{A}_\phi(K, \mathcal{M})$ (resp. $\mathcal{O}_\phi(K, \mathcal{M})$) be the subspace of $\mathcal{A}(K, \mathcal{M})$ (resp. $\mathcal{O}(K, \mathcal{M})$) consisting of sections of ϕ over K, i.e., maps $s : K \to \mathcal{M}$ such that $\phi \circ s = \mathrm{Id}_K$. We will say that a compact $K \subset \mathbb{C}$ has property A_3 if *for every complex manifold* $\mathcal{M}$ *and every holomorphic submersion* ϕ *such that* $\phi(\mathcal{M}) \supset K$*, and every finite set* $\mathcal{P} \subset K$*, every section* $\sigma \in \mathcal{A}_\phi(K, \mathcal{M})$ *can be uniformly approximated by sections in* $\mathcal{O}_\phi(K, \mathcal{M})$ *which interpolate the values of* σ *on* $\mathcal{P}$.

It is natural to ask the question: *which are the compact sets in the plane that satisfy property* A_2 *or property* A_3? Since from Bishop's localization theorem we know that approximation in the complex plane is local in nature, one can conjecture that using patching arguments in the manifold $\mathcal{M}$ we may be able to show that every set such that property A_2 holds in the special case when $\mathcal{M}$ is $\mathbb{C}$, has property A_2, or even A_3. We do not know if this program can be carried out. In this article, we confine ourselves to the much more modest goal of giving examples of sets K for which properties A_2 and A_3 hold (see Theorems 1, 2, and 3 below.)

We recall some definitions from elementary point-set topology. Let X be a topological space. For an integer $n \geq 0$, we say that $\dim(X) \leq n$ if the following holds: given any open cover $\mathcal{A}$ of X, there is a refinement $\mathcal{B}$ of $\mathcal{A}$ such that any point of X is contained in at most $n + 1$ of the sets in the cover $\mathcal{B}$. If $\dim(X) \leq n$ holds, but $\dim(X) \leq n - 1$ does not, we say that the topological dimension $\dim(X)$ of X is n. The dimension is clearly a topological invariant of X. Also, if $\dim(X) = n$, then each connected component of X (or more generally, any closed subset) has dimension less than or equal to n.

Theorem 1 *Let* $K \subset \mathbb{C}$ *be a connected compact set, such that* $\dim(K) = 0$ *or* 1. *If* $\dim(K) = 1$, *further suppose that given any cover* $\mathcal{A}$ *of* K *such that any point of* K *belongs to at most two sets in* $\mathcal{A}$, *there is a refinement* $\mathcal{B}$ *such that every non-empty intersection of two sets in* $\mathcal{B}$ *is contractible. Then* K *has property* A_3.

Examples of one-dimensional sets that satisfy the hypotheses are:

(1) arcs in $\mathbb{C}$, where By an *arc* α in a topological space X we mean a continuous *injective* map $\alpha : [0, 1] \to X$ from the unit interval. The injectivity avoids pathologies like space filling curves. By standard abuse of language, we will refer to the image $\alpha([0, 1])$ as the arc α.
(2) planar realizations of connected *graphs* i.e., finite simplicial complexes having only 0-simplices ("vertices") and 1-simplices ("edges".) We can think of these as unions of arcs in the plane, any two of which meet at most one endpoint of each.

We now go on to give examples of two-dimensional sets which have properties A_2 or A_3. Let $k \geq 0$ be an integer, ∞ or ω. We introduce a class of sets in the plane denoted by $\mathfrak{C}_k$. We will let $\mathfrak{C}_k$ denote the class of compact sets $K \subset \mathbb{C}$ such that there is an integer N and domains Ω_j, where $j = 1, \ldots, N$ with the following properties:

- For $k \neq 0$, each Ω_j is a domain with $\mathcal{C}^k$ boundary, and for $k = 0$, each Ω_j is a Jordan domain (an open set Ω in $\mathbb{C}$ is a *Jordan domain* if the boundary $\partial\Omega$ has finitely many connected components, each of which is homeomorphic to a circle.)
- The Ω_j's are pairwise disjoint: $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.
- $K = \bigcup_{i=1}^N \overline{\Omega_j}$. We will refer to each Ω_j as a *summand* of K.
- whenever $i \neq j$, the set

$$P_{ij} := \overline{\Omega_i} \cap \overline{\Omega_j} = \partial\Omega_i \cap \partial\Omega_j$$

is finite. Moreover, each $p \in P_{ij}$ belongs only to $\partial\Omega_i$ and $\partial\Omega_j$ (and not to any $\partial\Omega_k$ for $k \neq i, j$.)
- If $k \neq 0$, at each point of P_{ij}, the boundaries $\partial\Omega_i$ and $\partial\Omega_j$ are tangent to each other.

We can now provide a supply of sets which have property A_2.

Theorem 2 *Each compact set K of class $\mathfrak{C}_0$ has property A_2.*

The second result gives examples of sets with property A_3:

Theorem 3 *Each compact set K of class $\mathfrak{C}_2$ has property A_3.*

1.2 Some remarks

(1) Let us define the property A_1, which will be the special case of A_2 with $\mathcal{M} = \mathbb{C}$. More precisely, we will say that $K \subset \mathbb{C}$ has the property A_1, if, *for any finite $\mathcal{P} \subset K$, any function in $\mathcal{A}(K) := \mathcal{A}(K, \mathbb{C})$ can be uniformly approximated by functions in $\mathcal{O}(K) := \mathcal{O}(K, \mathbb{C})$ which interpolate the values of f on $\mathcal{P}$*. Obviously for a set K, $A_2 \Rightarrow A_1$.
It is easy to see that a set K has property A_1 iff $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$. Let $\mathcal{P}$ be a given finite subset of K. Note the fact that there is a constant $C > 0$ (depending only on $\mathcal{P}$ and K) such that for $g \in \mathcal{C}(K)$, if $L_{\mathcal{P}}(g)$ is the Lagrange polynomial which interpolates the values of g on $\mathcal{P}$, we have: $\|L_{\mathcal{P}}(g)\|_K < C\|g\|_K$. If $\tilde{f} \in \mathcal{O}(K)$ be such that $|f - \tilde{f}| < \frac{\epsilon}{C+1}$, then $|f - f_\epsilon| < \epsilon$, and $f(p) = f_\epsilon(p)$ for $p \in \mathcal{P}$, where $f_\epsilon = \tilde{f} + L_{\mathcal{P}}(f - \tilde{f})$.
The compact sets $K \subset \mathbb{C}$ such that $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$ (usually known as *sets of holomorphic approximation*) can be characterized by Vituškin's theorem [15]. A sufficient condition is that $\mathbb{C} \setminus K$ has finitely many connected components. Also, such approximation can be localized, in the sense that $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$ iff every point $z \in K$ has a neighborhood U_z in $\mathbb{C}$ such that $\mathcal{O}(K \cap \overline{U_z})$ is dense in $\mathcal{A}(K \cap \overline{U_z})$ (Bishop's localization theorem.)

(2) We have the following:

Lemma 1 *For a compact $K \subset \mathbb{C}$, $A_3 \Rightarrow A_2$.*

Proof Assume that K satisfies A_3, and let $\mathcal{P}$ and $\mathcal{M}$ have the same meaning as above, and let $f \in \mathcal{A}(K, \mathcal{M})$. Consider the complex manifold $\mathcal{N} = \mathcal{M} \times \mathbb{C}$

and let ϕ and π be the projections onto $\mathbb{C}$ and $\mathcal{M}$, respectively. If $F : K \to \mathcal{N}$ is defined by $F(z) = (f(z), z)$, then clearly $F \in \mathcal{A}_\phi(K, \mathcal{N})$ and since K has the property A_3, we can approximate F by maps $G \in \mathcal{O}_\phi(K, \mathcal{N})$ such that $G(p) = F(p) = (f(p), p)$ for each $p \in \mathcal{P}$. Then $g = \pi \circ G$ is in $\mathcal{O}(K, \mathcal{M})$ and an approximation to f with $g(p) = f(p)$ for each $p \in \mathcal{P}$, which shows that K has property A_2. □

(3) Holomorphic submersions $\phi : \mathcal{M} \to \mathbb{C}$ always exist if $\mathcal{M}$ is Stein [7]. However, in general such $\mathcal{M}$ may be highly nontrivial, see, e.g., [3].

(4) Some results have been obtained recently regarding the approximation and interpolation of manifold-valued maps.
The following was proved in [2].

Proposition 1 *Let $K = \overline{\Omega}$, where $\Omega \Subset \mathbb{C}$ is a Jordan domain. For any complex manifold $\mathcal{M}$, the subspace $\mathcal{O}(K, \mathcal{M})$ is dense in $\mathcal{A}(K, \mathcal{M})$.*

The following result is contained in what was proved by Drinovec-Drnovšek and Forstnerič in [5], Theorem 5.1.

Proposition 2 *Let $\Omega \Subset \mathbb{C}$ have $\mathcal{C}^2$ boundary, $\mathcal{M}$ be a complex manifold, and let $f : \overline{\Omega} \to \mathcal{M}$ be a map of class $\mathcal{C}^r$ $(r \geq 2)$ which is holomorphic in Ω. Given finitely many points $z_1, \ldots, z_l \in \Omega$, and an integer $k \in N$, there is a sequence of maps $f_\nu \in \mathcal{O}(K, \mathcal{M})$ such that f_ν agrees with f to order k at z_j for $j = 1, \ldots, l$ and $\nu \in \mathbb{N}$, and the sequence f_ν converges to f in $\mathcal{C}^r(\overline{\Omega})$ as $\nu \to \infty$.*

This can in fact be proved when $\Omega \Subset S$, where S is a non-compact Riemann surface, and $\mathcal{M}$ is a complex space. Observe that there is a stronger assumption on the smoothness of the map than in property A_2, and also a stronger conclusion regarding approximation and interpolation. However, the set of points of interpolation is restricted to the interior of $\overline{\Omega}$. The same authors subsequently proved the following:

Proposition 3 *If $\Omega \Subset \mathbb{C}$ is a domain with $\mathcal{C}^2$ boundary, $\mathcal{M}$ a complex manifold, and $\phi : \mathcal{M} \to \mathbb{C}$ a holomorphic submersion such that $\phi(\mathcal{M}) \supset \overline{\Omega}$, then $\mathcal{O}_\phi(\overline{\Omega}, \mathcal{M})$ is dense in $\mathcal{A}_\phi(\overline{\Omega}, \mathcal{M})$.*

This is a (very) special case of [6], Theorem 5.1 regarding the approximation of sections of submersions over strongly pseudoconvex sets in Stein manifolds, which may be thought of as a far-reaching generalization of the Henkin-Ramírez-Kerzman approximation theorem for functions on such domains ([9], Theorem 2.9.2.)

(5) The results proved here have analogs for the approximation and interpolation of $\mathcal{A}^k$ maps, i.e., maps which are $\mathcal{C}^k$ and holomorphic in the interior, when we can obtain approximation in the $\mathcal{C}^k$ topology, and interpolation can be done for the k-jets of the given map. The proofs are essentially the same. For notational simplicity we confine ourselves to the case of maps which are only continuous to the boundary.

1.3 Outline of Proofs

In Sect. 2 below, we prove a general result (Theorem 4) regarding approximation of sections of submersions over a set K which admits a decomposition $K = K_1 \cup K_2$ into a “good pair” (K_1, K_2) of compact sets, provided the image of the section has neighborhoods in $\mathbb{C}^n$. This is a direct generalization of the results in [2], Sect. 4 for maps into manifolds. This is the basic patching technique which is used in the proofs of Theorems 1 and 3.

In Sect. 3, we give proofs of Theorems 1 and 3. The proof of Theorem 1 is a direct application of Theorem 4, taking advantage of the one-dimensional character of the sets. For Theorem 3, we require a result (Theorem 5) which is found in [2], which asserts the existence of coordinate neighborhoods of arcs or a certain smoothness in complex manifolds. This in effect allows us to split up any set of class $\mathfrak{C}_2$ into a good pair, to which Theorem 4 can be applied. In fact, we require to apply Theorem 4 three times to obtain a section in a neighborhood of the given set.

In Sect. 4 we deduce Theorem 2 from Theorem 3. For this we use Lemma 12, which allows us to use conformal mapping continuous to the boundary to prove property A_2. The proof of Theorem 2 follows from a result from [11] regarding conformal mappings continuous to the boundary.

2 Approximation on Good Pairs

2.1 Some Definitions

We will call a set $K \subset \mathbb{C}$ *nicely contractible* if there is a homotopy $c : [0, 1] \times K \to K$ with the following properties:

(1) for each t, the map $z \mapsto c(t, z)$ is in $\mathcal{A}(K, \mathbb{C})$,
(2) $z \mapsto c(1, z)$ is the identity map on K.
(3) there is a $z_0 \in K$ such that $c(0, z) \equiv z_0$.
(4) for $t \neq 0$, $z \mapsto c(t, z)$ maps K° into K°.

Of course, convex sets are nicely contractible, as are strongly star-shaped sets (these are sets K such that the maps $c(t, z)$ may be taken as dilations with stretch factor t and center $z_0 \in K$). Another important class of nicely contractible sets are arcs. The property of nicely contractible sets which is used in the proof of Lemma 2 is the following: If K is nicely contractible and $\mathcal{M}$ is a connected complex manifold, then $\mathcal{A}(K, \mathcal{M})$ is in fact contractible. We can take the contraction to be the map ϕ_t from $\mathcal{A}(K, \mathcal{M})$ to itself given by $\phi_t(f)(z) = f(c(t, z))$.

Let K_1 and K_2 be compact subsets of $\mathbb{C}$. We will say that (K_1, K_2) is a *good pair* if the following hold:

(1) $\overline{K_1 \setminus K_2} \cap \overline{K_2 \setminus K_1} = \emptyset$.
(2) $K_{1,2} := K_1 \cap K_2$ has finitely many connected components, each of which is nicely contractible.

Let $\mathcal{M}$ be a complex manifold of complex-dimension n, and $\phi : \mathcal{M} \to \mathbb{C}$ be a submersion. We will say that an open set $U \subset \mathcal{M}$ is *ϕ-adapted*, if U is biholomorphic to an open set in $\mathbb{C}^n$, and there are holomorphic coordinates $(z_1, \ldots, z_n)$ on U such that with respect to these coordinates (and the standard coordinate on $\mathbb{C}$), the map ϕ takes the form $(z_1, \ldots, z_n) \mapsto z_n$. (This is of course the same as saying $z_n = \phi|_U$.)

2.2 The Main Result

We introduce the following notation and definitions:

(1) Let $K = K_1 \cup K_2$, where (K_1, K_2) is a good pair.
(2) Let $\mathcal{M}$ be a complex manifold and $\phi : \mathcal{M} \to \mathbb{C}$ be a holomorphic submersion such that $\phi(\mathcal{M}) \supset K$.
(3) Let $B \subset \mathbb{C}$ be compact and such that $B \cap K_1 = \emptyset$, and each function $g \in \mathcal{A}(K_2)$ can be approximated uniformly on K_2 by functions in $\mathcal{A}(K_2 \cup B)$; i.e., if $g \in \mathcal{A}(K_2)$ and $\epsilon > 0$, then there is a $g_\eta \in \mathcal{A}(K_2 \cup B)$ such that: $|g - g_\eta| < \eta$ on K_2.
(4) Let $\mathcal{P}$ be a finite subset of K.

We now state the following result regarding the approximation of sections of ϕ over K:

Theorem 4 *With $K_1, K_2, \mathcal{M}, \phi$, B, and $\mathcal{P}$ satisfying* (1) *through* (4) *above, let $s \in \mathcal{A}_\phi(K, \mathcal{M})$ be a section of ϕ such that each $s(K_j)$ has a ϕ-adapted neighborhood in $\mathcal{M}$ for $j = 1, 2$. Then, given $\eta > 0$, there is an $s_\eta \in \mathcal{A}_\phi(K, \mathcal{M})$ such that* $\mathrm{dist}(s, s_\eta) <$ *η on K, s_η extends as a holomorphic section of ϕ to a neighborhood B_η of $K_2 \cap B$, and for each $p \in \mathcal{P}$, we have $s_\eta(p) = s(p)$. Moreover, $s_\eta(K_2 \cup B_\eta)$ has a ϕ-adapted neighborhood in $\mathcal{M}$.*

The proof of Theorem 4 will be reduced to the solution of a non-linear patching problem in Euclidean space by the use of coordinate in neighborhoods of $s(K_j)$. We now describe the ingredients required in the proof . The most important is the following version (resembling that A. Douady in [4], pp. 47–48) of H. Cartan's lemma on holomorphic matrices.

Lemma 2 *Let $\mathfrak{G}$ be a complex connected Lie Group, and let (K_1, K_2) be a good pair, and let $g \in \mathcal{A}(K_{1,2}, \mathfrak{G})$. Then, for $j = 1, 2$ there are $g_j \in \mathcal{A}(K_j, \mathfrak{G})$ such that $g = g_2 \cdot g_1$ on $K_{1,2}$.*

For a proof, see [2], Lemma 4.4, where it is assumed that $\mathfrak{G} = GL_n(\mathbb{C})$, and each component of the intersection is star shaped, but the proof is valid for general $\mathfrak{G}$.

If $\mathcal{P} \subset \mathbb{C}$ is a finite set, we will let $\mathcal{A}^{\mathcal{P}}(K, \mathbb{C}^n)$ denote the closed subspace of the Banach space $\mathcal{A}(K, \mathbb{C}^n)$ consisting of those functions which vanish at each $p \in K \cap \mathcal{P}$. We will require the following version of the solution of the additive Cousin problem continuous to the boundary (the case with $\mathcal{P} = \emptyset$ is in fact used in the proof of Lemma 2).

Lemma 3 *Let K_1, K_2 be compact subsets of the plane such that $\overline{K_1 \setminus K_2} \cap \overline{K_2 \setminus K_1} = \emptyset$, and let $\mathcal{P}$ be a finite subset of the plane. There exist bounded linear* *maps* $T_j : \mathcal{A}^{\mathcal{P}}(K_{1,2}, \mathbb{C}) \to \mathcal{A}^{\mathcal{P}}(K_j, \mathbb{C})$ *such that for any function* f *in* $\mathcal{A}^{\mathcal{P}}(K_{1,2}, \mathbb{C})$ *we have on* $K_{1,2}$:

$$T_1 f + T_2 f = f, \tag{1}$$

Proof We reduce the problem to a $\overline{\partial}$ equation in the standard way. Let χ be a smooth cutoff which is 1 near $\overline{K_1 \setminus K_2}$ and 0 near $\overline{K_2 \setminus K_1}$. Let $\lambda := f.\frac{\partial \chi}{\partial \overline{z}}$, so that $\lambda \in \mathcal{A}(K_{1,2}, \mathbb{C})$. Let

$$\Lambda_f(z) = \frac{1}{2\pi i} \int_{K_{1,2}} \frac{\lambda(\zeta)}{\zeta - z} d\overline{\zeta} \wedge d\zeta.$$

Let q_f be the Lagrange interpolation polynomial with the property that $q_f(p) = \Lambda_f(p)$ for $p \in \mathcal{P}$. Observe that both Λ_f and q_f are linear in f and continuous in the sup norm when restricted to compact sets.

We can now define (assuming $(1 - \chi).f = 0$ where $\chi = 1$ even if f is not defined):

$$(T_1 f)(z) = (1 - \chi(z)) \,.\, f(z) + \Lambda_f(z) - q_f(z)$$

and (assuming $\chi.f = 0$ where $\chi = 0$ even if f is not defined):

$$(T_2 f)(z) = \chi(z) \,.\, f(z) - \Lambda_f(z) + q_f(z).$$

Since $T_j f$ is clearly holomorphic (resp. continuous) where f is, the result follows. □

We will use the following standard result regarding Banach spaces, which can be proved by iteration (see [10] pp. 397–398):

Lemma 4 *In a metric space* X, *let* $B_X(p, r)$ *denote the open ball in* X *of radius* r *centered at* p. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be Banach spaces and let* $\Phi : B_{\mathcal{E}}(p, r) \to \mathcal{F}$ *be a* $\mathcal{C}^1$ *map. Suppose there is a constant* $C > 0$ *such that*:

- *for each* $h \in B_{\mathcal{E}}(p, r)$, *the linear operator* $\Phi'(h) : \mathcal{E} \to \mathcal{F}$ *is surjective and the equation* $\Phi'(h)u = g$ *can be solved for* u *in* $\mathcal{E}$ *for all* g *in* $\mathcal{F}$ *with the estimate* $\|u\|_{\mathcal{E}} \leq C \|g\|_{\mathcal{F}}$.
- *for any* h_1 *and* h_2 *in* $B_{\mathcal{E}}(p, r)$ *we have* $\left\|\Phi'(h_1) - \Phi'(h_2)\right\| \leq \frac{1}{2C}$.

Then,

$$\Phi(B_{\mathcal{E}}(p, r)) \supset B_{\mathcal{F}}\left(\Phi(p), \frac{r}{2C}\right).$$

We will use the three lemmas above to give a proof of the following result (the main component of which is due to Rosay, see [13], and remarks in [12]) regarding the solution of a non-linear Cousin problem (see Lemma 4.5 of [2].) It is simply a translation of Theorem 4 to coordinates.

Lemma 5 *Let* ω *be an open subset of* $\mathbb{C}^n$ *and* $\mathfrak{F} : \omega \to \mathbb{C}^n$ *be a holomorphic immersion. Assume that* $\mathfrak{F}$ *preserves the last coordinate, i.e.,* $\mathfrak{F}$ *is of the form* $\mathfrak{F}(z_1, \ldots, z_n) = (F(z_1, \ldots, z_n), z_n)$ *for some map* $F : \omega \to \mathbb{C}^{n-1}$.

Let (K_1, K_2) *denote a good pair of compact subsets of* $\mathbb{C}$, *let* $\mathcal{P}$ *be a finite subset of* $K = K_1 \cup K_2$, *and suppose for each* $p \in K_2 \setminus K_1$ *we are given a point* $q(p) \in \mathbb{C}^{n-1}$. *Let* $u_1 \in \mathcal{A}(K_1, \mathbb{C}^n)$ *be such that*

- $u_1(K_{1,2}) \subset \omega$, *and*
- u_1 *is of the form* $u_1(z) = (t_1(z), z)$ *where* $t_1 : K_1 \to \mathbb{C}^{n-1}$.

Then, given any $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $u_2 \in \mathcal{A}(K_2, \mathbb{C}^n)$ *be such that*

- $\|u_2 - \mathfrak{F} \circ u_1\| < \delta$ *on* $K_2 \cap K_2$,
- *for* $p \in \mathcal{P} \cap (K_{1,2})$, *we have* $u_2(p) = \mathfrak{F}(u_1(p))$,
- *for* $p \in K_2 \setminus K_1$ *we have* $u_2(p) = (q(p), p) \in \mathbb{C}^n$, *and*
- $u_2(z) = (t_2(z), z)$, *where* $t_2 : K_2 \to \mathbb{C}^{n-1}$

then for $j = 1, 2$ *there exist* $v_j \in \mathcal{A}^{\mathcal{P}}(K_j, \mathbb{C}^n)$ *such that*

- $\|v_j\| < \epsilon$,
- $u_2 + v_2 = \mathfrak{F}(u_1 + v_1)$, *and*
- *the last component function of each of* v_1 *and* v_2 *is* 0.

Proof In order to apply Lemma 4 we choose the Banach spaces $\mathcal{E}$, $\mathcal{F}$ and the map Φ as follows:

- For $j = 1, 2$, let $\mathcal{B}_j$ be the closed subspace of the Banach space $\mathcal{A}^{\mathcal{P}}(K_j, \mathbb{C}^n)$ consisting of those maps whose last component function is 0, i.e., $\mathcal{B}_j = \mathcal{A}^{\mathcal{P}}(K_j, \mathbb{C}^{n-1}) \oplus 0$. We now let $\mathcal{E}$ the Banach space $\mathcal{B}_1 \oplus \mathcal{B}_2$, which we endow with the norm $\|\cdot\|_{\mathcal{E}} := \max(\|\cdot\|_{\mathcal{A}(K_1,\mathbb{C}^n)}, \|\cdot\|_{\mathcal{A}(K_2,\mathbb{C}^n)})$.
- Let $\mathcal{F}$ be the Banach space $\mathcal{A}^{\mathcal{P}}(K_{1,2}, \mathbb{C}^{n-1})$.
- Let $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ denote the projection on the first $n-1$ coordinates, and let the open subset $\mathcal{U}$ of $\mathcal{E}$ be given by $\{(w_1, w_2) : (u_1 + w_1)(K_{1,2}) \subset \omega\}$. (Observe that $\{0\} \times \mathcal{B}_2 \subset \mathcal{U}$.) Let $P = (P_1, \ldots, P_n)$ be an n-tuple of polynomials such that (i) $P_n(z) \equiv z$, (ii) for $p \in K_{1,2} \cap \mathcal{P}$, we have $P(p) = \mathfrak{F}(u_1(p))$, and (iii) for $p \in K_2 \setminus K_1$ we have $(P_1(p), \ldots, P_{n-1}(p)) = q(p)$.
  Let the map $\Phi : \mathcal{U} \to \mathcal{F}$ be given by

$$\Phi(w_1, w_2) := \pi \circ \left[(P + w_2)|_{K_{1,2}} - \mathfrak{F} \circ ((u_1 + w_1)|_{K_{1,2}})\right].$$

  (Observe that, since $\mathfrak{F}$ preserves the last coordinate, the last coordinate function of $(P + w_2)|_{K_{1,2}} - \mathfrak{F} \circ ((u_1 + w_1)|_{K_{1,2}})$ is actually 0. So the precomposition with π simply drops a coordinate which is identically 0.)

A computation shows that $\Phi'(w_1, w_2)$ is the bounded linear map from $\mathcal{E}$ to $\mathcal{F}$ given by:

$$(v_1, v_2) \mapsto \pi \circ \left[v_2|_{K_{1,2}} - \mathfrak{F}'((u_1 + w_1)|_{K_{1,2}})(v_1|_{K_{1,2}})\right].$$

Observe that w_2 plays no role in this expression, and therefore $\Phi'(w_1, w_2) \in BL(\mathcal{E}, \mathcal{F})$ is in fact a smooth function of w_1 alone, and we will henceforth denote it by: $\Phi'(w_1, *)$.

Let $\mathfrak{G} \subset GL_n(\mathbb{C})$ be the complex Lie subgroup of matrices of the form:

$$\begin{pmatrix} A & b \\ \mathbf{0} & 1 \end{pmatrix},$$

where $A \in GL_{n-1}(\mathbb{C})$, b is an $1 \times (n-1)$ column vector, and $\mathbf{0}$ is the zero row vector of size $(n-1) \times 1$. It is easy to see that $\mathfrak{G}$ is connected.

We construct a right inverse to $\Phi'(u_1, *)$. Let $\gamma = \mathfrak{F}' \circ \big(u_1|_{K_{1,2}}\big)$. Since $\mathfrak{F}$ preserves the last coordinate, $\gamma \in \mathcal{A}(K_{1,2}, \mathfrak{G})$, and thanks to Lemma 2 above, we may write $\gamma = \gamma_2 \cdot \gamma_1$, where $\gamma_j \in \mathcal{A}(K_j, \mathfrak{G})$. (We henceforth suppress the restriction signs.) Let $j : \mathcal{F} \to \mathcal{A}\big(K_{1,2}, \mathbb{C}^n\big)$ be the continuous inclusion induced by the map $\mathbb{C}^{n-1} \to \mathbb{C}^n$ given by $(z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{n-1}, 0)$. For $g \in \mathcal{F}$, let

$$S(g) = (-\gamma_1^{-1} T_1(\gamma_2^{-1} j(g)), \gamma_2 T_2(\gamma_2^{-1} j(g))),$$

where T_1, T_2 are as in equation 1 above (and extended to $\mathbb{C}^n$ componentwise). Observe that $-\gamma_1^{-1} T_1(\gamma_2^{-1} j(g))$ and $\gamma_2 T_2(\gamma_2^{-1} j(g))$ vanish at each $p \in \mathcal{P}$ and they belong to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. It is easy to see that $S : \mathcal{F} \to \mathcal{E}$ is a bounded linear operator, and a computation shows that $\Phi'(u_1, *) \circ S = \mathbb{I}_{\mathcal{F}}$. Choose $\theta > 0$ so small so that if $w_1 \in B_{\mathcal{F}}(u_1, \theta)$,

(1) $\left\|\Phi'(w_1, *) - \Phi'(u_1, *)\right\|_{\mathrm{op}} < \frac{1}{8\|S\|}$ (possible by continuity), and
(2) the equation $\Phi'(w_1, *)u = g$ can be solved with the estimate $\|u\| \le 2\,\|S\|\,\|g\|$, (possible from the fact that a small perturbation of a surjective linear operator is still surjective).

Consequently, if $\epsilon < \theta$ and $\tilde{u}_2 \in \mathcal{B}_2$, for the ball $B_{\mathcal{E}}((0, \tilde{u}_2), \epsilon)$ the hypothesis of Lemma 4 are verified with $C = 2\,\|S\|$. We have therefore,

$$\begin{aligned}\Phi\big(B_{\mathcal{E}}((0, \tilde{u}_2), \epsilon)\big) &\supset B_{\mathcal{F}}\left(\Phi(0, \tilde{u}_2), \frac{\epsilon}{2C}\right) \\ &= B_{\mathcal{F}}\left(P + \tilde{u}_2 - \mathfrak{F}(u_1), \frac{\epsilon}{2C}\right).\end{aligned}$$

So, if $\|P + \tilde{u}_2 - F(u_1)\| < \frac{\epsilon}{4C}$, we have $0 \in \Phi(B_{\mathcal{E}}((0, \tilde{u}_2), \epsilon))$. This is exactly the conclusion required, since any u_2 such that $u_2(p) = P(p)$ for each $p \in K_2 \cap \mathcal{P}$ can be written as $u_2 = P + \tilde{u}_2$ for some $\tilde{u}_2 \in \mathcal{B}_2$. □

We will now prove Theorem 4.

Proof We omit the restriction signs on maps for notational clarity. For $j = 1, 2$ let V_j be a ϕ-adapted neighborhood of $s(K_j)$, and let $\mathfrak{F}_j : V_j \to \mathbb{C}^n$ be a coordinate system such that $\mathfrak{F}_j(z) = (F_j(z), \phi(z))$. Let $\mathfrak{F} = \mathfrak{F}_2 \circ \mathfrak{F}_1^{-1}$ be the associated transition function. Then $\mathfrak{F}$ is a biholomorphism from the open set $\omega = \mathfrak{F}_1(V_1 \cap V_2)$ onto the open set $\mathfrak{F}_2(V_2 \cap V_1)$, and $\mathfrak{F}$ preserves the last coordinate, i.e., $\mathfrak{F}$ is of the form $\mathfrak{F}(z_1, \ldots, z_n) = (F(z_1, \ldots, z_n), z_n)$ for some map $F : \omega \to \mathbb{C}^{n-1}$. Any section of ϕ over K_j is represented in the coordinate system $\mathfrak{F}_j$ by a map of the form $t_j : K_j \to \mathbb{C}^n$, where $t_j(z) = (\tilde{t}_j(z), z)$, with $\tilde{t}_j$ a map from K_j to $\mathbb{C}^{n-1}$. Also, for $j = 1, 2$, given maps $t_j : K_j \to \mathbb{C}^n$ of the form $t_j = (\tilde{t}_j, z)$, they glue together to form a section over $K_1 \cup K_2$ (i.e. there is a section λ of ϕ over $K_1 \cup K_2$ such that $t_j = \mathfrak{F}_j \circ \lambda$) iff $t_2 = \mathfrak{F} \circ t_1$.

Since $B \cap K_1 = \emptyset$ by hypothesis, the pair of compact sets $(K_1, K_2 \cup B)$ is good. Let $\epsilon > 0$, and let $u_1 = \mathfrak{F}_1 \circ s$. Observe that $u_1(K_{1,2}) \subset \omega$, and u_1 is of the form $u_1(z) =$

$(t_1(z), z)$, where $t_1 : K_1 \to \mathbb{C}^{n-1}$. Then Lemma 5 gives a $\delta > 0$ corresponding to u_1, the good pair $(K_1, K_2 \cup B)$, ω and $\mathfrak{F}$.

Let $w_2 = \mathfrak{F}_2 \circ (s|_{K_2})$, then $w_2 \in \mathcal{A}(K_2, \mathbb{C}^n)$, and is of the form $w_2(z) = (\tilde{w}_2(z), z)$. Thanks to the hypothesis regarding uniform approximation of functions in $\mathcal{A}(K_2, \mathbb{C})$ by functions in $\mathcal{A}(K_2 \cup B, \mathbb{C})$, we can find a $u_2 \in \mathcal{A}(K_2 \cup B, \mathbb{C}^n)$ of the same form as w_2 such that $\|u_2 - \mathfrak{F}(u_1)\| < \delta$ on $K_{1,2}$ (Since $\mathfrak{F}(u_1) = w_2$ on $K_{1,2}$.) Further we may assume that for $p \in \mathcal{P} \cap K_2$, we have $u_2(p) = w_2(p)$ and that the last coordinate function of u_2 is z.

Then, by Lemma 5, there is a $v_1 \in \mathcal{A}^{\mathcal{P}}(K_1, \mathbb{C}^n)$ and a $v_2 \in \mathcal{A}^{\mathcal{P}}(K_2 \cup B, \mathbb{C}^n)$, such that $\|v_j\| < \epsilon$ and $u_2 + v_2 = \mathfrak{F}(u_1 + v_1)$, and the last coordinate functions of the v_j are 0's. Hence the maps $u_1 + v_1$ and $u_2 + v_2$ glue together to form a section of ϕ (which we call $\tilde{s}_\epsilon$) given by:

$$\tilde{s}_\epsilon := \begin{cases} \mathfrak{F}_1^{-1}(u_1 + v_1), & \text{on } K_1 \\ \mathfrak{F}_2^{-1}(u_2 + v_2), & \text{on } K_2 \text{ and near } K_2 \cap B. \end{cases}$$

Clearly, $\tilde{s}_\epsilon$ is in $\mathcal{A}_\phi(K_1 \cup K_2, \mathcal{M})$, and extends to a holomorphic section near $K_2 \cap B$. Moreover, $\text{dist}(\tilde{s}_\epsilon, s) = O(\epsilon)$. By construction, we have $s_\epsilon(p) = s(p)$. Therefore, given $\eta > 0$, we can find s_η with required properties. □

3 Sets with Property A_3

In this section we prove Theorems 1 and 3. In each of these we show that a certain set K has property A_3. As usual we will let $\mathcal{M}$ be a complex manifold, $\phi : \mathcal{M} \to \mathbb{C}$ a holomorphic submersion such that $\phi(\mathcal{M}) \supset K$, $\mathcal{P}$ a finite subset of K, and s a section of ϕ, $s \in \mathcal{A}_\phi(K, \mathcal{M})$. We let $\epsilon > 0$, and want to show that there is an $s_\epsilon \in \mathcal{O}_\phi(K, \mathcal{M})$ such that $\text{dist}(s, s_\epsilon) < \epsilon$ and $s_\epsilon(p) = s(p)$ for each $p \in \mathcal{P}$.

3.1 Zero-Dimensional Sets

Let $\dim(K) = 0$. If K is not finite, it can be written as a disjoint union of finitely many singletons and a closed subset C homeomorphic to the Cantor middle-third set. (See e.g., [1], pp. 108–109.) In particular, $\mathbb{C} \setminus K$ is connected, and K has property A_1.

Observe that $s \in \mathcal{A}_\phi(K, \mathcal{M})$ is simply a continuous section of ϕ over K. We can cover K by finitely many sets $\{U_j\}$ such that each $s(\overline{U_j})$ has a ϕ-adapted neighborhood in $\mathcal{M}$. We can choose a refinement $\{V_j\}$ of this cover, such that the V_j's are pairwise disjoint. Observe that $K_j := K \cap \overline{V_j}$ has property A_1. Now, with respect to the ϕ-adapted coordinate system around $s(K_j)$, the map s has a representation on K_j of the form $s(z) = (\tilde{s}(z), z)$, where $\tilde{s}$ is continuous and takes values in $\mathbb{C}^{n-1}$. Approximating $\tilde{s}$ by a holomorphic map in a neighborhood of K_j our result follows.

3.2 Proof of Theorem 1

Thanks to the previous section, we only need to consider one-dimensional sets. We first introduce some combinatorial preliminaries. Let vw be an edge in a graph Γ. We can construct a new graph Γ' with one more vertex by "splitting the edge vw." More formally, if V is the vertex set of Γ, and E its edge set, the vertex set of Γ' is $V \cup \{u\}$ (where $u \notin V$), and the edge set is $(E \setminus \{vw\}) \cup \{vu, uw\}$.

Recall that a graph is *n-colorable*, if there is an assignment of n colors to its vertices so that no two adjacent vertices have the same color. By a classical result of König (see [8], Theorem 12.1), the condition that a graph is 2-colorable is that it does not have a circuit (closed non-self intersecting path) of odd length. We now have the following:

Lemma 6 *Given any finite graph G, there are edges $e_1, \ldots, e_k$ such that the graph G' obtained after successively splitting these edges is 2-colorable.*

Proof Let V be the vector space over the field $\mathbb{Z}/2\mathbb{Z}$ with basis the set of edges of G. Given a circuit C in G, traversing the edges $x_1, x_2, \ldots, x_m$, associate with it an element c of V given by $c = x_1 + x_2 + \cdots + x_m$. Let Z be the subspace of V spanned by all c for each circuit C. Pick a basis $c_1, \ldots, c_k$. Since any circuit can be written as a linear combination of the c_j's it is sufficient to split any edge of those c_j which have an odd number of summands. Moreover, from the fact that the c_j are linearly independent over $\mathbb{Z}/2\mathbb{Z}$ it follows that each c_j has an edge not contained in any c_k, $k \neq j$. The result follows. □

Now we can prove Theorem 1. Note that $K^\circ = \emptyset$, so s is simply a continuous section of ϕ over K.

For $z \in K$, there is a neighborhood U_z of z in K such that $s(\overline{U_z})$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Select a finite subcover $\mathcal{F}$ of $\{U_z\}_{z \in K}$, and let $\mathcal{G}$ be a refinement of $\mathcal{F}$ which has only double intersections (this is possible since $\dim(K) = 1$, and K is connected). We can write $\mathcal{G} = \{V_1, \ldots, V_M\}$.

We associate to $\mathcal{G}$ a graph $N_{\mathcal{G}}$ by taking the nerve: the vertices V_i and V_j are joined by an edge in $N_{\mathcal{G}}$ iff $V_i \cap V_j \neq \emptyset$. Let V and W be open sets in $\mathcal{G}$ such that $V \cap W \neq \emptyset$. We define a new open cover $S_{VW}(\mathcal{G})$ of K in the following way. Replace the sets V and W by V', U and W', where $V' \subset V$, $W' \subset W$, and U is a neighborhood of $\overline{V \cap W}$ such that $V' \cup U \cup W' = V \cup W$, and $V \cap U \neq \emptyset$, $U \cap W \neq \emptyset$ but $V \cap W = \emptyset$. If U is chosen small enough, then $(\mathcal{G} \setminus \{V, W\}) \cup \{V', U, W'\}$ is again an open cover of K with only double intersections. This is our $\mathcal{G}' = S_{VW}(\mathcal{G})$. At the level of nerves $N_{\mathcal{G}'}$ is obtained by splitting the edge VW in the graph $N_{\mathcal{G}}$, as in Lemma 6. Thanks to Lemma 6 we can obtain after finitely many rounds of edge-splitting a new cover $\mathcal{H}$, with only double intersections such that the corresponding nerve $N_{\mathcal{H}}$ is 2-colorable. It also follows that for each $U \in \mathcal{H}$, the set $s(\overline{U})$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Further, by hypothesis, after passing to a refinement, we can assume that whenever an intersection $U \cap U'$ of two sets in $\mathcal{H}$ is non-empty, it is contractible. Let us call the colors used in coloring $N_{\mathcal{H}}$ red and blue. We can now define

$$K_1 = \bigcup_{\substack{U \in \mathcal{H} \\ U \text{ red}}} \overline{U}, \quad \text{and} \quad K_2 = \bigcup_{\substack{U \in \mathcal{H} \\ U \text{ blue}}} \overline{U}.$$

It is easy to verify that (K_1, K_2) is a good pair, and for $j = 1, 2$ the set $s(K_j)$ has a ϕ-adapted neighborhood in $\mathcal{M}$.

We also define sets $K_1' \subset K_1$ and $K_2' \subset K_2$ in the following way. For $U, U' \in \mathcal{H}$, let $\mathcal{W}_{UU'}$ be a simply connected neighborhood of $U \cap U'$ in $\mathbb{C}$ with $\mathcal{C}^2$ boundary. Furthermore, we can assume that $\overline{\mathcal{W}_{UU'} \cap K}$ is contractible and $s(\overline{\mathcal{W}_{UU'} \cap K})$ is contained in a ϕ-adapted open set of $\mathcal{M}$. We set

$$\mathcal{W} = \bigcup_{\substack{U,U' \in \mathcal{H} \\ U \cap U' \neq \emptyset}} \mathcal{W}_{UU'},$$

so that $\mathcal{W}$ is a neighborhood in $\mathbb{C}$ of the points in K which are contained in two sets of the cover $\mathcal{H}$.

Let $V \in \mathcal{H}$. We set:

$$V^s = V \setminus \overline{W}$$

So that $V^s \subset V$, and points of V^s do not belong to any other set of $\mathcal{H}$ apart from V. We set:

$$K_1' = \bigcup_{\substack{U \in \mathcal{H} \\ U \text{ red}}} \overline{U^s}, \quad \text{and} \quad K_2' = \bigcup_{\substack{U \in \mathcal{H} \\ U \text{ blue}}} \overline{U^s}.$$

We now apply Theorem 4 to the good pair (K_1, K_2), and obtain an $s_1 \in \mathcal{O}_\phi(K \cup \mathcal{B}_1, \mathcal{M})$, where $\mathcal{B}_1$ is a neighborhood of K_1' in $\mathbb{C}$, such that $\text{dist}(s, s_1) < \frac{\epsilon}{3}$, and $s(p) = s_1(p)$ for $p \in \mathcal{P}$. We can further assume that $\mathcal{B}_1$ is a disjoint union of simply connected neighborhoods of the sets V^s for V red, $\partial\mathcal{B}_1$ is smooth, and $\partial\mathcal{B}_1$ and $\partial\mathcal{W}$ meet transversely at each point of intersection.

Observe now that $(K_1 \cup \mathcal{B}_1, K_2)$ is a good pair, and we can apply Theorem 4 to it. We obtain a $s_2 \in \mathcal{A}_\phi(K \cup \mathcal{B}_1 \cup \mathcal{B}_1, \mathcal{M})$ with $\text{dist}(s_2, s_1) < \frac{\epsilon}{3}$, and $s_2(p) = s_1(p)$ for $p \in \mathcal{P}$. As before, $\mathcal{B}_2$ is a disjoint union of simply connected neighborhoods of the sets V^s for V blue, $\partial\mathcal{B}_2$ is smooth, and $\partial\mathcal{B}_2$ and $\partial\mathcal{W}$ meet transversely at each point of intersection.

We set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ so that $\partial\mathcal{B} = \partial\mathcal{B}_1 \cup \partial\mathcal{B}_2$ meets $\partial\mathcal{W}$ transversely, so that $\partial\mathcal{B} \cap \partial\mathcal{W}$ is a finite set. Let $\mathcal{U}$ be a simply connected neighborhood of K in $\mathbb{C}$ such that all points of $\partial\mathcal{B} \cap \partial\mathcal{W}$ lie outside $\mathcal{U}$ and $\partial\mathcal{U}$ meets both $\partial\mathcal{B}$ and $\partial\mathcal{W}$ transversely. Note that $K_1 \cup K_2 \subset \mathcal{W} \setminus \mathcal{B}$. Let

$$L_1 = \partial\mathcal{U} \cap \partial\mathcal{B},$$

and

$$L_2 = \partial\mathcal{U} \cap \partial\mathcal{W}.$$

It is easy to verify that (L_1, L_2) is a good pair, and each of $s(L_j)$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Setting $L = L_1 \cup L_2$, we apply Theorem 4 to (L_1, L_2) to obtain a map $s_3 \in \mathcal{A}_\phi(L \cup \mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is a neighborhood of $K_1 \cap K_2$ in $\mathbb{C}$, $\text{dist}(s_3, s_2) < \frac{\epsilon}{3}$, and $s_3(p) = s_2(p)$ for all $p \in \mathcal{P}$. Clearly, $s_3 \in \mathcal{O}_\phi(K, \mathcal{M})$, and we are done.

3.3 Proof of Theorem 3

In Sect. 3.3.1 we record a result that will be used in the proof. In Sect. 3.3.2 we use Theorems 5 and 4 to give a proof of Theorem 3 in the special case when $N = 1$, i.e.

K is the closure of a smoothly bounded domain. As in the proof of Theorem 1 this uses a "Triple bumping." Finally, in Sect. 3.3.3 we complete the proof by reducing the general case to the case considered in Sect. 3.3.2.

3.3.1 Arcs in Complex Manifolds

If X is a differentiable manifold, and α is an arc (continuous injective map from $[0, 1]$) which is at least $\mathcal{C}^1$, we will say α is *embedded* if for each $t \in [0, 1]$, we have $\alpha'(t) \neq 0$. A proof of the following result, required in the proof of Theorem 3, can be found in [2], Theorem 2.

Theorem 5 *Let $\mathcal{M}$ be a complex manifold, and α be an arc in $\mathcal{M}$. Assume that*

(1) *there is a complex-valued submersion ϕ defined in a neighborhood of α in $\mathcal{M}$ such that $\phi \circ \alpha$ is a $\mathcal{C}^1$ embedded arc in $\mathbb{C}$.*
(2) *there is a finite subset $P \subset [0, 1]$ such that α is $\mathcal{C}^3$ on $[0, 1] \setminus P$.*

Then α has a ϕ-adapted neighborhood in $\mathcal{M}$.

3.3.2 Case of a Smooth Domain

In this section we prove the following special case of Theorem 3 (Compare with Proposition 2):

Proposition 4 *Let $\Omega \Subset \mathbb{C}$ be a domain with $\mathcal{C}^2$ boundary. Then $\overline{\Omega}$ has property A_3.*

The first step in the proof is the following application of Sard's Theorem.

Lemma 7 *There is a unit vector $\mathbf{v}$ in the plane such that*:

- *every straight line in the plane parallel to $\mathbf{v}$ meets $\partial\Omega$ in only finitely many points.*
- *the number of straight lines parallel to $\mathbf{v}$ which are tangent to $\partial\Omega$ is finite.*

In fact, these hold for almost all unit vectors $\mathbf{v}$ in the unit circle S^1 (with respect to the standard measure on S^1).

Proof Fix an arbitrary orientation on $\partial\Omega$, and define the Gauss map $G : \partial\Omega \to S^1$ by mapping the point $z \in \partial\Omega$ to the unit tangent vector $G(z)$ to $\partial\Omega$ at the point z. This is a $\mathcal{C}^1$ map, and the set of its critical values is of measure 0 by Sard's Theorem. If $\mathbf{v}$ is any regular value of G, such that $-\mathbf{v}$ is also a regular value, it follows that the sets $G^{-1}(\mathbf{v})$ and $G^{-1}(-\mathbf{v})$ are discrete in $\partial\Omega$ (since G is a diffeomorphism near each of them). Since $\partial\Omega$ is compact, it follows that the sets $G^{-1}(\pm\mathbf{v})$ are finite. From this, the conclusions follow immediately. □

After a rotation if required, we will assume that $\mathbf{v}$ is vertical, i.e., $\mathbf{v} = \pm i$. Now we introduce some notation. For $c \in \mathbb{R}$, denote by $L(c)$ the vertical straight line $\Re(z) = c$ in $\mathbb{C}$, and for $a < b$, denote by $M[a, b]$ the vertical strip $\{z \in \mathbb{C}: a \leq \Re(z) \leq b\}$. Also, set $l(c) = L(c) \cap \overline{\Omega}$ and $m[a, b] = M[a, b] \cap \overline{\Omega}$. We will assume without loss of generality that $\Omega \subset M[0, 1]$.

Thanks to Lemma 7 and the choice $\mathbf{v} = \pm i$, it follows that for each c, the set $l(c)$ has only finitely many components, each of which is either a point, or a line segment. Also, $l(c) \cap \partial\Omega$ is finite. We now make the following observation.

Lemma 8 *There is a* $\delta > 0$ *such that for each* $c \in [0, 1]$, *the set* $s(m[c - \delta, c + \delta])$ *has a* ϕ*-adapted neighborhood in* $\mathcal{M}$.

Proof First we show that for each $c \in [0, 1]$, the set $s(l(c))$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Since s is injective (it has a left-inverse ϕ), it is sufficient to show that each component of $l(c)$ has a ϕ-adapted neighborhood in $\mathcal{M}$. For a component of $l(c)$ that reduces to a point, this is trivial. Therefore, consider a component which is a (vertical) line segment. We can think of this component as an arc in the plane, and parameterize it as $\lambda(t) = z_0 + iat$, where z_0 is the lower end point of the segment, and a is a positive real number. Then $s \circ \lambda$ is a continuous arc in $\mathcal{M}$, which is real analytic except at finitely many points, and ϕ is a submersion from $\mathcal{M}$ to $\mathbb{C}$ such that $\phi \circ (s \circ \lambda) = \lambda$. Thanks to Proposition 5 above, $s(\lambda)$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Therefore, $s(l(c))$ has a ϕ-adapted neighborhood in $\mathcal{M}$.

It follows that there is a δ_c such that $s(m[c - \delta_c, c + \delta_c])$ has a ϕ-adapted neighborhood in $\mathcal{M}$. The uniform choice of δ follows by compactness. □

We will also require the following simple fact (a proof may be found in [2], Observation 4.8).

Lemma 9 *Let* u *and* v *be real valued* $\mathcal{C}^1$ *functions defined on a neighborhood of* 0 *in* $\mathbb{R}$ *such that for each* x, *we have* $u(x) < 0 < v(x)$. *Then there is an* $\eta > 0$ *such that for* $0 < \theta \leq \eta$, *the vertical strip*

$$S := \left\{(x, y) \in \mathbb{R}^2 \colon x \in [-\theta, \theta], u(x) \leq y \leq v(x)\right\}$$

is star shaped with respect to the origin.

The proof of Proposition 3.3.2 will parallel that of Proposition 1 in that both require a "Triple bumping", i.e., three successive applications of Proposition 4. However, the good pairs are obtained by different methods.

Let $\mathcal{E} \subset [0, 1]$ be the set of c such that the line $L(c)$ is tangent to some component of $\partial\Omega$. $\mathcal{E}$ is finite by Lemma 7. If $c \notin \mathcal{E}$, $L(c)$ meets $\partial\Omega$ transversely at each point of intersection, so that (1) each component of $l(c)$ is a line segment, and (2) (by Lemma 9) there is a θ_c such that for $\epsilon \leq \theta_c$, each component of $m([c - \epsilon, c + \epsilon])$ is strongly star shaped.

By compactness, we can find $c_1 < c_2 < \cdots < c_M$, and $\eta_j > 0$, $j = 1, \ldots, M$, such that if $m_j = m[c_j - \eta_j, c_j + \eta_j]$, we have $\overline{\Omega} = \bigcup_{j=1}^{M} m_j$. We can assume that the $\eta_j < \frac{\delta}{100}$, where δ is as in Lemma 8. We see that for each j, $s(m_j)$ has a ϕ-adapted neighborhood in $\mathcal{M}$. We will further impose the following conditions on the m_j's

(1) Each point of $\overline{\Omega}$ is in at most two of the m_j's (this can be done by shrinking the η_j's.) Therefore, $m_j \cap m_k = \emptyset$ if $|j - k| > 1$.
(2) Each $c \in \mathcal{E}$ occurs in the list $\{c_j\}_{j=1}^{M}$.

(3) Each component of $m_j \cap m_{j+1}$ is strongly star shaped. Observe that by the previous step, $m_j \cap m_{j+1}$ does not contain any $l(c)$ for $c \in \mathcal{E}$. Therefore, this can be achieved by shrinking the η_j's.

Now let

$$K_1 = \bigcup_{j \text{ odd}} m_j, \quad \text{and} \quad K_2 = \bigcup_{j \text{ even}} m_j.$$

It is easy to see that (K_1, K_2) is a good pair. Since each $s(m_j)$ has a ϕ-adapted neighborhood, and K_1, K_2 are disjoint union of m_j's, it follows that each of $s(K_1)$ and $s(K_2)$ has a ϕ-adapted neighborhood in $\mathcal{M}$.

Let $I_1 \subset \partial\Omega \cap K_1$ and $I_2 \subset \partial\Omega \cap K_2$ be such that (1) $I_1 \cap K_2 = I_2 \cap K_2 = \emptyset$, and (2) Each connected component $\partial\Omega \setminus (I_1 \cup I_2)$ is contained in a vertical strip of width $\frac{\delta}{2}$, where δ is as in Lemma 8. (This is possible, since $\eta_j < \frac{\delta}{100}$.)

Let B be a neighborhood of I_1 such that $B \cap K_2 = \emptyset$. We apply Theorem 4 to the good pair (K_1, K_2) to obtain an $s_1 \in \mathcal{A}_\phi(\overline{\Omega} \cup B_1, \mathcal{M})$, where B_1 is a neighborhood of I_1 contained in B, such that $\text{dist}(s, s_1) < \frac{\epsilon}{3}$, and $s(p) = s_1(p)$ for $p \in \mathcal{P}$. Observe that for B_1 small enough, $s_1(K_1 \cup B_1)$ is contained in a ϕ-adapted open set of $\mathcal{M}$, and $(K_1 \cup B_1, K_2)$ is again a good pair. We now apply Theorem 4 again to this good pair to obtain an $s_2 \in \mathcal{A}_\phi(\overline{\Omega} \cup B_1 \cup B_2, \mathcal{M})$ where B_2 is a neighborhood of I_2, such that $\text{dist}(s_1, s_2) < \frac{\epsilon}{3}$, and $s_1(p) = s_2(p)$ for $p \in \mathcal{P}$.

Let $I_3 = \partial\Omega \setminus (I_1 \cup I_2)$. By construction, each connected component of I_3 is contained in a vertical strip of width $\frac{\delta}{2}$. Let Ω' be a domain with $\mathcal{C}^2$ boundary such that $\overline{\Omega} \cup B_1 \cup B_2 \supset \Omega' \supset \Omega \cup I_1 \cup I_2$ (i.e. Ω' is obtained by smoothly bumping Ω along I_1 and I_2.) We can assume that Ω' and Ω are so close that for δ as in Lemma 8, and any c, the set $s(M[c-\delta, c+\delta] \cap \overline{\Omega'})$ is contained in a ϕ-adapted open set in $\mathcal{M}$. We can now repeat the constructions that gave us K_1 and K_2 to obtain a good pair (L_1, L_2) such that (1) $L_1 \cup L_2 = \overline{\Omega'}$, (2) each of $s(L_1)$ and $s(L_2)$ has a ϕ-adapted neighborhood in $\mathcal{M}$, and (3) $I_3 \subset L_1 \setminus L_2$. We can now apply Theorem 4 to the good pair (L_1, L_2) to obtain an $s_3 \in \mathcal{A}_\phi(\overline{\Omega'} \cup C, \mathcal{M})$, where C is a neighborhood of I_3 in $\mathbb{C}$, such that $\text{dist}(s_3, s_2) < \frac{\epsilon}{3}$ and $s_3(p) = s_2(p)$, for each $p \in \mathcal{P}$. Then $s_3 \in \mathcal{O}_\phi(\overline{\Omega}, \mathcal{M})$, and we are done.

3.3.3 End of Proof of Theorem 3

Let K be of class $\mathfrak{C}_2$. Recall that $K = \bigcup_{i=1}^N \overline{\Omega_i}$, and for $i \neq j$, we have $\partial\Omega_i$ and $\partial\Omega_j$ meet at a set of finitely many points P_{ij}. Set $P = \bigcup_{i \neq j} P_{ij}$. We can refer to the points in P as *nonsmooth points* of ∂K.

The proof in this section is very similar to that in Sect. 3.3.2. The first step is to establish the following version of Lemma 7 for this case:

Lemma 10 *Let K be of class $\mathfrak{C}_2$. Then there is a unit vector* $\mathbf{v}$ *in the plane with the following properties*:

(1) *every straight line in the plane parallel to* $\mathbf{v}$ *meets ∂K in only finitely many points.*
(2) *The number of straight lines parallel to v which are tangent to ∂K at smooth points is finite.*

(3) *Let* $p \in P_{ij}$ *be a non-smooth point of* ∂K. *Then, the straight line through* p *parallel to* $\mathbf{v}$ *is transverse to both* $\partial\Omega_i$ *and* $\partial\Omega_j$ *at* p.

Proof Applying Lemma 7 separately to each $\partial\Omega_j$, we conclude that properties (1) and (2) hold for almost all unit vectors $\mathbf{v}$. Let p be a non-smooth point of ∂K, so that for some i, j, we have $p \in P_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Let $\mathbf{t}(p)$ be a common unit tangent vector to $\partial\Omega_i$ and $\partial\Omega_j$ at the point p. We can choose $\mathbf{v} \neq \pm\mathbf{t}(p)$ for all $p \in P_{ij}$ for all i and j. □

As in Sect. 3.3.2 we can assume that $\mathbf{v} = \pm i$. Let $L(c)$ and $M[a,b]$ have the same meaning as in the last section, and set $l'(c) = L(c) \cap K$, $m'[a,b] = M$. We can assume that $K \subset M[0,1]$.

We let $\mathcal{E}$ be the finite set of points $c \in [0,1]$ such that either (1) $L(c)$ is tangent to $\partial\Omega_i$ for some i, or (2) $L(c)$ passes through a nonsmooth point of ∂K. Arguing as in the previous section, we can find $c_1 < c_2 < \cdots < c_M$ and $\eta_j > 0$, $j = 1, \ldots, M$, such that if $m'_j = m'[c_j - \eta_j, c_j + \eta_j]$, we have $K = \bigcap_{j=1}^{M} m'_j$ and each $s(m'_j$ has a ϕ-adapted neighborhood in $\mathcal{M}$. We can further impose the following conditions, (the first three are just as in the last section, the last is a new condition):

(1) Each point in K is contained in at most two of the m'_j's, i.e., $m'_j \cap m'_k = \emptyset$ if $|j-k| > 1$.
(2) Each $c \in \mathcal{E}$ occurs among the $\{c_j\}_{j=1}^{M}$.
(3) Each component of $m'_j \cap m'_{j+1}$ is strongly star shaped (Note that, thanks to the last step, in this case we have ensured that $m'_j \cap m'_{j+1}$ does not contain any nonsmooth points.)
(4) Each nonsmooth point is contained in an m'_j with an even j. This can be ensured by introducing additional $c'_j s$. Observe that if a nonsmooth point $p \in m'_j$, then $p \notin m'_{j-1}$ and $p \notin m'_{j+1}$.

As in the proof of Proposition 4:

$$K_1 = \bigcup_{j \text{ odd}} m'_j, \quad \text{and} \quad K_2 = \bigcup_{j \text{ even}} m'_j.$$

It is easy to see that (K_1, K_2) is a good pair. Since each $s(m'_j)$ has a ϕ-adapted neighborhood, and K_1, K_2 are disjoint union of m'_j's, it follows that each of $s(K_1)$ and $s(K_2)$ has a ϕ-adapted neighborhood in $\mathcal{M}$. Moreover, the set P of nonsmooth points of ∂K is contained in $K_2 \setminus K_1$.

Let B be a neighborhood of the nonsmooth points P such that $B \cap K_1 = \emptyset$. Thanks to Theorem 4 we can find an $s' \in \mathcal{A}(K \cup B', \mathcal{M})$ (where B' is a neighborhood of P contained in B), such that $\operatorname{dist}(s, s') < \frac{\epsilon}{2}$, and $s'(p) = s(p)$ for each $p \in \mathcal{P}$.

Now we can find an open set Ω' with $\mathcal{C}^2$ boundary such that $\overline{\Omega'} \subset K \cup B'$ but $\Omega' \supset \Omega \cup P$. Then $s' \in \mathcal{A}(\overline{\Omega'}, \mathcal{M})$, and thanks to Proposition 4, we can find $s_\epsilon \in \mathcal{O}_\phi(\overline{\Omega'}, \mathcal{M})$ such that $\operatorname{dist}(s_\epsilon, s') < \frac{\epsilon}{2}$ and $s_\epsilon(p) = s'(p)$ for $p \in \mathcal{P}$. Since $s_\epsilon \in \mathcal{O}_\phi\,(K, \mathcal{M})$, the proof is complete.

4 Proof of Theorem 2

4.1 $\mathcal{A}$-equivalence

Let K and K' be compact subsets of $\mathbb{C}$. We will say that K and K' are $\mathcal{A}$-*equivalent* if there is a homeomorphism $\chi : K \to K'$ such that $\chi|_{K^\circ}$ is a conformal map of K° onto K'°. We will call χ an $\mathcal{A}$-*equivalence* from K to K'. A well-known example of $\mathcal{A}$-equivalence is the following ([14], Theorems IX.35 and IX.2):

Lemma 11 *Let Ω be a Jordan Domain, Then there is a domain ω in the plane bounded by* circles *such that $\overline{\Omega}$ and $\overline{\omega}$ are $\mathcal{A}$-equivalent.*

The significance of this notion in the current investigation is explained by the following observation.

Lemma 12 *Suppose that two compact sets K and L in $\mathbb{C}$ are $\mathcal{A}$-equivalent. If K has property A_2 and L has property A_1 then L has property A_2.*

Proof Let $\mathcal{M}$ be a complex manifold, $f \in \mathcal{A}(L, \mathcal{M})$, and $\mathcal{P}$ be a finite subset of K. We want to approximate f by maps f_n in $\mathcal{O}(L, \mathcal{M})$ such that $f_n(p) = f(p)$ for $p \in \mathcal{P}$. Let $\chi : K \to L$ be an $\mathcal{A}$-equivalence, let $\mathcal{Q} = \chi^{-1}(\mathcal{P})$, and let $g = f \circ \chi$. Then $g \in \mathcal{A}(K, \mathcal{M})$, and consequently there is a sequence $g_n \in \mathcal{O}(K, \mathcal{M})$ such that $g_n \to g$ uniformly, and $g_n(q) = g(q)$ for $q \in \mathcal{Q}$. Let $\zeta = \chi^{-1}$, so that $\zeta \in \mathcal{A}(L, \mathbb{C})$. Since L has property A_1, we can find $\zeta_n \in \mathcal{O}(L, \mathbb{C})$ such that $\zeta_n \to \zeta$ on L, with $\zeta_n(p) = \zeta(p)$ for $p \in \mathcal{P}$. Then $f_n := g_n \circ \zeta_n \in \mathcal{O}(L, \mathcal{M})$, $f_n \to f$ uniformly, and $f_n(p) = f(p)$ for $p \in \mathcal{P}$. □

4.2 Proof of Theorem 2

Thanks to Lemma 12 and Theorem 3, it is sufficient to prove the following result:

Theorem 6 *For each K in $\mathfrak{C}_0$ there is an L in $\mathfrak{C}_\omega$ such that K and L are $\mathcal{A}$-equivalent.*

We will require two results, the first from Combinatorics. A vertex v of a graph G is said to be a *cutpoint* of G if the graph $G^{\{v\}}$ obtained by removing from G the vertex v along with all edges incident at v has at least one more connected component than G has. (So, for example, if G is connected, $G^{\{v\}}$ is *not* connected.) We will need the following elementary fact.

Lemma 13 ([8], Theorem 3.4, p. 29) *Let G be a graph with more than one vertex. Then there are at least two vertices of G which are not cutpoints.*

The second result is the following boundary interpolation theorem for conformal maps, due to MacGregor and Tepper ([11], Theorem 1). $\Delta \subset \mathbb{C}$ is the open unit disc, and $\{z_1, z_2, \ldots, z_n\} \subset \partial\Delta$ and $\{w_1, w_2, \ldots, w_n\} \subset \partial\Delta$ are given finite subsets of the unit circle.

Proposition 5 *There is a function f which is analytic and univalent in the union of Δ and a neighborhood of $\{z_1, z_2, \ldots, z_n\}$ and continuous on $\overline{\Delta}$ such that $f(z_k) = w_k$ for $k = 1, \ldots, n$. Furthermore, $|f(z)| = 1$ if $|z| = 1$ and z is sufficiently near any of the points z_k, and also $f(\Delta) \subset \Delta$.*

For a compact connected set K in the plane, by the outer boundary we mean the boundary of the unbounded component of $\mathbb{C} \setminus K$ (this is also a component of ∂K.) We use Proposition 5 to prove the following lemma.

Lemma 14 *Let $\Omega \Subset \mathbb{C}$ be a Jordan domain, and let γ be its outer boundary. Suppose we are given a finite set of points $\{z_1, \ldots, z_n\}$ on γ and the same number of points $\{w_1, \ldots, w_n\}$ on the unit circle $\partial\Delta$. Then there is a continuous map $f : \overline{\Omega} \to \overline{\Delta}$ such that*

(1) *$f|_\Omega$ is conformal,*
(2) *$f(z_k) = w_k$, for $k = 1, \ldots, n$,*
(3) *let $W = f(\Omega)$. Then ∂W is $\mathcal{C}^\omega$, and at each w_k, ∂W is tangent to $\partial\Delta$.*

Proof By Lemma 11 there is a domain D bounded by circles and an $\mathcal{A}$-equivalence $f_0 : \overline{\Omega} \to \overline{D}$. After applying an inversion of the plane if required, we can assume further than the outer boundary γ of Ω is mapped onto the outer boundary of D, which we may assume is the unit circle $\partial\Delta$. Set $z'_k = f_0(z_k)$ for $k = 1, \ldots, n$.

Let D' be a simply connected open set in $\mathbb{C}$ with $\mathcal{C}^\infty$ boundary such that $\Delta \subset D'$, $\partial D' \cap \partial\Delta = \{z'_1, \ldots, z'_n\}$ where at each z'_k, the boundaries $\partial D'$ and $\partial\Delta$ are tangent to each other. Let $f_1 : D' \to \Delta$ be a conformal map of D' onto Δ. Since $\partial D'$ has $\mathcal{C}^\infty$ boundary, f_0 extends to a diffeomorphism of the closures. Set $z''_k = f_1(z'_k)$, and $D'' = f_1(D')$. Observe that $f_1 \circ f_0$ maps Ω to a subdomain Ω'' of Δ and the boundary $\partial\Omega''$ is tangent to $\partial\Delta$ at each point of intersection z''_k.

We now apply Proposition 5 to obtain a continuous $f_2 : \overline{\Delta} \to \overline{\Delta}$ such that $f_2(z''_k) = w_k$, f_2 is conformal on the union of Δ with a neighborhood of $\{z''_1, \ldots, z''_n\}$, and f_2 maps a piece of $\partial\Delta$ near each z''_k onto a piece of $\partial\Delta$ near w_k. It follows immediately that if $W = f_2(\Omega'')$, ∂W meets $\partial\Delta$ tangentially at each w_k. We set $f := f_2 \circ f_1 \circ f_0$. The properties claimed are easily verified. □

We now prove Theorem 6.

Proof It is clear that we only need to consider the case in which K is connected. We use induction on N, the number of summands of K.

When $N = 1$, the result is reduced to Lemma 11. Now suppose that the result has been proved for some $N \geq 1$, and let $K = \bigcup_{i=1}^{N+1} \overline{\Omega_j}$. Let G be a graph whose vertices v_i correspond to the sets $\overline{\Omega_i}$, and there is an edge connecting v_i and v_j iff $\overline{\Omega_i} \cap \overline{\Omega_j} \neq \emptyset$. Since we have assumed that K is connected, it follows that G is a connected graph. Thanks to Lemma 13 above, we can assume (after a renumbering of the vertices of G) that v_{N+1} is *not* a cutpoint of G. Let $K' = \bigcup_{i=1}^{N} \overline{\Omega_j}$, and let $P \subset K'$ be the finite set $K' \cap \overline{\Omega_{N+1}} = K' \cap \partial\Omega_{N+1}$. Then K' is connected, therefore is contained in exactly one connected component U of $\mathbb{C} \setminus \Omega_{N+1}$. Let $\gamma = \partial U$. Clearly γ is a connected component of $\partial\Omega_{N+1}$. It follows that the set $P \subset \gamma$. Moreover, as

$\overline{\Omega_{N+1}}$ is connected, it follows that Ω_{N+1} is contained in exactly one component of $\mathbb{C} \setminus K'$.

We claim that *we can assume that* γ *is the outer boundary of* Ω_{N+1}. To show this it is sufficient to show that for some $\mathcal{A}$-equivalence $\Phi : K \to \hat{K} \subset \mathbb{C}$, $\Phi(\gamma)$ is the outer boundary of $\Phi(\Omega_{N+1})$.

If γ not already the outer boundary of Ω_{N+1} let $z_0 \in U \setminus K'$, where $U \Subset \mathbb{C}$ is the component of $\mathbb{C} \setminus \Omega_{N+1}$ which contains K' (then $\gamma = \partial U$). Let $\rho > 0$ be small enough so that $B_{\mathbb{C}}(z_0, \rho) \Subset U \setminus K'$, and define the inversion $\Phi : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ by

$$\Phi(z) = \frac{\rho^2}{z - z_0}.$$

Then $\Phi(K)$ is contained in the ball $B_{\mathbb{C}}(0, \rho)$, and since z_0 is mapped to the point at infinity, it follows that $U \setminus K'$ is mapped to the unbounded component of $\mathbb{C} \setminus K$. Since $\gamma = \partial U$, we see that $\gamma \subset \partial(U \setminus K')$, so that $\Phi(\gamma)$ is the outer boundary of $\Phi(\Omega_{N+1})$.

Now, by induction hypothesis, there is an $L' \in \mathfrak{C}_\omega$, and an $\mathcal{A}$-equivalence $\chi' : K' \to L'$. Using Lemma 5 we will extend χ' to an $\mathcal{A}$-equivalence χ defined on $K = K' \cup \overline{\Omega_{N+1}}$.

Let us write $P = \{\zeta_1, \ldots, \zeta_n\}$, and let $\zeta_k' = \chi'(\zeta_k)$. Then the ζ_k''s lie at the boundary of a single connected component U of $\mathbb{C} \setminus L'$. Let U' be a simply connected domain, $U \subset U'$ such that $\partial U'$ is $\mathcal{C}^\omega$ and passes through each $\zeta_k' \in \partial U$, and further at each ζ_k', $\partial U'$ is tangent to ∂U, i.e. to ∂L. Let θ be a conformal map of U' onto the disc Δ. Then θ extends to a holomorphic map of a neighborhood of $\overline{U'}$. We set $w_k = \theta(\zeta_k')$.

Thanks to Lemma 14 above, there is a map $\lambda \in \mathcal{A}(\overline{\Omega_{N+1}}, \mathbb{C})$ such that $\lambda|_{\Omega_{N+1}}$ is conformal, $\lambda(\Omega_{N+1}) \subset \Delta$ $f(\zeta_k) = w_k$, $\partial(\lambda(\Omega_{N+1}))$ is real analytic and tangent to $\partial\Delta$ at each point w_k. We can now define

$$\chi := \begin{cases} \chi' & \text{on } K' \\ \theta^{-1} \circ \lambda & \text{on } \overline{\Omega_{N+1}}. \end{cases}$$

Let $\omega_{N+1} = \chi(\Omega_{N+1})$, and $L = L' \cup \overline{\omega_{N+1}}$. Then L is in $\mathfrak{C}_\omega$, and χ is an $\mathcal{A}$-equivalence between K and L. □

5 A Problem

We conclude this article by stating an open problem. An solution will lead to a clearer picture of sets with properties A_2 and A_3.

Is it possible to prove an analog of Theorem 4 for three sets K_1, K_2, K_3? That is, given $s : K \to \mathcal{M}$, (where $K = K_1 \cup K_2 \cup K_3$), such that $s(K_j)$ lies in a subset of $\mathcal{M}$ homeomorphic to an open set in $\mathbb{C}^n$, obtain an approximation to s, after assuming reasonable hypotheses. In particular, we should have $K_1 \cap K_2 \cap K_3 \neq \emptyset$.

Such a result will be necessary if we want to avoid the use of results like Theorem 5 two prove approximation results for two-dimensional sets. Observe that the use of Theorem 5 resulted in assumptions regarding the smoothness of the sets on which we want to do approximation.

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References

1. Arkhangel'skiǐ, A.V., Fedorchuk, V.V.: General Topology I. Encyclopedia of Mathematical Sciences, vol. 17. Springer, Berlin (1988)
2. Chakrabarti, D.: Coordinate neighborhoods of arcs and the approximation of maps into (almost) complex manifolds. Mich. Math. J. **55**(2), 299–333 (2007)
3. Demailly, J.-P.: Un exemple de fibré holomorphe non de Stein à fibre $\mathbf{C}^2$ ayant pour base le disque ou le plan. Invent. Math. **48**(3), 293–302 (1978)
4. Douady, A.: Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. Ann. Inst. Fourier (Grenoble), **16**(fasc. 1), 1–95 (1966)
5. Drinovec-Drnovšek, B., Forstnerič, F.: Holomorphic curves in complex spaces. Duke Math. J. **139**, 203–254 (2007)
6. Drinovec-Drnovšek, B., Forstnerič, F.: Approximation of holomorphic mappings on strongly pseudoconvex domains. Forum Math., to appear. Available online at arxiv.org, math.CV/0607185
7. Forstnerič, F.: Noncritical holomorphic functions on Stein manifolds. Acta Math. **191**(2), 143–189 (2003)
8. Harary, F.: Graph Theory. Addison–Wesley, Reading (1969)
9. Henkin, G.M., Leiterer, J.: Theory of Functions on Complex Manifolds. Akademie–Verlag, Berlin (1984)
10. Lang, S.: Real and Functional Analysis, 3rd edn. Springer, New York (1993)
11. MacGregor, T.H., Tepper, D.E.: Finite boundary interpolation by univalent functions. J. Approx. Theory, **52**(3), 315–321 (1988)
12. Rosay, J.-P.: Approximation of non-holomorphic maps, and Poletsky theory of discs. J. Korean Math. Soc. **40**(3), 423–434 (2003)
13. Rosay, J.-P.: Approximation of non-holomorphic maps, and Poletsky theory of discs. Unpublished Preprint
14. Tsuji, M.: Potential Theory in Modern Function Theory. Maruzen Co. Ltd., Tokyo (1959)
15. Vituškin, A.G.: Conditions on a set which are necessary and sufficient in order that any continuous function, analytic at its interior points, admit uniform approximation by rational fractions. Dokl. Akad. Nauk SSSR, **171**, 1255–1258 (1966). English translation in Sov. Mat. Dokl. **7**, 1622–1625 (1966)