LAURENT SERIES AS FOURIER SERIES

Anirban Dawn

A thesis submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics

Central Michigan University
Mount Pleasant, Michigan

May 2021

To my family and friends.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor Professor Debraj Chakrabarti for his guidance, continuous support, encouragement and his patience throughout my research work. I offer my sincere appreciation for the learning opportunities provided by him all across my Ph.D student life. I was privileged enough to have him as my mentor, and his lectures, his ideas and immense knowledge have a significant and lasting impact on my training as a mathematician.

Besides my advisor, I would like to thank the rest of my defense committee: Professor Dmitry Zakharov, and Professor Sönmez Şahutoğlu for their encouragement and valuable suggestions.

My sincere thanks to Professor Mainkar, Professor DeMeyer, Professor Zheng, Professor Salisbury, and Professor Gilsdorf for discussing with me many aspects of graduate student life, research, and for making numerous helpful suggestions.

I thank my fellow office mates Jordon, Nicole, Tyler, Shahid, Arkabrata, Pranav and Tanuj for many fruitful discussions, and for the times we were working together. It was fun!

Finally, my deepest appreciation to my family: maa, baba and my caring and supportive wife Debasmita. It was their continuous encouragement and support which boosted me in my difficult times.

ABSTRACT

LAURENT SERIES AS FOURIER SERIES

by Anirban Dawn

We study the classical power series expansions of complex analysis from the point of view of abstract Fourier analysis. Given a continuous representation of the n-dimensional torus group on a quasi-complete locally convex topological vector space, a notion of Fourier series expansion of an element of the topological vector space with respect to the representation is defined, generalizing the classical notion of a Fourier series of a function on the torus. The abstract Fourier series is shown to satisfy a version of Fejér's theorem on the Cesàro summability of the series. Conditions are given for actual convergence of the series.

The abstract results are used to study two classical examples of series expansions from complex analysis. First, a general form of Hartogs-Laurent expansion of a holomorphic function on a Hartogs domain is studied and its convergence in the Fréchet space of holomorphic functions on the domain is proved. This generalizes classical results of Hartogs, Cartan and Vladimirov. Then Laurent expansions of holomorphic functions smooth up to the boundary on Reinhardt domains are studied. It is shown that these series are absolutely convergent in the Fréchet space of holomorphic functions smooth up to the boundary.

As a last application, the Dolbeault cohomology groups of Reinhardt domains are studied. Using the natural action of the torus group on the cohomology groups, series expansions are obtained for reduced cohomology classes of Reinhardt domains.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE PAGE

CHAPTER I

INTRODUCTION

In this thesis we study classical power series expansions of holomorphic functions from the point of view of harmonic analysis on the n-dimensional torus group $\mathbb{T}^n = \{(\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n : |\zeta_j| = 1, 1 \leqslant j \leqslant n\}$. Though it is well understood that the Taylor and Laurent expansions of holomorphic functions on (complete) Reinhardt domains are a consequence of the symmetry of these domains under rotations in each complex coordinate, there was no development of the theory of series expansions in complex analysis in the literature from the abstract point of view taken here. As a consequence of the abstract theory developed in this thesis, we obtain not only strengthened versions of classical expansion theorems (Theorem IV.0.2) but also new convergence theorems for series expansions in spaces of holomorphic functions smooth up to the boundary (Theorem V.0.3). Our method is flexible enough to allow series expansions of other analytic objects admitting linear torus actions, such as (reduced) Dolbeault cohomology classes.

Chapter II contains the introductory material. We discuss the preliminaries of locally convex topological vector spaces, generating seminorms, and the theory of series in such spaces. In particular, we introduce notions of absolute and unconditional convergence of series in topological vector spaces generalizing classical notions in Banach spaces (see [22]).

In chapter III we introduce the notion of abstract Fourier series expansion of an element x in a locally convex topological vector space X, with respect to a continuous representation of the n-dimensional torus group. At this level of generality there is little one can say about the convergence of the Fourier series. However, we show that the Fourier series is *summable* to the original element x. In fact, we prove a version of Fejér's theorem which gives the convergence of the Cesàro means of the *square partial sums* of the Fourier series of x in the topology of X. We also show how to deduce the classical version of Fejér's theorem (see in [24]) from our results.

Chapter IV deals with an application of the theory outlined in chapter III. Let $n \geqslant 1$ and $1 \leqslant k \leqslant n$. An open set $\Omega \subset \mathbb{C}^n$ is said to be a *Hartogs open set* if $z = (z_1, \cdots, z_n) \in \Omega$ implies $(\lambda_1 z_1, \cdots, \lambda_k z_k, z_{k+1}, \cdots, z_n) \in \Omega$ for all $(\lambda_1, \cdots, \lambda_k) \in \mathbb{T}^k$. In this chapter we use our theory of abstract Fourier series to show that every holomorphic function on a Hartogs open set Ω in $\mathbb{C}^n$ admits a Hartogs-Laurent series. Special cases of this are known due to Hartogs [19], Cartan [6], and Vladimirov [31]. In these works it is assumed either the Hartogs open set is *complete* (a Hartogs open set $\Omega \subset \mathbb{C}^n$ is said to be complete if $z = (z_1, \cdots, z_n) \in \Omega$ implies $(\lambda_1 z_1, \cdots, \lambda_k z_k, z_{k+1}, \cdots, z_n) \in \Omega$ for all $|\lambda_j| \leqslant 1$, where $1 \leqslant j \leqslant k$) or the fiber of the projection in last $(n-k)$ variables of the domain is connected (see Section 15.2 in [31]). Our approach does not need any such completeness or connected assumption. We also show that the Hartogs-Laurent series converges absolutely and unconditionally to the original function in the Fréchet topology of $\mathcal{O}(\Omega)$ (where $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on an open set Ω in $\mathbb{C}^n$).

In chapter V we look at *Reinhardt open sets*. Reinhardt open sets are special cases of Hartogs open sets (when $n = k$, that is, the circular symmetry exists in each complex coordinate). A connected Reinhardt open set is called a Reinhardt domain. We show that if the domain Ω is Reinhardt, then the Hartogs-Laurent series is the classical Laurent series of holomorphic functions on Ω and the series converges absolutely and unconditionally in $\mathcal{O}(\Omega)$. We show that holomorphic functions on the Reinhardt domain Ω which lie in the space $\mathcal{A}^\infty(\Omega)$ of functions smooth up to the boundary have Laurent series which converge absolutely in the topology of $\mathcal{A}^\infty(\Omega)$, that is derivatives of all orders converge uniformly on compact subsets of $\overline{\Omega}$. Convergence results similar to this for other classical function spaces are well known. For $1 < p < \infty$, it is well known that the partial sums of the Taylor series of f in $H^p(\mathbb{D})$, the Hardy space on the unit disc in $\mathbb{C}$, converges to f in the $H^p(\mathbb{D})$ norm (see [17, p. 104-110]). It is also known (see [32, 10]) that for a bounded Reinhardt domain $\mathcal{R}$ in $\mathbb{C}^n$, the square partial sums of the Laurent series of f in $A^p(\mathcal{R})$ converges to the function

f in the $A^p(\mathcal{R})$ norm. Notice that for a general Reinhardt domain Ω, the convergence of the Laurent series in $\mathcal{A}^\infty(\Omega)$ and $\mathcal{O}(\Omega)$ is unconditional, which is not the case in $H^p(\mathbb{D})$ or $A^p(\mathcal{R})$.

This result is interesting because of the intrinsic importance of the space $\mathcal{A}^\infty(\Omega)$ in complex analysis. For example, it is known that each smoothly bounded pseudoconvex domain Ω is a so called $\mathcal{A}^\infty(\Omega)$-domain of holomorphy (see [7] and [18] for details). However, it is also known that pseudoconvex domains with non-smooth boundaries may not be $\mathcal{A}^\infty(\Omega)$-domain of holomorphy. This was first noticed for Hartogs triangle $\{(z,w):|z|<|w|<1\}\subset\mathbb{C}^2$ by Sibony (see [30]) and generalised by Chakrabarti to Reinhardt domains in $\mathbb{C}^n$ with 0 as a boundary point (see [9]). Towards the end of chapter 4 we introduce the principle of missing monomials and present a few applications.

In chapter VI we use the theory of abstract Fourier series outlined in chapter III to show that the Dolbeault cohomology classes of a Reinhardt domain in $\mathbb{C}^n$ can be represented in "Laurent type series". We also prove that the series converges to the original cohomology class absolutely and unconditionally in the quotient topology of the reduced Dolbeault cohomology. At the end we use our result to show series expansion of reduced Dolbeault cohomology classes of some Reinhardt domains in $\mathbb{C}^2$.

CHAPTER II

SEQUENCE AND SERIES IN TOPOLOGICAL VECTOR SPACES

In this chapter we recall some definitions and results related to topological vector spaces, seminorms and the theory of convergence of series in such spaces. This preliminary material will be used in the subsequent chapters.

II.0.1. Locally Convex Topological Vector Spaces.

Recall that a *topological vector space* over $\mathbb{C}$ is a vector space X with a Hausdorff topology τ with respect to which the addition operation $X \times X \to X$, given by $(x,y) \mapsto x+y$ and the scalar multiplication $\mathbb{C} \times X \to X$, given by $(\alpha, x) \mapsto \alpha x$ are continuous. A collection $\mathcal{B}$ of neighborhoods of a point x in X is called *local base at x* if every neighborhood of x contains a member of $\mathcal{B}$. The topological vector space X is called *locally convex* if there exists a local base at 0 whose members are convex open sets. For more details on locally convex topological vector spaces, see [29, Chapter 1] and [20, Chapter 2].

Next we recall some classical definitions.

Definition 1. A *seminorm* on a vector space X (over $\mathbb{C}$) is a non-negative real valued function p on X such that

(a) $p(x+y) \leqslant p(x) + p(y)$ and

(b) $p(\alpha x) = |\alpha| \, p(x)$

for all $x, y \in X$ and all scalars $\alpha \in \mathbb{C}$.

Definition 2 (Directed Sets, Nets)**.** A *directed set* is a set Γ together with a relation $\geqslant$ on Γ such that:

(a) $\geqslant$ is reflexive: $\alpha \geqslant \alpha$ for all $\alpha \in \Gamma$.

(b) $\geqslant$ is transitive: $\alpha \geqslant \beta$ and $\beta \geqslant \gamma$ implies $\alpha \geqslant \gamma$, and

(c) for any $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$.

A *net* $(x_\alpha)_{\alpha\in\Gamma}$ in a topological vector space X is a map $x : \Gamma \to X$ written as $x(\alpha) = x_\alpha$, where Γ is a directed set and $\alpha \in \Gamma$. Observe that the notion of net is a generalization of notion of sequence (directed set $\mathbb{N}$, the set of natural numbers, together with the usual $\geqslant$ relation). A net $(x_\alpha)_{\alpha\in\Gamma}$ *converges* to $x \in X$ if for every neighbourhood U of 0 in X, there exists $\beta \in \Gamma$ such that whenever $\alpha \geqslant \beta, x_\alpha - x \in U$.

Let P be a family of seminorms on a vector space X such that for every nonzero $x \in X$ there exists at least one $p \in \mathsf{P}$ such that $p(x) \neq 0$, the family P is said to be a *separating family of seminorms* on X. Let X be a locally convex topological vector space (abbreviated as LCTVS). It is a basic fact in the theory of LCTVS that the topology of X can be *defined* by a separating family of continuous seminorms P, in the sense that a net $(x_\alpha)_{\alpha\in\Gamma}$ in X converges to $x \in X$ if and only if the net $\Big(p(x_\alpha - x)\Big)_{\alpha\in\Gamma}$ of real numbers converges to 0 for each $p \in \mathsf{P}$. For more details see [29, Theorem 1.37, p. 27] and [4, Example 1.3.4, p. 18].

Let X be an LCTVS. A collection Q of continuous seminorms on X is said to be a *generating family* if and only if for every continuous seminorm p on X, there exists a finite subset $\{p_1, \cdots, p_m\}$ of Q and a number $C > 0$ such that

$$p(x) \leqslant C \cdot \max_{1\leqslant k\leqslant m} \{p_k(x)\} \quad \text{for all } x \in X. \tag{II.0.1}$$

Notice that convergence in an LCTVS can be characterised by generating families in the sense that, a net $(x_\alpha)_{\alpha\in\Gamma}$ in X converges to $x \in X$ if and only if the net $\Big(p(x_\alpha - x)\Big)_{\alpha\in\Gamma}$ of real numbers converges to 0 for each $p \in \mathsf{Q}$.

Cauchy nets and completeness.

Let (X, τ) be an LCTVS. A net $(x_\alpha)_{\alpha\in\Gamma}$ in X is said to be a *Cauchy net* if for every $\epsilon > 0$ and every continuous seminorm p on X, there exists $\gamma \in \Gamma$ such that whenever $\alpha, \beta \in \Gamma$ and $\alpha, \beta \geqslant \gamma$, $p(x_\alpha - x_\beta) < \epsilon$. Let P be a family of continuous separating seminorms on X that generates the topology τ. It follows from (II.0.1) that we can give an alternative definition of Cauchy net involving generating family of seminorms: a net $(x_\alpha)_{\alpha\in\Gamma}$ in X is said to be a *Cauchy net* if for every $\epsilon > 0$ and every $p \in \mathsf{P}$, there exists $\gamma \in \Gamma$ such that whenever $\alpha, \beta \in \Gamma$ and $\alpha, \beta \geqslant \gamma$, $p(x_\alpha - x_\beta) < \epsilon$. The space X is said to be *complete* if every Cauchy net of X converges.

II.0.2. Convergence of series in LCTVS

We introduce the following notions. For standard definitions in Banach spaces see [22].

Absolute and Unconditional convergence

Definition 3 (Absolute and unconditional convergence)**.** Let X be an LCTVS and Λ be a countable index set. Let $x_\alpha \in X$ for each $\alpha \in \Lambda$ and let $\sum_{\alpha\in\Lambda} x_\alpha$ be a formal infinite series in X.

(a) If $\Lambda = \mathbb{N} = \{0, 1, 2, \cdots\}$, then the series $\sum_{j=0}^{\infty} x_j$ in X is said to *converge* to an element x in X if for every continuous seminorm p on X,

$$p\left(\sum_{j=0}^{N} x_j - x\right) \to 0 \quad \text{as } N \to \infty. \tag{II.0.2}$$

(b) The formal series $\sum_{\alpha\in\Lambda} x_\alpha$ is said to converge *unconditionally* if for every bijection $\tau : \mathbb{N} \to \Lambda$, the series $\sum_{j=0}^{\infty} x_{\tau(j)}$ converges in X.

(c) The formal series is said to converge *absolutely* in X if there exists a bijection

$$\tau : \mathbb{N} \to \Lambda \tag{II.0.3}$$

such that for every continuous seminorm p on X, the series

$$\sum_{j=0}^{\infty} p(x_{\tau(j)}) \tag{II.0.4}$$

is a convergent series of non-negative real numbers.

Lemma II.0.1. *In a complete LCTVS, an absolutely convergent series is unconditionally convergent and the sum of each rearrangement is the same.*

Proof. Let $\sum_{\alpha \in \Lambda} x_\alpha$ be an absolutely convergent series in a complete LCTVS X. So, there exists a bijection $\tau : \mathbb{N} \to \Lambda$ such that for each continuous seminorm p on X, the series $\sum_{j=0}^{\infty} p(x_{\tau(j)})$ converges. Let $y_j = x_{\tau(j)}$ and $s_k = \sum_{j=0}^{k} y_j$. Since $\sum_{j=0}^{\infty} p(y_j)$ converges, for $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that whenever $m, \ell \in \mathbb{N}$ with $m \geqslant \ell \geqslant N_0$, $\sum_{j=\ell+1}^{m} p(y_j) < \epsilon$. Therefore for $m \geqslant \ell \geqslant N_0$,

$$p(s_m - s_\ell) = p\Big(\sum_{j=\ell+1}^{m} y_j \Big) \leqslant \sum_{j=\ell+1}^{m} p(y_j) < \epsilon. \tag{II.0.5}$$

It follows from (II.0.5) that the net $\{s_k\}$ is Cauchy in a complete LCTVS X, with directed set $(\mathbb{N}, \geqslant)$, and therefore converges. Let $s_k \to s$ as $k \to \infty$. In order to complete the proof, it suffices to show that for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{j=0}^{\infty} y_{\sigma(j)}$ converges to the same limit s. Let $s_k^\sigma = \sum_{j=0}^{k} y_{\sigma(j)}$. We show $s_k^\sigma \to s$ as $k \to \infty$. Choose $u \in \mathbb{N}$ such that the set of integers $\{0, 1, 2, \cdots, N_0\}$ is contained in the set $\{\sigma(0), \sigma(1), \cdots, \sigma(u)\}$. Then, if $k > u$, the elements $y_1, \cdots, y_{N_0}$ get cancelled in the difference $s_k - s_k^\sigma$ and we have $p(s_k - s_k^\sigma) < \epsilon$ by (II.0.5). This proves that the sequence $\{s_k\}$ and $\{s_k^\sigma\}$ converges to the same sum. So, $s_k^\sigma \to s$ as $k \to \infty$. □

The converse of Lemma II.0.1 is true only for finite dimentional normed spaces (see [22, Theorem 1.3.5, p. 10]); but not in general. Let $X = \ell_2$, the space of square summable sequences in the complex plane $\mathbb{C}$ and let $x_k = (0, \cdots, 0, k^{-1}, 0, \cdots) \in X$, where the nonzero coordinate is the k^{th}. Then the series $\sum_{k=1}^{\infty} x_k$ converges to the point $x = (1, 2^{-1}, 3^{-1}, \cdots)$ with respect to every rearrangement of its terms. However, the series does not converge absolutely, since $\sum_{k=1}^{\infty} \|x_k\|_2 = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Convergence in the net of partial sums

Let $(F(\Lambda), \subset)$ be the directed set of all finite subsets of a countable index set Λ with inclusion '$\subset$' as its relation. Let X be an LCTVS. The net $F(\Lambda) \to X$ defined by

$$I \mapsto \sum_{\alpha \in I} x_\alpha \tag{II.0.6}$$

is said to be the *net of partial sums* of the formal series $\sum_{\alpha \in \Lambda} x_\alpha$ in X.

Lemma II.0.2. *In a complete LCTVS, the net of partial sums of an absolutely convergent series converges.*

Proof. Let $\sum_{\alpha \in \Lambda} x_\alpha$ be an absolutely convergent series in a complete LCTVS X. So, there exists a bijection $\tau : \mathbb{N} \to \Lambda$ such that for each continuous seminorm p on X, the series of non-negative reals $\sum_{j=0}^{\infty} p(x_{\tau(j)})$ converges. Let $\epsilon > 0$ and let p be a continuous seminorm on X. Since $\sum_{j=0}^{\infty} p(x_{\tau(j)})$ converges, there exists $N \in \mathbb{N}$ such that for all $m, k \in \mathbb{N}$ with $m \geqslant k > N_0$,

$$\sum_{j=k}^{m} p(x_{\tau(j)}) < \epsilon/2. \tag{II.0.7}$$

Let $(F(\Lambda), \subset)$ be the directed set of all finite subsets of Λ with inclusion as its order. Let $I = \{\tau(0), \tau(1), \cdots, \tau(N_0)\}$, then $I \in F(\Lambda)$. Now, whenever $J, K \in F(\Lambda)$ with $J, K \supset I$,

$$\begin{aligned} p\Bigg(\sum_{\alpha \in J} x_\alpha - \sum_{\alpha \in K} x_\alpha\Bigg) \leqslant p\Bigg(\sum_{\alpha \in J \setminus I} x_\alpha - \sum_{\alpha \in K \setminus I} x_\alpha\Bigg) &\leqslant p\Bigg(\sum_{\alpha \in J \setminus I} x_\alpha\Bigg) + p\Bigg(\sum_{\alpha \in K \setminus I} x_\alpha\Bigg) \\ &\leqslant \sum_{\alpha \in J \setminus I} p(x_\alpha) + \sum_{\alpha \in K \setminus I} p(x_\alpha) \\ &\leqslant \epsilon/2 + \epsilon/2 = \epsilon, \text{ from (II.0.7)}. \end{aligned}$$

This shows that the net of partial sums $I \mapsto \sum_{\alpha \in I} x_\alpha$ of the series $\sum_{\alpha \in \Lambda} x_\alpha$ is Cauchy. Since X is complete, it is convergent. $\square$

II.0.3. Cesàro means of a sequence in LCTVS

In this section we generalize the notion of Cesàro means of a sequence of numbers. Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence in an LCTVS X. Define the Cesàro means $\{C_N\}_{N\in\mathbb{N}}$ of the sequence $\{S_k\}_{k\in\mathbb{N}}$ by,

$$C_N = \frac{1}{N+1}\sum_{k=0}^{N} S_k \quad \text{for all } N \in \mathbb{N}. \tag{II.0.8}$$

The following Lemma is well known for real numbers.

Lemma II.0.3. *Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence in an LCTVS X such that $S_k \to S$ in the topology of X as $N \to \infty$. Let $\{C_N\}_{N\in\mathbb{N}}$ be the Cesàro means of $\{S_k\}_{k\in\mathbb{N}}$. Then $C_N \to S$ in the topology of X as $N \to \infty$.*

Proof. Let p be a continuous seminorm on X and let $\epsilon > 0$. Since $S_N \to S$ in X, there exists $N_1 \in \mathbb{N}$ such that $p(S_N - S) < \epsilon/2$ for $N \geqslant N_1$. Set $s = \sum_{k=0}^{N_1} p(S_k - S)$ and choose a positive integer $M \geqslant N_1$ such that $s \leqslant \frac{\epsilon}{2}\cdot M$. Then, whenever $N \geqslant M$,

$$\begin{aligned} p(C_N - S) = p\left(\frac{1}{N+1}\sum_{k=0}^{N} S_k - S\right) &= \frac{1}{N+1}\; p\left(\sum_{k=0}^{N}(S_k - S)\right) \\ &\leqslant \frac{1}{N+1}\;\sum_{k=0}^{N} p(S_k - S) \\ &= \frac{1}{N+1}\left(\sum_{k=0}^{N_1} p(S_k - S) + \sum_{k=N_1+1}^{N} p(S_k - S)\right) \\ &\leqslant \frac{1}{N+1}\left(s + \frac{(N-N_1)\epsilon}{2}\right). \end{aligned} \tag{II.0.9}$$

Since $s \leqslant \frac{\epsilon}{2}\cdot M$ and $N - N_1 \leqslant N$,

$$\frac{1}{N+1}\left(s + \frac{(N-N_1)\epsilon}{2}\right) \leqslant \frac{1}{N+1}\left(\frac{M\epsilon}{2} + \frac{N\epsilon}{2}\right) = \frac{M+N}{2(N+1)}\cdot\epsilon < \epsilon,$$

where the last inequality is due to the fact that $M + N \leqslant 2N$. □

The converse of Lemma II.0.3 is not true in general. Consider the sequence of real numbers $\{(-1)^k\}_{k\in\mathbb{N}}$ and consider the sequence $\{S_N\}$ where $S_N = \sum_{k=0}^{N}(-1)^k$. Then S_N

does not converge as $N \to \infty$. However,

$$\left|C_N\right| = \left|\frac{1}{N+1}\sum_{k=0}^{N} S_k\right| = \frac{1}{N+1}\ \left|\sum_{k=0}^{N} S_k\right| \leqslant \frac{1}{N+1} \to 0 \ \text{ as } N \to \infty.$$

II.0.4. Seminorms on product and quotient topological vector spaces

In this section we prove a couple of lemmas which are used in Chapter VI. Recall that an LCTVS is said to be a *Fréchet space* if its topology is induced by a complete translation-invariant metric d. One can prove that the topology of a Fréchet space is generated by a countable collection of continuous seminorms and the translation invariant metric d can be represented using this collection of seminorms.

Lemma II.0.4. *For $j \in \mathbb{N}$, let X_j be a Fréchet space and let $P_j = \{p_{j,\ell}\}_{\ell\in\mathbb{N}}$ be a family of continuous seminorms on X_j that generates the locally convex topology on X_j. Let $X = \prod_{j\in\mathbb{N}} X_j$ and let $\Pi_j : X \to X_j$ be the projection map on the j^{th} component. Endow X with the product topology. Then X is a Fréchet space and the set of seminorms $P = \{p_{j,\ell} \circ \Pi_j\}_{j,\ell\in\mathbb{N}}$ generates the product topology on X.*

Proof. First we prove that P generates the product topology on X. Let $\epsilon > 0$ and let p be a continuous seminorm on X. Then the set $\{x = (x_j)_{j\in\mathbb{N}} \in X : p(x) < \epsilon\}$ is open in X. Since X is endowed with the product topology, there exists a product $\prod_j W_j$, where W_j is an open subset of X_j for j belonging to a finite subset $H \subset \mathbb{N}$ and $W_j = X_j$ for $j \in \mathbb{N} \setminus H$, such that

$$\prod_j W_j \subset \{x \in X : p(x) < \epsilon\}. \tag{II.0.10}$$

Now for every $j \in H$, since P_j generates the topology on X_j, there exists a finite set $M_j \subset \mathbb{N}$ such that

$$\bigcap_{\ell\in M_j} \left\{x_j : p_{j,\ell}(x_j) < \epsilon\right\} \subset W_j. \tag{II.0.11}$$

Therefore

$$\bigcap_{j\in H}\bigcap_{\ell\in M_j} \left\{x : (p_{j,\ell} \circ \Pi_j)(x) < \epsilon\right\} \subset \prod_j W_j \subset \left\{x : p(x) < \epsilon\right\}. \tag{II.0.12}$$

This proves the fact that P generates the product topology on X. Now we show that X is a Fréchet space. Let $\{V_{j,\ell}\}_{\ell\in\mathbb{N}}$ be a fundamental system of convex neighborhood of 0 of X_j. Then a fundamental system of neighborhood of 0 in X is given by all sets $\prod_{j\in\mathbb{N}} U_j$, where for j belonging to a finite subset of H of $\mathbb{N}$, the set U_j is equal to $V_{j,\ell}$ for some ℓ, and $U_j = X_j$ for $j \in \mathbb{N} \setminus H$. Since the set of generating seminorms P is countable, X is metrizable. For $x, y \in X$, define

$$d(x,y) = \sum_{j,\ell=1}^{\infty} \frac{1}{2^{j+\ell}} \frac{(p_{j,\ell} \circ \Pi_j)(x-y)}{1 + (p_{j,\ell} \circ \Pi_j)(x-y)}. \tag{II.0.13}$$

It is not difficult to check that d in (II.0.13) is a translation invariant metric. Let $x^k = (x_j^k)_j \in X$ and let $\{x^k\}_k \subset X$ be a Cauchy sequence. So, $d(x^k, x^m) \to 0$ as $k, m \to \infty$. Therefore from (II.0.13), for every $j, \ell \in \mathbb{N}$, $(p_{j,\ell} \circ \Pi_j)(x^k - x^m) \to 0$ as $k, m \to \infty$; that is, for every $j, \ell \in \mathbb{N}$, $p_{j,\ell}(x_j^k - x_j^m) \to 0$ as $k, m \to \infty$. Since X_j is complete for every j, there exists $x_j \in X_j$ such that for all $j, \ell \in \mathbb{N}$, $p_{j,\ell}(x_j^k - x_j) \to 0$ as $k \to \infty$. Let $x = (x_j)_j$, then $x \in X$ and for all $j, \ell \in \mathbb{N}$, $(p_{j,\ell} \circ \Pi_j)(x^k - x) \to 0$. This proves d in (II.0.13) is complete. $\square$

Lemma II.0.5. *Let p be a continuous seminorm on an LCTVS X and let Y be a closed subspace of X. Let ϕ be the quotient map from X to X/Y. Define $\overline{p} : X/Y \to [0,\infty)$ by*

$$\overline{p}(\overline{x}) = \inf_{\phi(x)=\overline{x}} p(x), \qquad \textit{where } \overline{x} = x + Y \in X/Y. \tag{II.0.14}$$

Then $\overline{p}$ is a seminorm on X/Y. Moreover, let P be a set of continuous seminorms that generates the locally convex topology of X. Then the set $\overline{P} = \{\overline{p} : p \in P\}$ generates the quotient topology of X/Y.

Proof. Let $\overline{x} \in X/Y$. Since p is a seminorm on X, $\overline{p}(\overline{x}) = \inf_{\phi(x)=\overline{x}} p(x) \geqslant 0$. Since ϕ is linear, then for a scalar $\alpha \in \mathbb{C}$,

$$\overline{p}(\alpha\overline{x}) = \inf_{\phi(\alpha x)=\alpha\overline{x}} p(\alpha x) = |\alpha| \inf_{\phi(x)=\overline{x}} p(x) = |\alpha| \cdot \overline{p}(\overline{x}).$$

Now, let $\overline{x}_1, \overline{x}_2 \in X/Y$. Then

$$\begin{aligned}
\overline{p}(\overline{x}_1 + \overline{x}_2) &= \inf_{\phi(x_1+x_2)=\overline{x}_1+\overline{x}_2} p(x_1 + x_2) \\
&\leqslant \inf_{\phi(x_1+x_2)=\overline{x}_1+\overline{x}_2} p(x_1) + p(x_2) \quad \text{(since } p \text{ is a seminorm)} \\
&= \inf_{\phi(x_1)+\phi(x_2)=\overline{x}_1+\overline{x}_2} p(x_1) + p(x_2) \quad \text{(since } \phi \text{ is linear)} \\
&= \inf_{\phi(x_1)=\overline{x}_1} p(x_1) + \inf_{\phi(x_2)=\overline{x}_2} p(x_2) = \overline{p}(\overline{x}_1) + \overline{p}(\overline{x}_2). \qquad \text{(II.0.15)}
\end{aligned}$$

This shows $\overline{p}$ is a seminorm on X/Y. It follows from the definition of $\overline{p}$ that the seminorm $\overline{p}$ is continuous at 0, using the reverse triangle inequality of seminorms it is continuous everywhere.

To prove that the set $\overline{P}$ generates the quotient topology of X/Y, we use the universal property of the quotient topology: Let X be an LCTVS and $\phi : X \to X/Y$ be the quotient map and let τ be a topology on X/Y which satisfies the following: for any topological space Z and any continuous map $g : X \to Z$ such that $g(x) = g(x')$ whenever $\phi(x) = \phi(x')$, there exists a unique continuous map $f : X/Y \to Z$ such that $g = f \circ \phi$. Then τ is the quotient topology on X/Y.

$$\begin{tikzcd}
X \arrow[d, "\phi"'] \arrow[dr, "g"] & \\
X/Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Let Z be a topological space and let $g : X \to Z$ be continuous, such that $g(x) = g(x')$ whenever $\phi(x) = \phi(x')$. Let τ be the topology on X/Y generated by the family of seminorm $\overline{P}$. To prove τ is the quotient topology, it is sufficient to show that there exists a unique continuous map $f : X/Y \to Z$ such that $g = f \circ \phi$.

Define $f(\overline{x}) = g(x)$, where $\overline{x} = \phi(x)$. This map is well-defined, since $g(x) = g(x')$ whenever $\overline{x} = \overline{x'}$. Let U be an open set in Z and let $\overline{x} \in f^{-1}(U)$. Therefore, $\phi^{-1}(\overline{x}) \in \phi^{-1}(f^{-1}(U))$, that is, $x \in g^{-1}(U)$. Since g is continuous, there exists a finite subset $\{p_1, \cdots, p_N\} \subset P$ such that

$$x \in \bigcap_{j=1}^{N} \left\{x : p_j(x) < \epsilon\right\} \subset g^{-1}(U). \tag{II.0.16}$$

Note that for $1 \leqslant j \leqslant N$, $\overline{p_j}(\overline{x}) = \inf_{\phi(x)=\overline{x}} p_j(x) \leqslant p_j(x)$. Therefore it follows from (II.0.16) that

$$\overline{x} \in \bigcap_{j=1}^{N} \left\{\overline{x} : \overline{p}_j(\overline{x}) < \epsilon\right\} \subset f^{-1}(U). \tag{II.0.17}$$

This shows that the map f is open. To show f is unique, let $h : X/Y \to Z$ be another map such that $g = h \circ \phi$. Then

$$f(\overline{x}) = f(\phi(x)) = g(x) = h(\phi(x)) = h(\overline{x}),$$

and therefore $f = h$. $\square$

II.0.5. An abstract criterion for convergence in LCTVS

In this section we introduce an abstract condition for convergence of series in an LCTVS, see Lemma II.0.6 below. The idea of the proof is inspired by a result in [10, Section 3.4.1]. Although we will not use this result in the subsequent chapters.

Definition 4. Suppose X and Y are LCTVS and let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of linear maps from X to Y. The sequence $\{T_k\}$ is said to be *equicontinuous* if for every continuous seminorm p on Y there exists a continuous seminorm q on X such that

$$p(T_k x) \leqslant q(x) \quad \text{for all } x \in X \text{ and for all } k \in \mathbb{N}.$$

Remark. If X is a Banach space with norm $\|\cdot\|$, the equicontinuity condition of the sequence $\{T_k\}$ implies that there exists a number $C > 0$ such that $\|T_k\|_{\text{op}} \leqslant C$ for all $k \in \mathbb{N}$, where $\|\cdot\|_{\text{op}}$ is the operator norm.

Lemma II.0.6. *Let X be an LCTVS and let $\{T_k\}_{k\in\mathbb{N}}$ be an equicontinuous sequence of linear maps from X to X. Suppose that there exists a dense subset D of X such that for all $x \in D$, $T_k(x) \to x$ in the topology of X, as $k \to \infty$. Then for all $x \in X$, $T_k(x) \to x$ in the topology of X, as $k \to \infty$.*

Proof. Let p be a continuous seminorm on X. By hypothesis, there exists a continuous seminorm q on X such that for every natural number k, $p \circ T_k \leqslant q$ on X. Observe that $p+q$ is also a continuous seminorm on X. Fix $x \in X$ and let $\epsilon > 0$. Since D is dense in X, there exists $y \in D$ such that,

$$(p+q)(x-y) < \epsilon/2. \tag{II.0.18}$$

Now, depending on the y above, choose $N_1 \in \mathbb{N}$ such that for all $k \geqslant N_1$,

$$p(T_k(y)-y) < \epsilon/2 \tag{II.0.19}$$

Therefore whenever $k \geqslant N_1$,

$$\begin{aligned}
p(T_k(x)-x) &= p(T_k(x)-T_k(y)+T_k(y)-y+y-x) \\
&\leqslant p(T_k(x)-T_k(y)) + p(T_k(y)-y) + p(y-x) \\
&\leqslant p(T_k(x-y)) + p(T_k(y)-y) + p(y-x), \quad \text{since } T_k \text{ is linear,} \\
&\leqslant q(x-y) + p(T_k(y)-y) + p(y-x), \quad \text{by hypothesis,} \\
&= (p+q)(x-y) + p(T_k(y)-y) < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

□

CHAPTER III

ABSTRACT FOURIER SERIES

Let $\mathbb{T}^n = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n : |\lambda_j| = 1 \text{ for every } 1 \leqslant j \leqslant n\}$ be the n-dimensional unit torus. Note that $\mathbb{T}^n$ is a compact subset of $\mathbb{C}^n$; and the space $\mathbb{T}^n$ with the subspace topology and the binary operation defined as $(\lambda, \xi) \mapsto \lambda \cdot \xi = (\lambda_1\xi_1, \cdots, \lambda_n\xi_n)$ for all $\lambda, \xi \in \mathbb{T}^n$ is a compact topological (commutative) group. In this chapter we develop a theory of abstract Fourier series with respect to a representation of the torus group $\mathbb{T}^n$ on an LCTVS X and our idea is inspired by the work of Johnson (see [21]).

III.0.1. Representation of torus on an LCTVS X.

A *representation* of a group G on a vector space X is a group homomorphism G to $\mathrm{GL}(X)$, where $\mathrm{GL}(X)$ denotes the group of *linear automorphisms* of X. If X is an LCTVS, we will restrict to the representations which are group homomorphisms G to $\mathrm{Aut}(X)$, where $\mathrm{Aut(X)} \subset \mathrm{GL}(X)$ is the group of linear *homeomorphisms* of X; suppose $\lambda \mapsto \sigma_\lambda$ be such a representation, $\lambda \in G$. We associate with this representation a map $\sigma : G \times X \to X$ by

$$\sigma(\lambda, x) = \sigma_\lambda(x) \quad \text{for } \lambda \in G \text{ and } x \in X. \tag{III.0.1}$$

Definition 5. The representation $\lambda \mapsto \sigma_\lambda$ of a topological group G on an LCTVS X is said to be *continuous* if the associated map $\sigma : G \times X \to X$ is continuous.

By abuse of language we will identify the map σ with the representation σ_λ and will call the map σ a representation of G on an LCTVS X.

Definition 6 (Invariant seminorms)**.** Let σ be a representation of $\mathbb{T}^n$ on an LCTVS X. A seminorm p on X is an *invariant seminorm* (with respect to σ) if $p(\sigma_\lambda(x)) = p(x)$ for all $x \in X$ and $\lambda \in \mathbb{T}^n$.

The following proposition provides a sufficient condition for the continuity of a representation of $\mathbb{T}^n$ on an LCTVS X.

Proposition III.0.1. *Let σ be a representation of $\mathbb{T}^n$ on an LCTVS X. Then the following conditions are equivalent.*

(a) σ is continuous.

(b) There exists a family P of continuous invariant seminorms on X which generates the topology, and for all $x \in X$ the function from $\mathbb{T}^n$ to X given by $\lambda \mapsto \sigma_\lambda(x)$ is continuous at $\lambda = \mathbb{1}$, where $\mathbb{1}$ is the identity in $\mathbb{T}^n$.

Proof. First we prove $(a) \Rightarrow (b)$. Since σ is continuous, so for all $x \in X$, the function $\lambda \mapsto \sigma_\lambda(x)$ is continuous on $\mathbb{T}^n$, in particular at $\lambda = \mathbb{1}$. We now prove that the family of continuous invariant seminorms generates the topology of X. Let q be a continuous seminorm on X. Following the inequality in (II.0.1) it is sufficient to show that there exists a continuous invariant seminorm p on X such that $q(x) \leqslant p(x)$ for all $x \in X$. Define $p(x) = \sup_{\lambda \in \mathbb{T}^n} q(\sigma_\lambda(x))$. It is easy to show that p satisfies all the properties of a seminorm. To show p is invariant, we use the definition of p and the homomorphism property of the map $\lambda \to \sigma_\lambda$. We get $p(\sigma_\lambda(x)) = \sup_{\lambda \in \mathbb{T}^n} q(\sigma_\lambda(\sigma_\lambda(x))) = \sup_{\lambda \in \mathbb{T}^n} q(\sigma_{\lambda.\lambda}(x)) = \sup_{\lambda \cdot \lambda \in \mathbb{T}^n} q(\sigma_{\lambda.\lambda}(x)) = \sup_{\xi \in \mathbb{T}^n} q(\sigma_\xi(x)) = p(x)$. It remains to show that p is continuous. We claim that $|p(x) - p(y)| \leqslant p(x-y)$ for all $x, y \in X$. Indeed,

$$p(x) = p(x - y + y) \leqslant p(x-y) + p(y)$$

so that $p(x) - p(y) \leqslant p(x-y)$. This also holds if we switch x and y. Since $p(x-y) = p(y-x)$, our claim follows. Since $|p(x) - p(y)| \leqslant p(x-y)$ for all $x, y \in X$, it follows that the seminorm p is continuous on X if and only if it is continuous at 0_X. Let $\epsilon > 0$. We show that there exists a neighbourhood V of 0_X such that for all $\lambda \in \mathbb{T}^n$, $q \circ \sigma_\lambda < \epsilon$ on V. Let $\xi \in \mathbb{T}^n$. Since q and σ are continuous, there exists a neighbourhood U_ξ of ξ and a neighbourhood V_ξ of 0_X such that,

$$q(\sigma_\lambda(x)) < \epsilon \quad \text{for all } x \in V_\xi \text{ and } \lambda \in U_\xi$$

The collection $\{U_\xi\}_{\xi\in\mathbb{T}^n}$ forms an open cover of $\mathbb{T}^n$. Since $\mathbb{T}^n$ is compact, let $\{U_{\xi_1},...,U_{\xi_k}\}$ be a finite subcover of $\mathbb{T}^n$ corresponding to the open cover. Let $V = \bigcap_{j=1}^{k} V_{\xi_j}$, then V is also a neighbourhood of 0_X. Then for all $x \in V$ and $\lambda \in \mathbb{T}^n$, we have $q(\sigma_\lambda(x)) < \epsilon$.

Now let us prove $(b) \Rightarrow (a)$. Let $(\Gamma, \geqslant)$ be a directed set and let $(\lambda_\alpha)_{\alpha\in\Gamma}$ and $(x_\alpha)_{\alpha\in\Gamma}$ be nets in $\mathbb{T}^n$ and X respectively with $(\lambda_\alpha, x_\alpha) \to (\lambda, x)$ in $\mathbb{T}^n \times X$, we need to show $\sigma_{\lambda_\alpha}(x_\alpha) \to \sigma_\lambda(x)$ in X. Let $\epsilon > 0$. We show that there exists a $\beta \in \Gamma$ such that for all continuous invariant seminorm p on X,

$$p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_\lambda(x)) < \epsilon \quad \text{whenever } \alpha \geqslant \beta.$$

Since $x_\alpha \to x$ in X (taking constant net in $\mathbb{T}^n$), there exists $\alpha_1 \in \Gamma$ such that for all $\alpha \geqslant \alpha_1$,

$$p(x_\alpha - x) < \epsilon/2. \tag{III.0.2}$$

Since p is an invariant seminorm and σ_{λ_α} is linear,

$$p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_{\lambda_\alpha}(x)) = p(\sigma_{\lambda_\alpha}(x_\alpha - x)) = p(x_\alpha - x). \tag{III.0.3}$$

Therefore, if $\alpha \geqslant \alpha_1$,

$$p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_{\lambda_\alpha}(x)) < \epsilon/2. \tag{III.0.4}$$

Also, since p is invariant and the map $\mu \mapsto \sigma_\mu$ is a group homomorphism on $\mathbb{T}^n$,

$$p(\sigma_{\lambda_\alpha}(x) - \sigma_\lambda(x)) = p(\sigma_{\lambda^{-1}}(\sigma_{\lambda_\alpha}(x) - \sigma_\lambda(x))) = p(\sigma_{\lambda^{-1}\cdot\lambda_\alpha}(x) - \sigma_{\mathbb{1}}(x)). \tag{III.0.5}$$

Since $\lambda_\alpha \to \lambda$ (taking constant net on X), it follows from the continuity of the map $\mu \mapsto \lambda^{-1}\cdot\mu$ on $\mathbb{T}^n$ that, $\lambda^{-1}\cdot\lambda_\alpha \to \mathbb{1}$ in $\mathbb{T}^n$. Since $\lambda \mapsto \sigma_\lambda(x)$ is continuous at $\lambda = \mathbb{1}$, there exists $\alpha_2 \in \Gamma$ such that whenever $\alpha \geqslant \alpha_2$,

$$p(\sigma_{\lambda_\alpha}(x) - \sigma_\lambda(x)) = p(\sigma_{\lambda^{-1}\cdot\lambda_\alpha}(x) - \sigma_{\mathbb{1}}(x)) < \epsilon/2. \tag{III.0.6}$$

Let $\beta \in \Gamma$ be such that $\beta \geqslant \alpha_1$ and $\beta \geqslant \alpha_2$ (since, in a directed set, every pair of elements has an upper bound, such a β exists). So, it follows from (III.0.4) and (III.0.6) that whenever $\alpha \geqslant \beta$,

$$p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_\lambda(x)) \leqslant p(\sigma_{\lambda_\alpha}(x_\alpha) - \sigma_{\lambda_\alpha}(x)) + p(\sigma_{\lambda_\alpha}(x) - \sigma_\lambda(x)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

$\square$

III.0.2. Quasi-completeness and the Pettis Integral.

Let X be an LCTVS. In this section we introduce an integral of an X-valued continuous function on a compact Hausdorff space, well-known as Pettis integral.

Definition 7 (Bounded nets and quasi-completeness)**.** Let $(\Gamma, \geqslant)$ be a directed set. A net $(x_\alpha)_{\alpha \in \Gamma}$ in an LCTVS X is said to be *bounded* if for each continuous seminorm p on X, there exists a number $B < \infty$ such that $p(x_\alpha) < B$ for all $\alpha \in \Gamma$. The LCTVS X is said to be *quasi-complete* if every bounded Cauchy net of X converges. For example, complete topological spaces, such as Fréchet spaces, are automatically quasi-complete.

The following is an example of an LCTVS which is quasi-complete but not complete. Let $\mathscr{D}(\Omega)$ be the space of all infinitely differentiable functions defined on the open subset $\Omega \subset \mathbb{R}^n$ and having compact support (the space $\mathscr{D}(\Omega)$ is commonly known as the space of *test functions* on Ω). The topological dual of $\mathscr{D}(\Omega)$ is known as the space of *Schwartz distributions*. The space of Schwartz distributions with the weak* topology is quasi-complete but not complete.

Definition 8. Let K be a compact Hausdorff space and let μ be a Borel measure on K. Let f be a continuous map from K to an LCTVS X. An element $x \in X$ is called a *Pettis integral* of f on K with respect to μ if for all $\phi \in X'$,

$$\phi(x) = \int_K (\phi \circ f)\, d\mu, \tag{III.0.7}$$

where X' denotes the space of continuous linear functionals on X and the right hand side of (III.0.7) is an integral of a continuous function.

It can be shown that if X is quasi-complete, then there exists a unique $x \in X$ such that (III.0.7) holds and we denote the Pettis integral of f on K with respect to μ by $\int_K f\, d\mu$. See [5, p. INT III.32-39]) for more detailed discussion on Pettis integral, including a proof of the above existence result. In our applications we have considered only complete LCTVS.

III.0.3. Haar Measure on Locally Compact Groups

Let G be a locally compact topological group and let μ be a regular Borel measure on G such that for all $g \in G$ and all Borel measurable subsets $E \subset G$, $\mu(gE) = \mu(E)$. Then μ is said to be a left-invariant *Haar measure* on G (right-invariant Haar measure is defined similarly). It is an important result in topological group theory that for every locally compact group there exists a non-trivial Haar measure and it is unique up to a multiplicative constant, i.e., if μ_1, μ_2 be two different (left-invariant) Haar measures on G, then there exists $c > 0$ such that for every Borel measurable subset E of G, $\mu_1(E) = c.\mu_2(E)$. For more information on Haar measure, see [11].

Now we describe the construction of the Haar measure on the torus group $\mathbb{T}^n$. This is the only Haar measure we will be using in the thesis. Let ϕ be a continuous linear functional on $\mathcal{C}(\mathbb{T}^n)$ (here $\mathcal{C}(\mathbb{T}^n)$ is the space of continuous functions on $\mathbb{T}^n$) defined by,

$$\phi(f) = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} f(e^{i\theta_1}, \cdots, e^{i\theta_n})\ d\theta_n \cdots d\theta_1. \tag{III.0.8}$$

By a theorem of F. Riesz (see [28, p. 34]), there exists a unique Radon measure μ on $\mathbb{T}^n$ (with the normalization condition $\mu(\mathbb{T}^n) = 1$) such that

$$\int_{\mathbb{T}^n} f\ d\mu = \frac{1}{(2\pi)^n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} f(e^{i\theta_1}, \cdots, e^{i\theta_n})\ d\theta_n \cdots d\theta_1. \tag{III.0.9}$$

Let $\lambda \in \mathbb{T}^n$ and let $\sigma_\lambda : \mathcal{C}(\mathbb{T}^n) \to \mathcal{C}(\mathbb{T}^n)$ be the representation defined by $\sigma_\lambda(f)(\xi) = f(\lambda \cdot \xi)$, where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{T}^n$ and $\lambda \cdot \xi = (\lambda_1 \cdot \xi_1, \cdots, \lambda_n \cdot \xi_n)$. Since ϕ in (III.0.8) is invariant under σ_λ, that is, $\phi(\sigma_\lambda(f)) = \phi(f)$, it follows from (III.0.9) that

$$\int_{\mathbb{T}^n} f(\lambda)\ d(\mu(\lambda\cdot\xi)) = \int_{\mathbb{T}^n} f(\lambda)\ d(\mu(\lambda)). \tag{III.0.10}$$

Therefore, the measure μ satisfying (III.0.9) is the Haar measure on $\mathbb{T}^n$.

III.0.4. Fejér Theorem

Let X be a quasi-complete LCTVS and let σ be a continuous representation of $\mathbb{T}^n$ on X. For each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$, let

$$\Pi_\alpha^\sigma(x) = \int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_\lambda(x)\, d\lambda, \tag{III.0.11}$$

be the Pettis integral of the continuous function $\lambda \mapsto \lambda^{-\alpha}\sigma_\lambda(x)$ on $\mathbb{T}^n$ with respect to the normalised Haar measure μ, where we write $d\mu(\lambda) = d\lambda$. More details on the significance of the Π_α^σ can be found in Section III.0.5 below. Given an $x \in X$, the *Fourier series* of x with respect to the continuous representation σ is defined to be the formal series

$$x \sim \sum_{\alpha\in\mathbb{Z}^n} \Pi_\alpha^\sigma(x), \tag{III.0.12}$$

For a positive integer N, define the N^{th} *square partial sum* of the Fourier series in (III.0.12) by

$$S_N^\sigma(x) = \sum_{|\alpha|_\infty \leqslant N} \Pi_\alpha^\sigma(x), \tag{III.0.13}$$

where $|\alpha|_\infty := \max\left\{ |\alpha_j|, 1 \leqslant j \leqslant n\right\}$. We are ready to state an abstract version of Fejér theorem.

Theorem III.0.1. *Let σ be a continuous representation of $\mathbb{T}^n$ on an LCTVS X and let $x \in X$. Then the Cesàro means of the square partial sums of the Fourier series of x (with respect to σ) converge to x in the topology of X.*

Proof. Write the Cesàro means of the square partial sums of the Fourier series of x as,

$$\begin{aligned}
C_N^\sigma(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k^\sigma(x) &= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{|\alpha|_\infty\leqslant k}\Pi_\alpha^\sigma(x)\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{|\alpha|_\infty\leqslant k}\left(\int_{\mathbb{T}^n}\lambda^{-\alpha}\sigma_\lambda(x)\ d\lambda\right),\\
&= \frac{1}{N+1}\sum_{k=0}^{N}\int_{\mathbb{T}^n}\left(\sum_{|\alpha|_\infty\leqslant k}\lambda^{-\alpha}\right)\sigma_\lambda(x)\ d\lambda\\
&= \frac{1}{N+1}\int_{\mathbb{T}^n}\sum_{k=0}^{N}\left(\sum_{|\alpha|_\infty\leqslant k}\lambda^{-\alpha}\right)\sigma_\lambda(x)\ d\lambda,\\
&= \int_{\mathbb{T}^n} F_N(\lambda)\,\sigma_\lambda(x)\ d\lambda
\end{aligned} \tag{III.0.14}$$

where $F_N(\lambda) = \frac{1}{N+1}\sum_{k-0}^{N}\left(\sum_{|\alpha|_\infty\leqslant k}\lambda^{-\alpha}\right)$ is called the Fejér kernel. We introduce polar coordinates by letting $\lambda = (e^{i\theta_1},\cdots,e^{i\theta_n})$. Then

$$\begin{aligned}
F_N(\lambda) = F_N(e^{i\theta_1},\cdots,e^{i\theta_n}) &= \frac{1}{N+1}\sum_{k-0}^{N}\left(\prod_{j=1}^{n}\Big(\sum_{\alpha_j=-k}^{k}e^{-i\alpha_j\theta_j}\Big)\right)\\
&= \frac{1}{N+1}\prod_{j=1}^{n}\left(\sum_{k-0}^{N}\Big(\sum_{\alpha_j=-k}^{k}e^{-i\alpha_j\theta_j}\Big)\right)
\end{aligned} \tag{III.0.15}$$

Note that for $(\theta_1,\cdots,\theta_n)\neq(0,\cdots,0)$

$$\begin{aligned}
\sum_{\alpha_j=-k}^{k}e^{-i\alpha_j\theta_j} = e^{-i\theta_j k}\big(1+e^{i\theta_j}+\cdots+e^{i2k\theta_j}\big) &= e^{-ik\theta_j}\left(\sum_{\alpha_j=0}^{2k}e^{i\alpha_j\theta_j}\right)\\
&= e^{-ik\theta_j}\left(\frac{e^{i(2k+1)\theta_j}-1}{e^{i\theta_j}-1}\right)
\end{aligned} \tag{III.0.16}$$

Multiplying numerator and denominator of (III.0.16) by $e^{-i\frac{1}{2}\theta_j}$ we get,

$$\sum_{\alpha_j=-k}^{k}e^{-i\alpha_j\theta_j} = \frac{e^{i(k+\frac{1}{2})\theta_j}-e^{-i(k+\frac{1}{2})\theta_j}}{e^{i\frac{1}{2}\theta_j}-e^{-i\frac{1}{2}\theta_j}} = \frac{\sin(k+\frac{1}{2})\theta_j}{\sin(\frac{\theta_j}{2})} \tag{III.0.17}$$

So, from (III.0.15) we get,

$$
\begin{aligned}
(N+1)\cdot F_N(e^{i\theta_1},\cdots,e^{i\theta_n}) &= \prod_{j=1}^{n}\left(\sum_{k=0}^{N}\frac{\sin(k+\frac{1}{2})\theta_j}{\sin(\frac{\theta_j}{2})}\right)\\
&= \prod_{j=1}^{n}\frac{1}{\sin(\frac{\theta_j}{2})}\cdot \mathrm{Im}\left(\sum_{k=0}^{N} e^{i(k+\frac{1}{2})\theta_j}\right)\\
&= \prod_{j=1}^{n}\frac{1}{\sin(\frac{\theta_j}{2})}\cdot \mathrm{Im}\left(\frac{e^{i(N+1)\theta_j}-1}{e^{i\frac{1}{2}\theta_j}-e^{-i\frac{1}{2}\theta_j}}\right)\\
&= \prod_{j=1}^{n}\frac{1}{\sin(\frac{\theta_j}{2})}\cdot \frac{1-\cos(N+1)\theta_j}{2\sin(\frac{\theta_j}{2})}\\
&= \prod_{j=1}^{n}\frac{\sin^2\left(\left(\frac{N+1}{2}\right)\theta_j\right)}{\sin^2\left(\frac{\theta_j}{2}\right)}. \qquad \text{(III.0.18)}
\end{aligned}
$$

Now we write the Fejér kernel in a simplified and useful form as,

$$
F_N(e^{i\theta_1},\cdots,e^{i\theta_n}) = \frac{1}{N+1}\prod_{j=1}^{n}\frac{\sin^2\left(\frac{N+1}{2}\right)\theta_j}{\sin^2\left(\frac{\theta_j}{2}\right)}. \qquad \text{(III.0.19)}
$$

The Fejér kernel has the following properties,

(a) $F_N \geqslant 0$ for all N.

(b) $\int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda = 1$.

(c) For each $\delta > 0$, $F_N \to 0$ uniformly on $\mathbb{T}^n \backslash B(\mathbb{1},\delta)$, where $B(\mathbb{1},\delta)$ be the n-dimensional sphere centered at $\mathbb{1} = (1,1,...,1)$ and radius δ.

Note that (a) follows directly from (III.0.19). To prove (b) we use the result

$$
\int_0^{2\pi} e^{-im\theta}\ d\theta = \begin{cases} 0 & \text{if } m \neq 0, m \in \mathbb{Z} \\ 2\pi & \text{if } m = 0. \end{cases} \qquad \text{(III.0.20)}
$$

By (III.0.15),

$$
\int_{\mathbb{T}^n} F_N(\lambda)\ d\lambda = \frac{1}{(2\pi)^n}\int_{\theta_1=0}^{2\pi}\cdots\int_{\theta_n=0}^{2\pi}\left(\frac{1}{N+1}\prod_{j=1}^{n}\left(\sum_{k-0}^{N}\left(\sum_{\alpha_j=-k}^{k} e^{-i\alpha_j\theta_j}\right)\right)\right)\ d\theta_n\cdots d\theta_1.
$$

$$= \frac{1}{(2\pi)^n} \cdot \frac{1}{N+1} \sum_{k=0}^{N} \Bigg(\prod_{j=1}^{n} \Bigg(\sum_{\alpha_j=-k}^{k} \int_{\theta_j=0}^{2\pi} e^{-i\alpha_j\theta_j} \ d\theta_j \Bigg)\Bigg)$$

$$= \frac{1}{(2\pi)^n} \cdot \frac{1}{N+1} \sum_{k=0}^{N} (2\pi)^n \quad \text{(applying the result in (III.0.20) for each integral)}$$

$$= \frac{1}{(2\pi)^n} \cdot (2\pi)^n \cdot \frac{1}{N+1} \sum_{k=0}^{N} 1 = 1. \tag{III.0.21}$$

To prove (c), let $\delta > 0$. Then there exists $\theta > 0$ such that the set $U = \Big\{(e^{ix_1}, \cdots, e^{ix_n}) \in \mathbb{T}^n : |x_j| < \theta\Big\}$ is a subset of $B(\mathbb{1}, \delta) \cap \mathbb{T}^n$. Let $\lambda = (e^{i\zeta_1}, \cdots, e^{i\zeta_n}) \in \mathbb{T}^n \backslash B(\mathbb{1}, \delta)$ be arbitrary. Then $|\zeta_j| \geqslant \theta$ for all j, which gives $1/\sin^2\left(\frac{\zeta_j}{2}\right) < 1/\sin^2\left(\frac{\theta}{2}\right)$ for all j. Therefore

$$0 \leqslant F_N(\lambda) < \frac{1}{N+1} \prod_{j=1}^{n} \frac{1}{\sin^2\left(\frac{\theta}{2}\right)}. \tag{III.0.22}$$

Taking supremum over all such λ we get,

$$0 \leqslant \sup_{\lambda \in \left(\mathbb{T}^n \backslash B(\mathbb{1}, \delta)\right)} F_N(\lambda) < \frac{1}{N+1} \cdot \frac{1}{\sin^{2n}\left(\frac{\theta}{2}\right)}. \tag{III.0.23}$$

This proves that as $N \to \infty$, $F_N \to 0$ uniformly on $\mathbb{T}^n \backslash B(\mathbb{1}, \delta)$. Let p be a continuous invariant seminorm on X. Then

$$p(C_N(x) - x) = p\Big(\int_{\mathbb{T}^n} F_N(\lambda)\, \sigma_\lambda(x) \ d\lambda - x \cdot \int_{\mathbb{T}^n} F_N(\lambda) \ d\lambda\Big), \tag{III.0.24}$$

where we have used the fact that $\int_{\mathbb{T}^n} F_N(\lambda) \ d\lambda = 1$. Now let $x_0 = x \cdot \int_{\mathbb{T}^n} F_N(\lambda) \ d\lambda - \int_{\mathbb{T}^n} x \cdot F_N(\lambda) \ d\lambda$, then $x_0 \in X$. We claim that $x_0 = 0$. If not, then by the Hahn-Banach theorem, there exists $\phi \in X'$ such that $\phi(x_0) \neq 0$. However, for all $\phi \in X'$,

$$\begin{aligned} \phi(x_0) &= \phi\Big(x \cdot \int_{\mathbb{T}^n} F_N(\lambda) \ d\lambda - \int_{\mathbb{T}^n} x \cdot F_N(\lambda) \ d\lambda\Big) \\ &= \phi\Big(x \cdot \int_{\mathbb{T}^n} F_N(\lambda) \ d\lambda\Big) - \phi\Big(\int_{\mathbb{T}^n} x \cdot F_N(\lambda) \ d\lambda\Big) \ \text{(using linearity)} \end{aligned} \tag{III.0.25}$$

$$= \Big(\int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda\Big)\cdot\phi(x) - \int\limits_{\mathbb{T}^n} \phi\big(x\cdot F_N(\lambda)\big)\ d\lambda \tag{III.0.26}$$
$$= \Big(\int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda\Big)\cdot\phi(x) - \Big(\int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda\Big)\cdot\phi(x) = 0,$$

where, to get (III.0.26) we have used the property of vector valued (X valued) integral of the function $\lambda \mapsto x\cdot F_N(\lambda)$ (see (III.0.7)). Hence, a contradiction. Therefore $x\cdot \int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda = \int\limits_{\mathbb{T}^n} x\cdot F_N(\lambda)\ d\lambda$. Now, from (III.0.24) we write,

$$\begin{aligned} p(C_N(x)-x) &= p\Big(\int\limits_{\mathbb{T}^n} F_N(\lambda)\cdot\sigma_\lambda(x)\ d\lambda - x\cdot\int\limits_{\mathbb{T}^n} F_N(\lambda)\ d\lambda\Big) \\ &= p\Big(\int\limits_{\mathbb{T}^n} F_N(\lambda)\cdot\sigma_\lambda(x)\ d\lambda - \int\limits_{\mathbb{T}^n} x\cdot F_N(\lambda)\ d\lambda\Big) \\ &= p\Big(\int\limits_{\mathbb{T}^n} \big(\sigma_\lambda(x)-x\big)\cdot F_N(\lambda)\ d\lambda\Big) \leqslant \int\limits_{\mathbb{T}^n} F_N(\lambda)\cdot p(\sigma_\lambda(x)-x)\ d\lambda, \end{aligned} \tag{III.0.27}$$

where the last inequality is due to Proposition 6 in [5, p. INT III.37] and the positivity of $F_N(\lambda)$. Let $\epsilon > 0$ be given. Since $\lambda \mapsto \sigma_\lambda(x)$ is continuous and p is a continuous seminorm, there exists $\delta > 0$ such that

$$p(\sigma_\lambda(x) - \sigma_{\mathbb{1}}(x)) = p(\sigma_\lambda(x)-x) < \epsilon/2 \quad \text{whenever } \lambda \in (\mathbb{T}^n \cap B(\mathbb{1},\delta)). \tag{III.0.28}$$

On the region $\mathbb{T}^n \backslash B(\mathbb{1},\delta)$ we can bound p as follows,

$$p(\sigma_\lambda(x)-x) \leqslant p(\sigma_\lambda(x)) + p(x) = 2\cdot p(x).$$

By property (c) of F_N above, there exists $N_1 \in \mathbb{N}$ such that whenever $N \geqslant N_1$,

$$F_N(\lambda) < \frac{\epsilon}{4\cdot p(x)} \quad \text{for all } \lambda \in \mathbb{T}^n\backslash B(\mathbb{1},\delta). \tag{III.0.29}$$

Now, from equation (III.0.27), whenever $N \geqslant N_1$,

$$p(C_N(x)-x) \leqslant \int\limits_{\mathbb{T}^n} F_N(\lambda)\cdot p(\sigma_\lambda(x)-x)\ d\lambda$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{T}^n \cap B(\mathbb{1},\delta)} F_N(\lambda) \cdot p(\sigma_\lambda(x) - x) \ d\lambda + \int\limits_{\mathbb{T}^n \setminus B(\mathbb{1},\delta)} F_N(\lambda) \cdot p(\sigma_\lambda(x) - x) \ d\lambda \\
&< \epsilon/2 \int\limits_{\mathbb{T}^n} F_n(\lambda) \ d\lambda + \frac{\epsilon}{4 \cdot p(x)} \cdot 2 \cdot p(x) \quad \text{by (III.0.28) and (III.0.29),} \\
&= \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

□

We now deduce the classical Fejér theorem from the abstract version above.

Classcial Fejér Theorem

Let $X = \mathcal{C}(\mathbb{T})$, the space of continuous functions on the unit circle $\mathbb{T}$. Note that $\mathcal{C}(\mathbb{T})$ is a Banach space with the sup norm. Let σ be the representation of $\mathbb{T}$ on X defined by,

$$
\sigma(\lambda, f)(\zeta) = f(\lambda\zeta) \quad \text{for all } \lambda, \zeta \in \mathbb{T} \text{ and } f \in \mathcal{C}(\mathbb{T}). \tag{III.0.30}
$$

We call the representation in (III.0.30) the *standard representation* of $\mathbb{T}$ on X. The next lemma shows that if $X = \mathcal{C}(\mathbb{T})$ and if σ be the standard representation of $\mathbb{T}$ on X, then the abstract Fourier series with respect to the standard representation is the classical Fourier series (thus the classical Fourier series is a special case of the abstract Fourier series defined in (III.0.12)).

Lemma III.0.1. *Let $X = \mathcal{C}(\mathbb{T})$ and $f \in X$. Let σ be the standard representation of $\mathbb{T}$ on X. Then the abstract Fourier series of f with respect to σ defined in* (III.0.12) *is the classical Fourier series of f.*

Proof. Let $\eta = e^{i\phi} \in \mathbb{T}$. Now, for $\alpha \in \mathbb{Z}$ and the given representation σ, the α-th Fourier mode of the abstract Fourier series

$$
\Pi_\alpha^\sigma(f)(\eta) = \int\limits_{\mathbb{T}} \lambda^{-\alpha} f(\lambda\eta) \, d\lambda
$$

$$= \frac{1}{2\pi}\int\limits_{\mathbb{T}} e^{-i\alpha\theta} f(e^{i\theta}\eta)\, d\theta, \quad \text{by Equation (III.0.9) and by taking } \lambda = e^{i\theta},$$
$$= \frac{1}{2\pi}\int\limits_{\mathbb{T}} e^{-i\alpha\theta} f(e^{i(\theta+\phi)})\, d\theta, \quad \text{since } \eta = e^{i\phi},$$
$$= \frac{1}{2\pi}\int\limits_{\mathbb{T}} e^{-i\alpha(t-\phi)} f(e^{it})\, dt, \quad \text{letting } t = \theta+\phi, \text{ so } dt = d\theta,$$
$$= \left(\frac{1}{2\pi}\int\limits_{\mathbb{T}} e^{-i\alpha t} f(e^{it})\, dt\right)\cdot e^{i\alpha\phi}$$
$$= c_\alpha(f)\cdot\eta^\alpha, \quad \text{where } c_\alpha(f) = \frac{1}{2\pi}\int\limits_{\mathbb{T}} e^{-i\alpha t} f(e^{it})\, dt,$$

and $c_\alpha(f)$ is the α-th Fourier coefficient of the classical Fourier series f. □

The next result is now a corollary of Theorem III.0.1.

Corollary III.0.2. *(Classical Fejér Theorem) The Cesàro means of the partial sums of the Fourier series of a continuous function on $\mathbb{T}$ converge to the function uniformly on $\mathbb{T}$.*

Proof. It follows from Theorem III.0.1 that we only need to show that the standard representation σ of $\mathbb{T}$ on $\mathcal{C}(\mathbb{T})$ is continuous. For the continuity of σ, we follow Proposition III.0.1 and show that the supnorm over $\mathbb{T}$ is invariant (with respect to the representation σ), and for a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{T}$, if $\lambda_j \to \mathbb{1}$ in $\mathbb{T}$, then for all $f \in \mathcal{C}(\mathbb{T})$, $\sigma_{\lambda_j}(f) \to f$ uniformly on $\mathbb{T}$, where $\sigma(\lambda_j, f) = \sigma_{\lambda_j}(f)$. Now, for $\lambda \in \mathbb{T}$,

$$\|\sigma_\lambda(f)\|_{\text{sup}} = \sup_{\zeta\in\mathbb{T}} |f(\lambda\zeta)| = \sup_{\lambda\zeta\in\mathbb{T}} |f(\lambda\zeta)| = \|f\|_{\text{sup}}$$

This proves that the supnorm is invariant. Now let $\epsilon > 0$ and $f \in \mathcal{C}(\mathbb{T})$. Since f is uniformly continuous on $\mathbb{T}$ (continuous function on a compact set), for the given ϵ there exists a $\delta > 0$ such that for all $\eta, \xi \in \mathbb{T}$,

$$\text{whenever } |\eta - \xi| < \delta, \quad |f(\eta) - f(\xi)| < \epsilon/2. \tag{III.0.31}$$

Since $\lambda_j \to \mathbb{1}$ in $\mathbb{T}$, for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that when $j \geqslant N$,

$$|\lambda_j - \mathbb{1}| < \delta. \tag{III.0.32}$$

So, for all $\zeta \in \mathbb{T}$, whenever $j \geqslant N$,

$$|\lambda_j \zeta - \zeta| = |\zeta| \,.\, |\lambda_j - \mathbb{1}| = |\lambda_j - \mathbb{1}| < \delta. \tag{III.0.33}$$

Therefore, whenever $j \geqslant N$, by (III.0.31) and (III.0.33),

$$\left\| \sigma_{\lambda_j}(f) - f \right\|_{\sup} = \sup_{\zeta \in \mathbb{T}} |f(\lambda_j \zeta) - f(\zeta)| \leqslant \epsilon/2 < \epsilon. \tag{III.0.34}$$

This shows $\sigma_{\lambda_j}(f) \to f$ uniformly on $\mathbb{T}$, that is, in the topology of $\mathcal{C}(\mathbb{T})$. □

Fejer theorem for $L^p(\mathbb{T}^n)$

For $1 \leqslant p < \infty$, let $X = L^p(\mathbb{T}^n)$. By the Riesz-Fischer Theorem X is a Banach space with p-norm $\|\cdot\|_p$ defined as

$$\|f\|_p^p = \int_{\mathbb{T}^n} |f(\lambda)|^p \, dV(\lambda), \tag{III.0.35}$$

where $f \in L^p(\mathbb{T}^n)$ and dV is the volume (Lebesgue) measure on $\mathbb{T}^n$. For an element $f \in L^p(\mathbb{T}^n)$, note that the integral in the right hand side of (III.0.35) is the Haar integral of the measurable function $|f|^p$ with the Haar measure μ_V where $\mu_V(\lambda) = dV(\lambda)$. Let σ be the standard representation of $\mathbb{T}^n$ on X, that is, for $\lambda, \zeta \in \mathbb{T}^n$ and $f \in L^p(\mathbb{T}^n)$,

$$\sigma(\lambda, f)(\zeta) = f(\lambda \cdot \zeta). \tag{III.0.36}$$

Lemma III.0.2. *Let $X = L^p(\mathbb{T}^n)$ and $f \in X$. Let σ be the standard representation of $\mathbb{T}^n$ on X. Then the abstract Fourier series of f is the classical Fourier series of f.*

Proof. The proof is similar to the proof of Lemma III.0.1. □

Next we present the Fejér theorem for $L^p(\mathbb{T}^n)$ as a corollary of Theorem III.0.1.

Corollary III.0.3. *(Fejér Theorem for $L^p(\mathbb{T}^n)$) For $1 \leqslant p < \infty$, the Cesàro means of the square partial sums of the Fourier series of a function in $L^p(\mathbb{T}^n)$ converge to the function in the p-norm.*

Proof. We only need to prove the continuity of the standard representation of $\mathbb{T}^n$ on $L^p(\mathbb{T}^n)$ in the p-norm defined in (III.0.35). It is sufficient to show that the p-norm is invariant with respect to the standard representation, and if $\{\lambda_j\} \subset \mathbb{T}^n$ be such that $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, then for all $f \in L^p(\mathbb{T}^n)$, $\sigma_{\lambda_j}(f) \to f$ in the p-norm defined in (III.0.35), where $\sigma_{\lambda_j}(f)(\zeta) = \sigma(\lambda_j, f)(\zeta) = f(\lambda \cdot \zeta)$, $\zeta \in \mathbb{T}^n$. Let $\lambda \in \mathbb{T}^n$ and let χ_E be the characteristic function of a measurable subset E of $\mathbb{T}^n$. For the Haar measure $dV(\lambda)$,

$$\int_E \chi_E(\zeta\lambda)\, dV(\lambda) = \int_{\zeta^{-1}E} dV(\lambda) = \mu_V(\zeta^{-1}E) = \mu_V(E) = \int_E \chi_E(\lambda)\, dV(\lambda).$$

Let $f \in L^p(\mathbb{T}^n)$. Then the measurable function $|f|^p$ can be approximated pointwise by an increasing sequence of simple functions $\{\phi_n\}$ from below. Using monotone convergence theorem we get,

$$\int_{\mathbb{T}^n} |f(\lambda\zeta)|^p \;\, dV(\lambda) = \lim_{n\to\infty} \int_{\mathbb{T}^n} \phi_n(\lambda\zeta) \;\, dV(\lambda) = \lim_{n\to\infty} \int_{\mathbb{T}^n} \phi_n(\lambda) \;\, dV(\lambda) = \int_{\mathbb{T}^n} |f(\lambda)|^p \;\, dV(\lambda).$$

Therefore,

$$\|\sigma_\lambda(f)\|_p^p = \int_{\mathbb{T}^n} |f(\lambda\zeta)|^p \;\, dV(\lambda) = \int_{\mathbb{T}^n} |f(\lambda)|^p \;\, dV(\lambda) = \|f\|_p^p, \tag{III.0.37}$$

Taking p-th root on both sides, we get $\|\sigma_\lambda(f)\|_p = \|f\|_p$. This shows that the p-norm is invariant.

Denote by $\mathcal{C}_c(\mathbb{T}^n)$ the space of continuous compactly supported functions on $\mathbb{T}^n$. Let $\epsilon > 0$. By the approximation property by continuous compactly supported functions, there exists $g_\epsilon \in \mathcal{C}_c(\mathbb{T}^n)$ such that

$$\|f - g_\epsilon\|_p < \epsilon/3 \tag{III.0.38}$$

In fact, this g_ϵ is uniformly continuous on $\mathbb{T}^n$. So, there exists a $\delta > 0$ such that

$$\text{whenever } |z-w| < \delta, \quad |g_\epsilon(z) - g_\epsilon(w)| < \left(\frac{\epsilon/3}{(2\pi)^n}\right)^{1/p} \tag{III.0.39}$$

Since $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that when $j \geqslant N$,

$$|\lambda_j - \mathbb{1}| < \delta. \tag{III.0.40}$$

So, whenever $j \geqslant N$, then for all $\zeta \in \mathbb{T}^n$,

$$|\lambda_j \cdot \zeta - \zeta| = |\zeta| \,.\, |\lambda_j - \mathbb{1}| = |\lambda_j - \mathbb{1}| < \delta. \tag{III.0.41}$$

So, for all $j \geqslant N$,

$$\begin{aligned} &\left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_p \\ &= \int_{\mathbb{T}^n} |g_\epsilon(\lambda_j \cdot \zeta) - g_\epsilon(\zeta)|^p \, dV(\zeta) \\ &< \epsilon/3, \quad \text{by (III.0.39), (III.0.40) and (III.0.41).} \end{aligned} \tag{III.0.42}$$

Therefore whenever $j \geqslant N$,

$$\begin{aligned} &\left\|\sigma_{\lambda_j}(f) - f\right\|_p \\ &= \left\|\sigma_{\lambda_j}(f - g_\epsilon) + \sigma_{\lambda_j}(g_\epsilon) - g_\epsilon + g_\epsilon - f\right\|_p, \quad \text{since } \sigma_{\lambda_j} \text{ is linear,} \\ &\leqslant \left\|\sigma_{\lambda_j}(f - g_\epsilon)\right\|_p + \left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_p + \|g_\epsilon - f\|_p \\ &\leqslant \|f - g_\epsilon\|_p + \left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_p + \|g_\epsilon - f\|_p < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

□

III.0.5. Fourier Modes and Trigonometric polynomials

Throughout this section X denotes a quasi-complete LCTVS and σ denotes a continuous representation of $\mathbb{T}^n$ on X. For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{T}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$, write

$\lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}$ in multi-index notation. Define

$$X_\alpha^\sigma = \{x \in X : \sigma_\lambda(x) = \lambda^\alpha x \text{ for all } \lambda \in \mathbb{T}^n\} \tag{III.0.43}$$

Lemma III.0.3. *X_α^σ is a closed invariant (invariant with respect to σ) subspace of X.*

Proof. Observe that for a fixed $\lambda \in \mathbb{T}^n$, the function $x \mapsto \sigma_\lambda(x) - \lambda^\alpha x$ is continuous. Now

$$X_\alpha^\sigma = \bigcap_{\lambda\in\mathbb{T}^n} \big\{x \in X : \sigma_\lambda(x) - \lambda^\alpha x = 0\big\}$$

is an arbitrary intersection of closed sets, and therefore closed. Let $x \in X_\alpha^\sigma$ and $\lambda \in \mathbb{T}^n$. To prove the invariance, we show that $\sigma_\lambda(x) \in X_\alpha^\sigma$. Note that for all $\mu \in \mathbb{T}^n$,

$$\sigma_\mu\big(\sigma_\lambda(x)\big) = \sigma_{\mu\cdot\lambda}(x) = (\mu\cdot\lambda)^\alpha x = \mu^\alpha \cdot \lambda^\alpha x = \mu^\alpha \sigma_\lambda(x). \tag{III.0.44}$$

□

Proposition III.0.2. *The map Π_α^σ in* (III.0.11) *is a continuous linear projection from X onto X_α^σ.*

Proof. Linearity of Π_α^σ follows from the linearity of σ_λ. Recall from Proposition III.0.1 that there exists a family P of continuous invariant seminorms that generates the locally convex topology of X. To prove the continuity of Π_α^σ, it is sufficient to show there exists a constant $c > 0$ such that for all $p \in \mathsf{P}$,

$$p(\Pi_\alpha^\sigma(x)) \leqslant c.\, p(x) \quad \text{for all } x \in X. \tag{III.0.45}$$

Observe that

$$p(\Pi_\alpha^\sigma(x)) = p\left(\int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_\lambda(x)\, d\lambda\right) \leqslant \int_{\mathbb{T}^n} p\Big(\lambda^{-\alpha}\sigma_\lambda(x)\Big)\, d\lambda = \int_{\mathbb{T}^n} p(x)\, d\lambda = p(x), \tag{III.0.46}$$

where the inequality is due to Proposition 6 in [5, p. INT III.37]. Now, if $x \in X_\alpha^\sigma$, then

$$\Pi_\alpha^\sigma(x) = \int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_\lambda(x)\, d\lambda = \int_{\mathbb{T}^n} \lambda^{-\alpha}\cdot\lambda^\alpha x\, d\lambda = x\cdot\left(\int_{\mathbb{T}^n} d\lambda\right) = x. \tag{III.0.47}$$

Note that by (III.0.47), $X_\alpha^\sigma \subset \Pi_\alpha^\sigma(X)$. To prove that the range of Π_α^σ is X_α^σ, we show $\Pi_\alpha^\sigma(X) \subset X_\alpha^\sigma$. Let $\eta \in \mathbb{T}^n$. It is sufficient to prove that $\sigma_\eta(\Pi_\alpha^\sigma(x)) = \eta^\alpha . \Pi_\alpha^\sigma(x)$ for all $x \in X$. Note that

$$\begin{aligned}
\sigma_\eta(\Pi_\alpha^\sigma(x)) &= \sigma_\eta\left(\int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_\lambda(x)\, d\lambda\right) \\
&= \int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_{\eta\cdot\lambda}(x)\, d\lambda \\
&\qquad\qquad \left(\text{since } \int_{\mathbb{T}^n} \lambda^{-\alpha}\sigma_\lambda(x)\, d\lambda \text{ is the Pettis integral and } \sigma_\eta \circ \sigma_\lambda = \sigma_{\eta\cdot\lambda}\right) \\
&= \eta^\alpha \int_{\mathbb{T}^n} (\eta\cdot\lambda)^{-\alpha}\sigma_{\eta\cdot\lambda}(x)\, d\lambda = \eta^\alpha \cdot \Pi_\alpha^\sigma(x).
\end{aligned}$$

□

We define the *Trigonometric polynomials* of X with respect to σ by,

$$TP_\sigma(X) = \operatorname{span}\left\{X_\alpha^\sigma\right\} \quad \text{where } \alpha \in \mathbb{Z}^n; \tag{III.0.48}$$

that is, if $x \in TP_\sigma(X)$ if and only if x is a finite sum of elements of X each of which belongs to X_α^σ for some α. Observe that $TP_\sigma(X)$ is a linear subspace of X.

Theorem III.0.4. *Let X be a Hausdorff, quasi-complete LCTVS and let σ be a continuous representation of $\mathbb{T}^n$ on X. Then $TP_\sigma(X)$ is dense in X.*

Proof. Let $x \in X$. Observe that the sequence of Cesàro means of the partial sums of the Fourier series of x contained in $TP_\sigma(X)$, that is, $\left\{C_N(x)\right\} \subset TP_\sigma(X)$. The result now follows from Theorem III.0.1. □

Proposition III.0.3. *Let X be a quasi-complete LCTVS and let σ be a representation of $\mathbb{T}^n$ on X. Then the following are equivalent.*

(a) σ is continuous.

(b) There exists a family P *of continuous invariant seminorms on* X *which generates the topology of* X *and the abstract trigonometric polynomials with respect to* σ *are dense in* X.

Proof. The direction $(a) \Rightarrow (b)$ follows from Proposition III.0.1 and Theorem III.0.4.

For $(b) \Rightarrow (a)$, Let $\epsilon > 0$, let $p \in \mathsf{P}$ and $x \in X$. It is sufficient to show that if the sequence $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, then $\sigma_{\lambda_j}(x) \to x$ in X. Since the abstract trigonometric polynomials are dense, there exists $y \in TP_\sigma(X)$ such that

$$p(x-y) < \frac{\epsilon}{3}. \tag{III.0.49}$$

Now $y \in TP_\sigma(X)$, that is $y = \sum_{|\alpha|_\infty \leqslant N} y_\alpha$, where $\alpha \in \mathbb{Z}^n, y_\alpha \in X_\alpha$ and $N \in \mathbb{N}$. So,

$$\begin{aligned} p(\sigma_{\lambda_j}(y)-y) &= p\Bigg(\sum_{|\alpha|_\infty \leqslant N} \sigma_{\lambda_j}(y_\alpha) - \sum_{|\alpha|_\infty \leqslant N} y_\alpha\Bigg) \\ &= p\Bigg(\sum_{|\alpha|_\infty \leqslant N} \lambda_j^\alpha y_\alpha - \sum_{|\alpha|_\infty \leqslant N} y_\alpha\Bigg) \quad \text{since } y_\alpha \in X_\alpha, \\ &= p\Bigg(\sum_{|\alpha|_\infty \leqslant N} (\lambda_j^\alpha - 1) y_\alpha\Bigg) \leqslant \sum_{|\alpha|_\infty \leqslant N} \big|\lambda_j^\alpha - 1\big| p(y_\alpha), \end{aligned} \tag{III.0.50}$$

where the last inequality holds since p is a seminorm. If $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, then it is easy to see that for all $\alpha \in \mathbb{Z}^n, \lambda_j^\alpha \to 1$ in $\mathbb{T}$. Let $S = \sum_{|\alpha|_\infty \leqslant N} p(y_\alpha)$, so $S > 0$. Then there exists a $N_1 \in \mathbb{N}$ such that for all $j \geqslant N_1$,

$$\big|\lambda_j^\alpha - 1\big| < \frac{\epsilon}{3S}. \tag{III.0.51}$$

So, it follows from (III.0.50) and (III.0.51) that whenever $j \geqslant N_1$,

$$p(\sigma_{\lambda_j}(y)-y) < \frac{\epsilon}{3}. \tag{III.0.52}$$

Now, whenever $j \geqslant N_1$,

$$p(\sigma_{\lambda_j}(x)-x) = p(\sigma_{\lambda_j}(x) - \sigma_{\lambda_j}(y) + \sigma_{\lambda_j}(y) - y + y - x)$$

$$
\begin{aligned}
&\leqslant p(\sigma_{\lambda_j}(x)-\sigma_{\lambda_j}(y))+p(y-x)+p(\sigma_{\lambda_j}(y)-y) \\
&= p(x-y)+p(x-y)+p(\sigma_{\lambda_j}(y)-y), \qquad \text{(III.0.53)}
\end{aligned}
$$

and the proof follows by an '$\epsilon/3$ argument'. □

We will see some applications of this section in Chapter V.

CHAPTER IV

SERIES EXPANSION IN HARTOGS OPEN SETS

In this chapter we use our theory of abstract Fourier series to expand holomorphic functions on Hartogs open sets (defined below) in Hartogs-Laurent series. The main result of this chapter is Theorem IV.0.2. Let $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Write $z \in \mathbb{C}^n$ as $z = (z', \widetilde{z})$ where $z' = (z_1, \cdots, z_k) \in \mathbb{C}^k$ and $\widetilde{z} = (z_{k+1}, \cdots, z_n) \in \mathbb{C}^{n-k}$.

Definition 9. An open set $\Omega \subset \mathbb{C}^n$ is said to be a *Hartogs open set* if $z = (z', \widetilde{z}) \in \Omega$ implies $(\lambda_1 z_1, \cdots, \lambda_k z_k, \widetilde{z}) \in \Omega$ for all $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{T}^k$. To simplify our notation, we write $\lambda \odot z = (\lambda_1 z_1, \cdots, \lambda_k z_k, \widetilde{z})$. If, in addition, Ω is connected, then Ω is said to be a *Hartogs domain*.

Denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$. An example of Hartogs domain in $\mathbb{C}^2$ is the so called *worm domain*, originally introduced by Diederich and Fornaess in 1976, see [14]. The worm domain is an example of a smoothly bounded pseudoconvex domain that does not have a Stein neignboirhood basis. It was proved by Barrett (see [3]) that if Ω is the worm domain of Diederich and Fornaess, then for a sufficiently large positive number k, the Bergman projection $P : L^2(\Omega) \to L^2(\Omega) \bigcap \mathcal{O}(\Omega)$ fails to preserve $W^{k,2}(\Omega)$, where $W^{k,2}(\Omega)$ is the Sobolev space consisting of functions whose derivatives of order up to k are in $L^2(\Omega)$. Depending on the result of Barrett, it was proved by Christ that for such a domain Ω, the Bergman projection fails to preserve $\mathcal{C}^\infty(\overline{\Omega})$, see [12]. Moreover, Barrett also gave an example of a smoothly bounded non-pseudoconvex Hartogs domain D in $\mathbb{C}^2$ such that the Bergman projection $P : L^p(\Omega) \to L^p(\Omega) \bigcap \mathcal{O}(\Omega)$ does not preserve $\mathcal{C}_0^\infty(D)$ for sufficiently large p, see [2]. These results were extended to L^p Sobolev spaces in [1].

IV.0.1. Covering of a Hartogs open set by polyannuli

For $1 \leqslant j \leqslant k$, let r_j, R_j be real numbers such that $0 < r_j < R_j < \infty$. Take V to be a relatively compact open ball in $\mathbb{C}^{n-k}$. Let $\mathcal{P} \subset \mathbb{C}^n$ be such that

$$\mathcal{P} := \big\{(z', \widetilde{z}) \in \mathbb{C}^n : r_j < |z_j| < R_j \ \text{ or } \ |z_j| < R_j \quad \text{for } 1 \leqslant j \leqslant k \text{ and } \widetilde{z} \in V\big\}. \tag{IV.0.1}$$

The set $\mathcal{P}$ is said to be a *polyannulus* in $\mathbb{C}^n$.

Lemma IV.0.1. *Every Hartogs open set in $\mathbb{C}^n$ is a countable union of polyannuli.*

Proof. Let $\Omega \subset \mathbb{C}^n$ be an open set with Hartogs symmetry and $a = (a', \widetilde{a}) \in \Omega$. Since Ω is open, there exists an open set $W \subset \mathbb{C}^k$ and a relatively compact open ball $V_r(p) = \{w \in \mathbb{C}^{n-k} : |w - p| < r\}$ in $\mathbb{C}^{n-k}$, centered at p and radius $r > 0$, where r is a rational number and $p = (p_1, \cdots, p_{n-k}) \in \mathbb{C}^{n-k}$ and each p_j is rational as well; such that $a = (a', \widetilde{a}) \in W \times V_r(p)$. Let $\tau : \Omega \to \mathbb{R}^k_+ \times \mathbb{C}^{n-k}$ be defined by,

$$\tau(z_1, \cdots, z_k, \widetilde{z}) = (|z_1|, \cdots, |z_k|, \widetilde{z}). \tag{IV.0.2}$$

Since τ is an open map, $\tau(W \times V_r(p))$ is open in $\mathbb{R}^k_+ \times \mathbb{C}^{n-k}$. So, there exists rational numbers $0 < r_j < R_j < \infty$ such that $r_j < |a_j| < R_j$ or $|a_j| < R_j$ for all $1 \leqslant j \leqslant k$ and the point $\tau(a) \in \big(\prod_{j=1}^k A_j \times V_r(p)\big) \subset \tau(W \times V_r(p)) \subset \tau(\Omega)$, where

$$A_j = \begin{cases} (r_j, R_j) \ \text{ if } a_j \neq 0 \\ [0, R_j) \ \text{ if } a_j = 0. \end{cases}$$

Therefore, $a = (a', \widetilde{a}) \in \tau^{-1}\big(\prod_{j=1}^k A_j \times V_r(p)\big)$, which is the polyannulus $\mathcal{P}_a = \big\{z = (z', \widetilde{z}) \in \Omega : r_j < |z_j| < R_j \ \text{ or } \ |z_j| < R_j \quad \text{for } 1 \leqslant j \leqslant k \text{ and } \widetilde{z} \in V_r(p)\big\}$. So, $\Omega \subset \bigcup \mathcal{P}_a$, and the union is a countable one. Conversely, each polyannulus $\mathcal{P}_a$ is constructed in such a way that $\mathcal{P}_a \subset \Omega$, and therefore the union $\bigcup \mathcal{P}_a \subset \Omega$. □

IV.0.2. The representation of the torus group on the space of holomorphic functions

Denote by $\mathcal{C}(\Omega)$ the space of continuous functions on a Hartogs open set Ω topologized by the Fréchet topology (that is, the topology of uniform convergence on compact subsets). By Lemma IV.0.1, Ω can be covered by a countable collection of polyannuli, we enumerate them as $\{\mathcal{P}_j : j \in \mathbb{N}\}$. Note that for each natural number j, $\mathcal{P}_j$ is a relatively compact Hartogs open set in Ω and the function

$$f \mapsto \|f\|_{\overline{\mathcal{P}_j}} \tag{IV.0.3}$$

where $\|f\|_{\overline{\mathcal{P}_j}} = \sup_{z\in\overline{\mathcal{P}_j}} |f(z)|$ is a seminorm on $\mathcal{C}(\Omega)$. The Fréchet topology of $\mathcal{C}(\Omega)$ is generated by the family of seminorms

$$\left\{ \|\cdot\|_{\overline{\mathcal{P}_j}} : j \in \mathbb{N}\right\}. \tag{IV.0.4}$$

Let $\sigma : \mathbb{T}^k \times \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ be defined by

$$\sigma(\lambda, f)(z) = f(\lambda \odot z), \tag{IV.0.5}$$

where $\lambda \in \mathbb{T}^k$, $f \in \mathcal{C}(\Omega)$ and $z \in \Omega$. Then the following is true.

Proposition IV.0.1. *The map σ in* (IV.0.5) *is a continuous representation of $\mathbb{T}^k$ in $\mathcal{C}(\Omega)$.*

Proof. First we show that σ in (IV.0.5) defines a representation of $\mathbb{T}^k$ in $\mathcal{C}(\Omega)$. For $\lambda \in \mathbb{T}^k$, $f \in \mathcal{C}(\Omega)$ and $z = (z', \widetilde{z}) \in \Omega$, write

$$\sigma(\lambda, f)(z) = \sigma_\lambda(f)(z) = f(\lambda \odot z). \tag{IV.0.6}$$

We first show that $\sigma_\lambda \in \mathrm{Aut}(\mathcal{C}(\Omega))$. It is easy to see that σ_λ is linear and one to one. Also, for every $f \in \mathcal{C}(\Omega)$, define $g(z) = f(\lambda^{-1} \odot z)$. Then $g \in \mathcal{C}(\Omega)$ and $\sigma_\lambda(g) = f$. This proves σ_λ is bijective, hence σ_λ^{-1} exists. Recall that the family of seminorms $\left\{ \|\cdot\|_{\overline{\mathcal{P}_j}} : j \in \mathbb{N}\right\}$ in (IV.0.4) that generates the Fréchet topology of $\mathcal{C}(\Omega)$. Observe that for every $f \in \mathcal{C}(\Omega)$,

$$\|\sigma_\lambda(f)\|_{\overline{\mathcal{P}_j}} = \sup_{z\in\overline{\mathcal{P}_j}} |f(\lambda \odot z)| = \sup_{(\lambda\odot z)\in\overline{\mathcal{P}_j}} |f(\lambda \odot z)| = \|f\|_{\overline{\mathcal{P}_j}}. \tag{IV.0.7}$$

The expression in (IV.0.7) shows that σ_λ is continuous at 0 in $\mathcal{C}(\Omega)$ and therefore continuous everywhere by sublinearity. By the open mapping theorem, σ_λ^{-1} is also continuous. Thus $\sigma_\lambda \in \mathrm{Aut}(\mathcal{C}(\Omega))$. Let $\lambda, \mu \in \mathbb{T}^k$. Now, for all $f \in \mathcal{C}(\Omega)$ and $z \in \Omega$,

$$\begin{aligned}\sigma_{\lambda\cdot\mu}(f)(z) &= f(\lambda_1\mu_1 z_1, \cdots, \lambda_k\mu_k z_k, \widetilde{z}) \\ &= \sigma_\lambda\big(f(\mu_1 z_1, \cdots, \mu_k z_k, \widetilde{z})\big) = (\sigma_\lambda \circ \sigma_\mu)(f)(z) \end{aligned} \tag{IV.0.8}$$

This proves the map $\lambda \mapsto \sigma_\lambda$ is a group homomorphism from $\mathbb{T}^k$ on $\mathrm{Aut}(\mathcal{C}(\Omega))$. We have now proved that the map σ in (IV.0.5) is a representation of $\mathbb{T}^k$ on $\mathcal{C}(\Omega)$.

Now we use Proposition III.0.1 to show the continuity of σ. The existence of a continuous invariant family of seminorms that generates the Fréchet topology of $\mathcal{C}(\Omega)$ is shown in (IV.0.4). It remains to show that for all continuous function f on Ω, the map $\lambda \mapsto \sigma_\lambda(f)$ is continuous at $\lambda = \mathbb{1} \in \mathbb{T}^k$. Let $\epsilon > 0$, $f \in \mathcal{C}(\Omega)$ and $j \in \mathbb{N}$. Since f is uniformly continuous on $\overline{\mathcal{P}_j}$, there exists $\delta > 0$ such that

$$\text{whenever } |z - w| < \delta, \quad |f(z) - f(w)| < \epsilon/2. \tag{IV.0.9}$$

Let

$$M = \max_{(z_1,\cdots,z_k,\widetilde{z})\in\overline{\mathcal{P}_j}} \max_{1\leqslant j\leqslant k} |z_j|. \tag{IV.0.10}$$

Choose $\delta_1 = \delta/M$ if $M \neq 0$ and $\delta_1 = \delta$ if $M = 0$. Then whenever $|\lambda - \mathbb{1}| < \delta_1$, $|(\lambda \odot z) - z| < \delta$, and so

$$\|\sigma_\lambda(f) - f\|_{\overline{\mathcal{P}_j}} = \sup_{\overline{\mathcal{P}_j}} |f(\lambda \odot z) - f(z)| \leqslant \epsilon/2 < \epsilon.$$

□

Since $\mathcal{O}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$ (with the induced Fréchet topology), the following result holds.

Corollary IV.0.1. *The map σ in* (IV.0.5) *is a continuous representation of $\mathbb{T}^k$ in $\mathcal{O}(\Omega)$.*

IV.0.3. The Hartogs-Laurent series of holomorphic functions

Since the representation σ of $\mathbb{T}^k$ in $\mathcal{O}(\Omega)$ defined in (IV.0.5) is continuous, it follows from Section III.0.4 that every $f \in \mathcal{O}(\Omega)$ admits an abstract Fourier series with respect to σ,

$$f \sim \sum_{\alpha \in \mathbb{Z}^k} \Pi^{\sigma}_{\alpha}(f). \tag{IV.0.11}$$

We call the representation σ in (IV.0.5) the *standard representation* of $\mathbb{T}^k$ on $\mathcal{O}(\Omega)$ and call the abstract Fourier series in (IV.0.11) with respect to the standard representation the *Hartogs-Laurent series.* In this section we will prove the following.

Theorem IV.0.2. *Let $\Omega \subset \mathbb{C}^n$ be a Hartogs open set. Let σ be the standard representation of $\mathbb{T}^k$ on $\mathcal{O}(\Omega)$ and $f \in \mathcal{O}(\Omega)$. Then the following results hold.*

(a) For each $\alpha \in \mathbb{Z}^k$, there exists a function $c_\alpha(f) \in \mathcal{O}(\Omega)$ such that

$$\Pi^{\sigma}_{\alpha}(f) = c_\alpha(f)\ e_\alpha \ \text{ on } \Omega, \tag{IV.0.12}$$

where for $z = (z', \widetilde{z}) = (z_1, \cdots, z_k, \widetilde{z}) \in \Omega$, e_α is the monomial function of the first k variables, that is $e_\alpha(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$.

(b) The function $c_\alpha(f) \in \mathcal{O}(\Omega)$ is locally independent of the first k variables $z_1, \cdots, z_k$, that is, for every $p \in \Omega$, there exists an open neighborhood U of p such that for $z = (z', \widetilde{z}) \in U$,

$$c_\alpha(f)(z) = g(\widetilde{z}),$$

where $g \in \mathcal{O}(\widetilde{U})$, where $\widetilde{U}$ is the projection of U onto the last $(n-k)$ components.

(c) If, moreover, Ω intersects the set $\{z_j = 0\}$ for some $j \in \{1, \cdots, k\}$ and if $\widehat{\Omega}$ is a connected component of Ω for which $\widehat{\Omega} \bigcap \{z_j = 0\} \neq \emptyset$, then $c_\alpha(f) = 0$ on $\widehat{\Omega}$ whenever $\alpha_j < 0$.

(d) The Hartogs-Laurent series of f as in (IV.0.11) *converges unconditionally to f in the topology of $\mathcal{O}(\Omega)$.*

For $1 \leqslant j \leqslant k$, let r_j, R_j be real numbers such that $0 < r_j < R_j < \infty$. Take V to be a relatively compact open ball in $\mathbb{C}^{n-k}$. Let $\mathcal{P}$ be a polyannulus in Ω defined by

$$\mathcal{P} := \left\{ z = (z', \widetilde{z}) \in \Omega : r_j < |z_j| < R_j \ \text{ or } \ |z_j| < R_j \quad \text{for } 1 \leqslant j \leqslant k \text{ and } \widetilde{z} \in V \right\}, \qquad \text{(IV.0.13)}$$

Let δ be the distance from the closure of $\mathcal{P}$ to Ω. Since $\overline{\mathcal{P}}$ is compact and Ω is open, $\delta > 0$. Let $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$. For $1 \leqslant j \leqslant k$, let ρ_j be defined as

$$\rho_j = \begin{cases} R_j + \delta/2 & \text{if } \alpha_j \geqslant 0 \\ r_j - \delta/2 & \text{if } \alpha_j < 0. \end{cases} \qquad \text{(IV.0.14)}$$

Then $\rho = (\rho_1, \cdots, \rho_k) \in \mathbb{R}^k$. We first prove part $(a), (b)$ and (c) of Theorem IV.0.2. The following proposition is the key to proving part (d).

Proposition IV.0.2. *Let $\Omega \subset \mathbb{C}^n$ be an open set with Hartogs symmetry, σ be the standard representation of $\mathbb{T}^k$ on $\mathcal{O}(\Omega)$ and $f \in \mathcal{O}(\Omega)$. For $w = (w_1, \cdots, w_k) \in \mathbb{C}^k$, define the torus*

$$T(w) = \left\{ (\zeta_1, \cdots, \zeta_k) \in \mathbb{C}^k : |\zeta_j| = |w_j| \ \textit{ for } 1 \leqslant j \leqslant k \right\}. \qquad \text{(IV.0.15)}$$

Let $\mathcal{P}$ be a polyannulus in Ω as in (IV.0.13)*. Then for all $\alpha \in \mathbb{Z}^k$, $\Pi^\sigma_\alpha(f) = c_\alpha(f)\ e_\alpha$ on $\mathcal{P}$, where the function $c_\alpha(f)$ can be represented on $\mathcal{P}$ by a repeated contour integral*

$$c_\alpha(f)(z_1, \cdots, z_k, \widetilde{z}) = \frac{1}{(2\pi i)^k} \int\limits_{T(\rho)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \frac{d\zeta}{\zeta_1 \cdots \zeta_k} \qquad \text{(IV.0.16)}$$

for all $z = (z', \widetilde{z}) \in \mathcal{P}$*, where* ρ *is as in* (IV.0.14) *and the multi-index notations* $\zeta^\alpha = \zeta^{\alpha_1} \cdots \zeta^{\alpha_k}$ *and* $d\zeta = d\zeta_k \cdots d\zeta_1$.

Proof. Let

$$W = \big\{(w_1, \cdots, w_k, \widetilde{w}) \in \Omega : w_j = 0 \text{ for some } j, 1 \leqslant j \leqslant k\big\}. \tag{IV.0.17}$$

Let $z = (z', \widetilde{z}) \in \mathcal{P} \backslash W$. Using the definition of $\Pi^\sigma_\alpha(f)$ from (III.0.11),

$$\begin{aligned} \Pi^\sigma_\alpha(f)(z) &= \int\limits_{\mathbb{T}^k} \lambda^{-\alpha} f(\lambda \odot z)\, d\lambda \\ &= \frac{1}{(2\pi)^k} \int\limits_{\theta_1, \cdots, \theta_k = 0}^{2\pi} \frac{1}{\exp(i\langle \alpha, \theta \rangle)} f(e^{i\theta_1} z_1, \cdots, e^{i\theta_k} z_k, \widetilde{z})\, d\theta_k d\theta_{k-1} \cdots d\theta_1 \end{aligned} \tag{IV.0.18}$$

where $\lambda_j = e^{i\theta_j}$ for $1 \leqslant j \leqslant k$ and the 'vector like' notation $\langle \alpha, \theta \rangle = \sum_{j=1}^k \alpha_j \theta_j$. Let I denote the integral in (IV.0.18). Then I can be written as an iterated integral,

$$I = \frac{1}{2\pi} \int\limits_{\theta_1 = 0}^{2\pi} \frac{1}{e^{i\alpha_1 \theta_1}} \quad \frac{1}{2\pi} \int\limits_{\theta_2 = 0}^{2\pi} \frac{1}{e^{i\alpha_2 \theta_2}} \cdots \frac{1}{2\pi} \int\limits_{\theta_k = 0}^{2\pi} \frac{f(z_1 e^{i\theta_1}, \cdots, z_k e^{i\theta_k}, \widetilde{z})}{e^{i\alpha_k \theta_k}} \; d\theta_k \cdots d\theta_1. \tag{IV.0.19}$$

Denote by I_k the innermost integral in (IV.0.19), that is,

$$I_k = \frac{1}{2\pi} \int\limits_{\theta_k = 0}^{2\pi} \frac{f(z_1 e^{i\theta_1}, \cdots, z_k e^{i\theta_k}, \widetilde{z})}{e^{i\alpha_k \theta_k}} \; d\theta_k. \tag{IV.0.20}$$

Take $\zeta_k = z_k e^{i\theta_k}$, so $d\theta_k = \frac{d\zeta_k}{i.\zeta_k}$. For $w \in \mathbb{C}$, let $S(w) = \{\zeta \in \mathbb{C} : |\zeta| = |w|\}$ be the circle of radius $|w|$. Now, I_k has the form

$$I_k = \frac{z_k^{\alpha_k}}{2\pi i} \int\limits_{S(z_k)} \frac{f(z_1 e^{i\theta_1}, \cdots, \zeta_k, \widetilde{z})}{\zeta_k^{\alpha_k + 1}} \; d\zeta_k. \tag{IV.0.21}$$

Let $\Phi_j : \mathbb{C}^k \to \mathbb{C}$ be the projection map on the j-th variable, where $1 \leqslant j \leqslant k$. Observe that the function

$$g(\zeta) = \frac{f(z_1 e^{i\theta_1}, \cdots, z_{k-1} e^{i\theta_{k-1}}, \zeta, \widetilde{z})}{\zeta^{\alpha_k + 1}} \tag{IV.0.22}$$

is holomorphic on the open set $\Phi_k(\Omega\backslash W)$. Recall that we set ρ_j, $1 \leqslant j \leqslant k$, in the expression (IV.0.14) and note that it depends on the polyannulus $\mathcal{P}$. Since the circles $S(z_k)$ and $S(\rho_k)$ are homotopic in $\Phi_k(\Omega\backslash W)$, it follows from the Cauchy theorem (homotopy version) that

$$I_k = \frac{z_k^{\alpha_k}}{2\pi i} \int\limits_{S(\rho_k)} \frac{f(z_1 e^{i\theta_1}, \cdots, \zeta_k, \widetilde{z})}{\zeta_k^{\alpha_k+1}} \; d\zeta_k. \tag{IV.0.23}$$

Now we change the order of integrations in (IV.0.19) and use the Cauchy theorem in each of the remaining $(k-1)$ variables. Eventually we get

$$I = \frac{z_1^{\alpha_1} \cdots z_k^{\alpha_k}}{(2\pi i)^k} \int\limits_{S(\rho_1)\times\cdots\times S(\rho_k)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \; \frac{d\zeta_1 \cdots d\zeta_k}{\zeta_1 \cdots \zeta_k}. \tag{IV.0.24}$$

Note that $S(\rho_1) \times \cdots \times S(\rho_k) = T(\rho)$. Therefore it follows from (IV.0.18), (IV.0.19) and (IV.0.24) that, on $\mathcal{P}\backslash W$

$$\Pi_\alpha^\sigma(f) = c_\alpha(f) \; e_\alpha \tag{IV.0.25}$$

where $e_\alpha(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ and $c_\alpha(f)$ has the integral representation on $\mathcal{P}\backslash W$,

$$c_\alpha(f)(z', \widetilde{z}) = \frac{1}{(2\pi i)^k} \int\limits_{T(\rho)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \frac{d\zeta}{\zeta_1 \cdots \zeta_k}. \tag{IV.0.26}$$

Recall that V is an open relatively compact subset of $\mathbb{C}^{n-k}$ in the definition of the polyannulus $\mathcal{P}$ in (IV.0.13). Since the torus $T(\rho)$ is independent of the z' variable, for a fixed $\widetilde{z} \in V$, the expression (IV.0.26) remains valid for all z' in as long as $z \in \mathcal{P}\backslash W$, so the function $c_\alpha(f)(\cdot, \widetilde{z})$ is independent of z' on the set $\{z' : (z', \widetilde{z}) \in \mathcal{P}\backslash W\}$. In fact, for a fixed $\widetilde{z} \in V$, the function $c_\alpha(f)$ is a constant on $\mathcal{P}\backslash W$ and can be extended as a constant to W, if $\mathcal{P}$ intersects with W at all. Therefore for all $z = (z', \widetilde{z}) \in \mathcal{P}$,

$$c_\alpha(f)(z', \widetilde{z}) = \frac{1}{(2\pi i)^k} \int\limits_{T(\rho)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \frac{d\zeta}{\zeta_1 \cdots \zeta_k}. \tag{IV.0.27}$$

□

Proof of Theorem IV.0.2. First, let us prove part $(a), (b)$ and (c). Let W be as in (IV.0.17) and write $Y = \mathcal{O}(\Omega \backslash W)$. Let ρ be the standard representation of $\mathbb{T}^k$ on $\mathcal{O}(\Omega \backslash W)$. Note that $\Omega \backslash W$ is a Hartogs open set, by Corollary IV.0.1, ρ is continuous. Let $f_0 \in Y_0^{\rho}$. Then by (III.0.43), $f_0(\lambda \odot z) = f_0(z)$ for all $z \in \Omega \backslash W$ and for all $\lambda \in \mathbb{T}^k$, that is,

$$f_0(\lambda_1 z_1, \cdots, \lambda_k z_k, \widetilde{z}) = f_0(z_1, \cdots, z_k, \widetilde{z}) \quad \text{for all } \lambda \in \mathbb{T}^k \text{ and } z = (z', \widetilde{z}) \in \Omega \backslash W. \qquad \text{(IV.0.28)}$$

Choose $\lambda = \left(\frac{|z_1|}{z_1}, \cdots, \frac{|z_k|}{z_k}\right)$. Then for all $z = (z', \widetilde{z}) \in \Omega \backslash W$,

$$f_0(|z_1|, \cdots, |z_k|, \widetilde{z}) = f_0(z_1, \cdots, z_k, \widetilde{z}). \qquad \text{(IV.0.29)}$$

Let $1 \leqslant j \leqslant k$. Use polar co-ordinates $z_j = r_j e^{i\theta_j}$. It follows from (IV.0.29) that $\dfrac{\partial f_0}{\partial \theta_j} = 0$. Now, using the chain rule,

$$\frac{\partial f_0}{\partial \overline{z}_j} = \frac{1}{2} \cdot \frac{\partial f_0}{\partial r_j} \cdot \left(\frac{z_j}{|z_j|}\right). \qquad \text{(IV.0.30)}$$

Since f_0 is holomorphic on $\Omega \backslash W$, $\dfrac{\partial f_0}{\partial \overline{z}_j} = 0$. Since $z_j \neq 0$, it follows that $\dfrac{\partial f_0}{\partial r_j} = 0$. This proves that on the set $\Omega \backslash W$, the function f_0 is independent of the first k variables, that is, for every $p \in \Omega \backslash W$, there exists an open neighborhood U of p such that for $z = (z', \widetilde{z}) \in U$,

$$f_0(z) = g(\widetilde{z}),$$

where $g \in \mathcal{O}(\widetilde{U})$, where $\widetilde{U}$ is the projection of U onto the last $(n-k)$ components.

Let $\varphi : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega \backslash W)$ be the restriction map of holomorphic functions on Ω onto the subset $\Omega \backslash W$. Since $f \in \mathcal{O}(\Omega)$, it follows from Proposition III.0.2 that for $\alpha \in \mathbb{Z}^k$, $\Pi_\alpha^\rho(\varphi(f)) \in Y_\alpha^\rho$, where Y_α^ρ is the α-th Fourier mode of Y with respect to the representation ρ and Π_α^ρ is defined as in (III.0.11). Define $c_\alpha(\varphi(f))$ as

$$c_\alpha(\varphi(f)) = \frac{\Pi_\alpha^\rho(\varphi(f))}{e_\alpha}, \qquad \text{(IV.0.31)}$$

where e_α is the monomial function of the first k variables, that is $e_\alpha(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. Observe that for all $\lambda \in \mathbb{T}^k$ and $z \in \Omega \backslash W$,

$$c_\alpha(\varphi(f))(\lambda \odot z) = \frac{\lambda^\alpha \ \Pi_\alpha^\rho(\varphi(f))(z)}{\lambda^\alpha \ (z')^\alpha} = \frac{\Pi_\alpha^\rho(\varphi(f))(z)}{(z')^\alpha} = c_\alpha(\varphi(f))(z), \tag{IV.0.32}$$

where $(z')^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. Therefore, on the set $\Omega \backslash W$, $\Pi_\alpha^\rho(\varphi(f)) = c_\alpha(\varphi(f)) \cdot e_\alpha$ and by the previous case, $c_\alpha(\varphi(f))$ is locally independent of the first k variables on the set $\Omega \backslash W$. Since $f \in \mathcal{O}(\Omega)$ and since σ is a continuous representation of $\mathbb{T}^k$ on $\mathcal{O}(\Omega)$, by the abstract Fejér theorem, f admits an abstract Fourier series with respect to σ

$$f \sim \sum_{\alpha \in \mathbb{Z}^k} \Pi_\alpha^\sigma(f). \tag{IV.0.33}$$

We claim $\varphi(\Pi_\alpha^\sigma(f)) = \Pi_\alpha^\rho(\varphi(f))$. Let $z \in \Omega \backslash W$ and let δ_z be the evaluation function at z. Note that

$$\begin{aligned}\delta_z(\Pi_\alpha^\rho(\varphi(f))) = \delta_z\Big(\int_{\mathbb{T}^k} \lambda^{-\alpha} \rho_\lambda(\varphi(f))\, d\lambda\Big) = \int_{\mathbb{T}^k} \lambda^{-\alpha}\ \varphi(f)(\lambda \odot z)\, d\lambda &= \varphi(\Pi_\alpha^\sigma(f))(z) \\ &= \delta_z(\varphi(\Pi_\alpha^\sigma(f))).\end{aligned}$$

Then, by analytic continuation, the function $\Pi_\alpha^\rho(\varphi(f)) = c_\alpha(\varphi(f)) \cdot e_\alpha$ extends to all of Ω as $\Pi_\alpha^\sigma(f) = c_\alpha(f) \cdot e_\alpha$, and the extended $c_\alpha(f)$ is holomorphic on Ω and is locally independent of the first k variables, that is, for every $p \in \Omega$, there exists an open neighborhood U of p such that for $z = (z', \widetilde{z}) \in U$,

$$c_\alpha(f)(z) = g(\widetilde{z}),$$

where $g \in \mathcal{O}(\widetilde{U})$, where $\widetilde{U}$ is the projection of U onto the last $(n-k)$ components. This proves part (a) and (b).

To prove part (c), let $\widehat{\Omega} \bigcap \{z_j = 0\} \neq \varnothing$, where $\widehat{\Omega}$ is a connected component of Ω. Let $p = (p', \widetilde{p}) \in \widehat{\Omega} \bigcap \{z_j = 0\}$. If $\alpha_j < 0$, then $c_\alpha(f)(p) = \Pi_\alpha^\sigma(f)(p) \cdot e_{-\alpha}(p) = 0$ (since $\Pi_\alpha^\sigma(f)$ is holomorphic on Ω, $\Pi_\alpha^\sigma(f)(p)$ must be bounded, and $e_{-\alpha}(p) = 0$). Now, it follows from part (b) that there exists an open neighborhood $U \subset \widehat{\Omega}$ of p such that if we fix $\widetilde{p} \in \widetilde{U}$, where $\widetilde{U}$ is the projection of U onto the last $(n-k)$ components, then $c_\alpha(f)(\omega', \widetilde{p}) = 0$ for all $\omega' \in \{z' \in \mathbb{C}^k : (z', \widetilde{z}) \in U\}$. Now, if $\widetilde{p}$ varies over $\widetilde{U}$, we get $c_\alpha(f) = 0$ on U. Since U is open in $\widehat{\Omega}$, by the identity theorem, $c_\alpha(f) = 0$ on $\widehat{\Omega}$.

Now we will prove part (d). Let $z \in \Omega$. Since Ω is a union of countable numbers of polyannuli, there exists a polyannulus $\mathcal{P} \subset \Omega$ such that $z \in \mathcal{P}$. Now, from part (a),

$$\Pi_\alpha^\sigma(f)(z) = c_\alpha(f)(z) \cdot e_\alpha(z), \tag{IV.0.34}$$

where, from Proposition IV.0.2,

$$c_\alpha(f)(z', \widetilde{z}) = \frac{1}{(2\pi i)^k} \int\limits_{T(\rho)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \frac{d\zeta}{\zeta_1 \cdots \zeta_k}. \tag{IV.0.35}$$

For $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$, we write $|\alpha|_\infty := \max\left\{ |\alpha_j|, 1 \leqslant j \leqslant k \right\}$. For an integer $N \geqslant 0$, let $Q_N = \{\alpha \in \mathbb{Z}^k : |\alpha|_\infty \leqslant N\}$ denote the "box" of edge $2N$, so that Q_N has $(2N+1)^k$ points. Construct a bijection $\tau : \mathbb{N} \to \mathbb{Z}^k$ by successively enumerating $Q_0, Q_1 \setminus Q_0, Q_2 \setminus Q_1$ and so on. Therefore τ satisfies

$$\begin{cases} \tau(0) = 0, \\ \tau(1), \cdots, \tau(3^k) \text{ is an enumeration of } Q_1 \setminus Q_0, \\ \text{in general } \left\{ \tau(j) : (2N-1)^k + 1 \leqslant j \leqslant (2N+1)^k \right\} = Q_N \setminus Q_{N-1}. \end{cases} \tag{IV.0.36}$$

Let $M \geqslant 1$ be an integer and let M_1 be the largest integer such that $(2M_1+1)^k \leqslant M$, so that $M_1 = \left\lfloor \frac{\sqrt[k]{M}-1}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Then it is clear that

$$Q_{M_1} \subset \left\{ \tau(0), \tau(1), \cdots, \tau(M) \right\} \subset Q_{M_1+1}. \tag{IV.0.37}$$

To prove part (d), first we show that the series in (IV.0.11) is absolutely convergent in $\mathcal{O}(\Omega)$. Let τ be the bijection as in (IV.0.36). It suffices to show that for every polyannulus $\mathcal{P}$ in Ω,

$$\sum_{j=1}^{\infty} \left\| c_{\tau(j)}(f)\, e_{\tau(j)} \right\|_{\overline{\mathcal{P}}} < \infty. \tag{IV.0.38}$$

Let $\mathcal{P}$ be a polyannulus in Ω as in (IV.0.13) and let W be defined as in (IV.0.17). Consider the following cases.

Case 1. Let $\overline{\mathcal{P}} \bigcap W = \varnothing$.

Let $z \in \mathcal{P}$. Use multi-index notation $z^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. Then,

$$c_\alpha(f)(z) \cdot z^\alpha = \frac{z^\alpha}{(2\pi i)^k} \int\limits_{T(\rho)} \frac{f(\zeta_1, \cdots, \zeta_k, \widetilde{z})}{\zeta^\alpha} \frac{d\zeta}{\zeta_1 \cdots \zeta_k}. \tag{IV.0.39}$$

Taking absolute values,

$$|c_\alpha(f)(z) \cdot z^\alpha| \leqslant \frac{1}{(2\pi)^k} \|f\|_{\overline{\mathcal{P}}} \frac{(2\pi)^k \rho_1 \cdots \rho_k}{\rho_1 \cdots \rho_k} \prod_{j=1}^{k} \left(\frac{|z_j|}{\rho_j} \right)^{\alpha_j} \tag{IV.0.40}$$

and therefore

$$\sup_{z \in \mathcal{P}} |c_\alpha(f)(z) \cdot z^\alpha| \leqslant \|f\|_{\overline{\mathcal{P}}} \cdot \prod_{j=1}^{k} \left(\frac{m_j}{\rho_j} \right)^{\alpha_j} \tag{IV.0.41}$$

where

$$m_j = \begin{cases} \max\limits_{z \in \overline{\mathcal{P}}} |z| & \text{if } \alpha_j \geqslant 0 \\ \min\limits_{z \in \overline{\mathcal{P}}} |z| & \text{if } \alpha_j < 0. \end{cases} \tag{IV.0.42}$$

Write $c_\alpha = c_\alpha(f)$ when f is understood. So, $c_\alpha e_\alpha$ is bounded on $\mathcal{P}$ as follows,

$$\|c_\alpha e_\alpha\|_{\overline{\mathcal{P}}} \leqslant \|f\|_{\overline{\mathcal{P}}} \cdot \prod_{j=1}^{k} \left(\frac{m_j}{\rho_j} \right)^{\alpha_j}. \tag{IV.0.43}$$

Note that,

$$\lim_{N \to \infty} \sum_{|\alpha|_\infty \leqslant N} \left(\frac{m_j}{\rho_j} \right)^{\alpha_j} = \prod_{j=1}^{k} \lim_{N \to \infty} \left(\sum_{\alpha_j = 0}^{N} \left(\frac{m_j}{\rho_j} \right)^{\alpha_j} + \sum_{\alpha_j = -1}^{-N} \left(\frac{m_j}{\rho_j} \right)^{\alpha_j} \right)$$

$$= \prod_{j=1}^{k} \left(\frac{R_j + \delta/2}{R_j + \delta/2 - m_j} + \frac{r_j - \delta/2}{m_j - r_j + \delta/2} \right) < \infty. \tag{IV.0.44}$$

Let $m \geqslant 1$ be any integer. It follows from (IV.0.37) that

$$\sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{m}-1}{2} \right\rfloor} \|c_\alpha e_\alpha\|_{\overline{\mathcal{P}}} < \sum_{j=0}^{m} \left\| c_{\tau(j)} e_{\tau(j)} \right\|_{\overline{\mathcal{P}}} < \sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{m}-1}{2} \right\rfloor + 1} \|c_\alpha e_\alpha\|_{\overline{\mathcal{P}}}. \tag{IV.0.45}$$

Note that by (IV.0.43) and (IV.0.44),

$$\lim_{m \to \infty} \sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{m}-1}{2} \right\rfloor} \|c_\alpha e_\alpha\|_{\overline{\mathcal{P}}} < \infty, \tag{IV.0.46}$$

and therefore

$$\lim_{m \to \infty} \sum_{j=0}^{m} \left\| c_{\tau(j)} e_{\tau(j)} \right\|_{\overline{\mathcal{P}}} = \sum_{j=0}^{\infty} \left\| c_{\tau(j)} e_{\tau(j)} \right\|_{\overline{\mathcal{P}}} < \infty. \tag{IV.0.47}$$

This completes the proof of Case 1.

Case 2. Let $\mathcal{P} \bigcap W \neq \varnothing$.

Let $p = (p_1, \cdots, p_k, \widetilde{p}) \in \mathcal{P} \bigcap W$, so there exists at least one $j \in \{1, \cdots, k\}$ such that $p_j = 0$ and let $S = \{\alpha \in \mathbb{Z}^k : \alpha_j < 0\}$. It follows from part (b) that $c_\alpha(f)(p) \cdot p^\alpha = 0$ for all $\alpha \in S$. If $\alpha \notin S$, $c_\alpha(f)(p) \cdot p^\alpha$ is bounded by the previous case.

We have now proved that the series in (IV.0.11) converges absolutely (and hence unconditionally, see Lemma II.0.1) in $\mathcal{O}(\Omega)$. So, the series

$$\sum_{j=0}^{N} c_{\tau(j)}(f)\, e_{\tau(j)} \tag{IV.0.48}$$

converges. Let $g = \lim_{N \to \infty} \sum_{j=0}^{N} c_{\tau(j)}(f)\, e_{\tau(j)}$ in $\mathcal{O}(\Omega)$. Therefore by Lemma II.0.3 the Cesàro means of the partial sums of the series in (IV.0.48) converges to g as well. But, by the Fejér theorem (Theorem III.0.1), the Cesàro means of the partial sums converge to f. Therefore $f = g$. $\square$

CHAPTER V

SERIES EXPANSION ON REINHARDT OPEN SETS

In this chapter we discuss convergence results in spaces of holomorphic functions on Reinhardt open sets. Reinhardt open sets are special cases for Hartogs open sets (when $k = n$, that is a circular symmetry exists in all the n variables). We begin with the formal definition.

Definition 10. A open set Ω in $\mathbb{C}^n$ is called a *Reinhardt open set* if $z = (z_1, ..., z_n) \in \Omega$ implies $(\lambda_1 z_1, ..., \lambda_n z_n) \in \Omega$ for all $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{T}^n$. A connected Reinhardt open set is called a *Reinhardt domain*. In this case, the notation $\lambda \odot z$ means the n-tuple $(\lambda_1 z_1, ..., \lambda_n z_n)$.

V.0.1. Classical Laurent series of holomorphic functions on Reinhardt open sets

Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt open set and let $f \in \mathcal{O}(\Omega)$. The next result is immediate from Theorem IV.0.2.

Theorem V.0.1. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt open set. Let σ be the standard representation of $\mathbb{T}^n$ on $\mathcal{O}(\Omega)$ and $f \in \mathcal{O}(\Omega)$. Then the Hartogs-Laurent series of the function f in* (IV.0.11) *with respect to the standard representation of $\mathbb{T}^n$ on $\mathcal{O}(\Omega)$, is the classical Laurent series of f and the series converges unconditionally to f in the topology of $\mathcal{O}(\Omega)$.*

Proof. Since Reinhardt open sets are special cases of Hartogs open sets (when $k = n$), we follow proof Theorem IV.0.2 by taking $k = n$, and we finally get that for each $\alpha \in \mathbb{Z}^n$, there exists a holomorphic function $c_\alpha(f)$ on Ω which is locally constant (constant in each connected component) such that

$$\Pi_\alpha^\sigma(f) = c_\alpha(f)\ e_\alpha \ \text{ on } \Omega, \tag{V.0.1}$$

where $c_\alpha(f)$ is locally constant on Ω, and therefore constant (since Ω is connected), and for every $z \in \Omega$, $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the monomial function of n variables. $\square$

Corollary V.0.2. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain and let σ, f are as in the theorem above. Then $c_\alpha(f)$ is constant for each $\alpha \in \mathbb{Z}^n$.*

Proof. Note that $c_\alpha(f)$ is a locally constant on a connected set Ω, and therefore constant. □

V.0.2. The Laurent series of holomorphic functions smooth up to the boundary

In this section it is shown that the Laurent series of a holomorphic function smooth up to the boundary on a Reinhardt domain in $\mathbb{C}^n$ converges unconditionally to the function in the Fréchet topology of the space. This result can be found in [13].

The topology of the space of holomorphic function smooth up to the boundary

For a domain (open and connected set) Ω in $\mathbb{C}^n$, denote by $\mathcal{A}^\infty(\Omega)$ the space of holomorphic functions smooth up to the boundary of Ω, i.e. the space of holomorphic functions whose derivatives of all orders can be extended continuously up to the boundary. For a sequence of functions $\{f_j\} \subset \mathcal{A}^\infty(\Omega)$, $f_j \to f$ in $\mathcal{A}^\infty(\Omega)$ means that for every compact subset $K \subset \overline{\Omega}$, $f_j \to f$ uniformly on K along with all partial derivatives. In particular, if Ω is bounded, then $f_j \to f$ in $\mathcal{A}^\infty(\Omega)$ means that $f_j \to f$ uniformly on $\overline{\Omega}$ along with all partial derivatives.

We now describe the topology of $\mathcal{A}^\infty(\Omega)$. First, assume Ω is bounded. Then $\mathcal{A}^\infty(\Omega) = \bigcap_{m\in\mathbb{N}} \mathcal{A}^m(\Omega)$, where for every $m \in \mathbb{N}$, $\mathcal{A}^m(\Omega) := \mathcal{C}^m(\overline{\Omega}) \bigcap \mathcal{O}(\Omega)$ and $\mathcal{C}^m(\overline{\Omega})$ denotes the space of m-times continuously differentiable $\mathbb{C}$-valued functions whose derivatives up to order m can be extended continuously up to the boundary of Ω. The space $\mathcal{A}^\infty(\Omega)$ is a Fréchet space and its Fréchet topology is generated by the $\mathcal{C}^m$-seminorms given by,

$$\|f\|_{m,\Omega} := \sup\left\{ |D^\alpha f(z)| : z \in \Omega, [\alpha] \leqslant m \right\} \tag{V.0.2}$$

where m ranges over $\mathbb{N}$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index with length $[\alpha] = \sum_{j=1}^{n} \alpha_j$, and

$$D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}. \tag{V.0.3}$$

Now assume Ω is unbounded. For $\nu \in \mathbb{N}$, let $\Omega_\nu = \Omega \bigcap P_\nu$ where $P_\nu = \{z \in \mathbb{C}^n : |z_j| < \nu \text{ for all } 1 \leqslant j \leqslant n\}$ is the polydisc of radius ν. Then Ω_ν is bounded for each ν and we write $\Omega = \bigcup_{\nu=0}^{\infty} \Omega_\nu$. The Fréchet topology of $\mathcal{A}^\infty(\Omega)$ is generated by the collection of seminorms $\{\|f\|_{m,\Omega_\nu} : m, \nu \in \mathbb{N}\}$, where

$$\|f\|_{m,\Omega_\nu} := \sup\Big\{ |D^\alpha f(z)| : z \in \Omega_\nu, [\alpha] \leqslant m \Big\}. \tag{V.0.4}$$

Note that for a sequence of functions $\{f_N\} \subset \mathcal{A}^\infty(\Omega)$, $f_N \to f$ in $\mathcal{A}^\infty(\Omega)$ as $N \to \infty$ if and only if $f_N \to f$ in $\mathcal{A}^\infty(\Omega_\nu)$ for every $\nu \in \mathbb{N}$, as $N \to \infty$.

Now we describe another collection of seminorms that generates the same locally convex topology of $\mathcal{A}^\infty(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is bounded. For $\alpha \in \mathbb{Z}^n$, recall that we defined

$$|\alpha|_\infty := \max\big\{ |\alpha_j|, 1 \leqslant j \leqslant n \big\}. \tag{V.0.5}$$

For $m \in \mathbb{N}$, define

$$\widetilde{\mathcal{A}}^m(\Omega) := \Big\{ f \in \mathcal{A}^m(\Omega) : D^\alpha(f) \in \mathcal{A}^0(\Omega) \text{ where } |\alpha|_\infty \leqslant m \Big\}, \tag{V.0.6}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index in $\mathbb{N}^n$ and D^α is defined in (V.0.3). Note that $\widetilde{\mathcal{A}}^m(\Omega)$ is a Banach space with the norm,

$$|||f|||_{m,\Omega} = \sup\Big\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leqslant m \Big\}. \tag{V.0.7}$$

When $n = 1$, $\widetilde{\mathcal{A}}^m(\Omega)$ coincides with $\mathcal{A}^m(\Omega)$. Observe that for $n \geqslant 2$, $\mathcal{A}^{nm}(\Omega) \subsetneq \widetilde{\mathcal{A}}^m(\Omega) \subsetneq \mathcal{A}^m(\Omega)$. Moreover, for each $m \in \mathbb{N}$, the inclusion maps $\mathcal{A}^{nm}(\Omega) \xhookrightarrow{i} \widetilde{\mathcal{A}}^m(\Omega) \xhookrightarrow{i} \mathcal{A}^m(\Omega)$ are bounded with norm 1. The next result is now immediate.

Lemma V.0.1. *For a bounded $\Omega \subset \mathbb{C}^n$, the collection of seminorms $\{|||\cdot|||_{m,\Omega} : m \in \mathbb{N}\}$ generates the same Fréchet topology of $\mathcal{A}^\infty(\Omega)$ as the collection $\{\|\cdot\|_{m,\Omega} : m \in \mathbb{N}\}$, the $\mathcal{C}^m$-seminorms of Ω.*

Proof. Let $m \in \mathbb{N}$. Note that for every $f \in \mathcal{A}^\infty(\Omega)$,

$$\begin{aligned}
|||f|||_{m,\Omega} &= \sup\left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leqslant m \right\} \\
&= \sup\left\{ |D^\alpha f(z)| : z \in \Omega, \alpha_j \leqslant m \text{ for all } j = 1, 2, \cdots, n \right\} \\
&\leqslant \sup\left\{ |D^\alpha f(z)| : z \in \Omega, [\alpha] \leqslant nm \right\} = \|f\|_{nm,\Omega}.
\end{aligned}$$

Also observe that for every $f \in \mathcal{A}^\infty(\Omega)$,

$$\begin{aligned}
\|f\|_{m,\Omega} &= \sup\left\{ |D^\alpha f(z)| : z \in \Omega, [\alpha] \leqslant m \right\} \\
&\leqslant \sup\left\{ |D^\alpha f(z)| : z \in \Omega, \alpha_j \leqslant m \text{ for all } j = 1, 2, \cdots, n \right\} \\
&= \sup\left\{ |D^\alpha f(z)| : z \in \Omega, |\alpha|_\infty \leqslant m \right\} = |||f|||_{m,\Omega}.
\end{aligned}$$

□

The Laurent series of holomorphic functions smooth up to the boundary on Reinhardt domain

Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and let $f \in \mathcal{A}^\infty(\Omega)$. Since, $\mathcal{A}^\infty(\Omega) \subset \mathcal{O}(\Omega)$, it follows from Theorem V.0.1 that the function f admits a Laurent series

$$f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha(f)\ e_\alpha \text{ on } \Omega, \tag{V.0.8}$$

where for $z = (z_1, \cdots, z_n) \in \Omega$, $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the monomial function of n variables. We have proved that series in (V.0.8) converges unconditionally in $\mathcal{O}(\Omega)$. Now, we ask if the series converges unconditionally in $\mathcal{A}^\infty(\Omega)$ as well (that is, in the stronger topology). The next result answers our question.

Theorem V.0.3. *Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and $f \in \mathcal{A}^\infty(\Omega)$. Then the Laurent series of f in* (V.0.8) *converges unconditionally to the function f in the topology of $\mathcal{A}^\infty(\Omega)$.*

We write $c_\alpha(f)$ as c_α when f is understood. We first prove the result where Ω is bounded. Recall that for a bounded Ω, the collection of seminorms $\{|||\cdot|||_{m,\Omega} : m \in \mathbb{N}\}$ generates the Fréchet topology of $\mathcal{A}^\infty(\Omega)$, where for each m, the seminorm $|||\cdot|||_{m,\Omega}$ is as in (V.0.7). To prove Theorem V.0.3 for a bounded domain Ω, first we prove that the series in (V.0.8) is absolutely convergent in $\mathcal{A}^\infty(\Omega)$ (and therefore unconditionally in $\mathcal{A}^\infty(\Omega)$, see Lemma II.0.1). Consider the special case $k = n$ in the construction of the bijection τ in (IV.0.36), that makes τ to be a bijection from $\mathbb{N}$ to $\mathbb{Z}^n$. It is sufficient to show that for every $m \in \mathbb{N}$, $\sum_{j=0}^{\infty} \left|\left|\left| c_{\sigma(j)} e_{\sigma(j)} \right|\right|\right|_{m,\Omega} < \infty$. The result for the unbounded case follows easily from here.

The following proposition is the key to prove Theorem V.0.3.

Proposition V.0.1. *Let $\mathcal{P}$ be a polyannulus in $\mathbb{C}^n$, that is*

$$\mathcal{P} := \Big\{ z \in \mathbb{C}^n : r_j < |z_j| < R_j \ \text{ or } \ |z_j| < R_j \quad \text{for } 1 \leqslant j \leqslant n \Big\}, \tag{V.0.9}$$

where $0 < r_j < R_j < \infty$ are real numbers. For an integer ℓ, consider

$$\mu_\ell = \begin{cases} \dfrac{1}{|\ell(\ell-1)|} & \text{if } \ell \neq 0, 1 \\ 1 & \text{if } \ell = 0, 1. \end{cases} \tag{V.0.10}$$

For $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$, let $M_\beta = \prod_{j=1}^n \mu_{\beta_j}$. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}^n$ and suppose $f = \sum_{\eta \in \mathbb{Z}^n} c_\eta e_\eta$ is the Laurent series expansion of $f \in \mathcal{A}^\infty(\mathcal{P})$, where e_η is the monomial function of exponent η. Then

$$\|D^\gamma(c_\alpha e_\alpha)\|_{\mathcal{P}} \leqslant \left(M_{\alpha-\gamma} \cdot \prod_{j=1}^n (1 + R_j^2) \right) \cdot |||D^\gamma f|||_{2,\mathcal{P}}. \tag{V.0.11}$$

where R_j's are as in (V.0.9), $\|\cdot\|_{\mathcal{P}}$ *is the supremum norm over* $\mathcal{P}$, *and* $|||\cdot|||_{2,\mathcal{P}}$ *is as in* (V.0.7). *Also, the derivative* $D^\gamma f$ *is as in* (V.0.3).

Proof. Let W be defined as in (IV.0.17), that is,

$$W = \left\{(w_1, \cdots, w_n) \in \mathbb{C}^n : w_j = 0 \text{ for some } j, 1 \leqslant j \leqslant n\right\}$$

and $z \in \mathcal{P} \backslash W$. One can write the coefficient c_α of the Laurent series of f using the Cauchy formula:

$$c_\alpha(z_1, \cdots, z_n) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=|z_1|} \cdots \int_{|\zeta_n|=|z_n|} \frac{f(\zeta_1, \cdots, \zeta_n)}{\zeta^\alpha} \frac{d\zeta_n}{\zeta_n} \cdots \frac{d\zeta_1}{\zeta_1}, \tag{V.0.12}$$

where the multi-index notation $\zeta^\alpha = \zeta^{\alpha_1} \cdots \zeta^{\alpha_n}$. Note that if A_j is a disc for some j, then $c_\alpha = 0$ whenever $\alpha_j < 0$ (see part (c) of Theorem IV.0.2). Fix some multi-index notations: $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $z \odot e^{i\theta} = (z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n})$ and use "vector-like" notation: $\langle \alpha, \theta \rangle = \alpha_1\theta_1 + \cdots + \alpha_n\theta_n$. Parametrize the contours in (V.0.12) by $\zeta_j = z_j e^{i\theta_j}$ for every $1 \leqslant j \leqslant n$. We get the repeated contour integral

$$c_\alpha(z_1, \cdots, z_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(z \odot e^{i\theta})}{z^\alpha \exp(i\langle \alpha, \theta \rangle)} \; d\theta_n \cdots d\theta_1. \tag{V.0.13}$$

Consider the case $\gamma = 0$, where $0 = (0, \cdots, 0) \in \mathbb{N}^n$. Introduce the multi-index $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$ (depending on α) as

$$\beta_j = \begin{cases} 2 & \text{if } \alpha_j \neq 0, 1 \\ 0 & \text{if } \alpha_j = 0, 1. \end{cases}$$

We claim that

$$c_\alpha z^\alpha = \frac{z^\beta}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left(\prod_{j=1}^n U(\alpha_j, \theta_j) \right) \frac{\partial^{[\beta]} f}{\partial \zeta_1^{\beta_1} \cdots \partial \zeta_n^{\beta_n}} (z \odot e^{i\theta}) \; d\theta_n \cdots d\theta_1, \tag{V.0.14}$$

where for each $1 \leqslant j \leqslant n$

$$U(\alpha_j, \theta_j) = \begin{cases} \dfrac{e^{-i(\alpha_j - 2)\theta_j}}{\alpha_j(\alpha_j - 1)} & \text{if } \alpha_j \neq 0, 1 \\ e^{-i\alpha_j\theta_j} & \text{if } \alpha_j = 0, 1. \end{cases} \tag{V.0.15}$$

To prove (V.0.14), let us write (V.0.13) as

$$c_\alpha z^\alpha = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \left(\int_0^{2\pi} \frac{f(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n})}{\exp(i\alpha_n\theta_n)} d\theta_n \right) \frac{d\theta_{n-1} \cdots d\theta_1}{\exp(i(\alpha_1\theta_1 + \cdots + \alpha_{n-1}\theta_{n-1}))}. \tag{V.0.16}$$

Let I_n be the integral inside the parentheses in (V.0.16). That is

$$I_n = \int_0^{2\pi} e^{-i\alpha_n\theta_n}\ f(z \odot e^{i\theta})\, d\theta_n. \tag{V.0.17}$$

If $\alpha_n = 0, 1$, we set $\beta_n = 0$ and one can write an expression for I_n (in terms of β_n) from (V.0.17) as,

$$I_n = z_n^{\beta_n} \int_0^{2\pi} e^{-i\alpha_n\theta_n}\ \frac{\partial^{\beta_n} f}{\partial \zeta_n^{\beta_n}}(z \odot e^{i\theta})\, d\theta_n. \tag{V.0.18}$$

If $\alpha_n \neq 0, 1$, we integrate I_n in (V.0.17) by parts with respect to θ_n as follows: take $u = u(\theta_n) = f(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n})$ and $dv = e^{-i\alpha_n\theta_n}\, d\theta_n$ in the formula $\int_0^{2\pi} u dv = [uv]_0^{2\pi} - \int_0^{2\pi} v du$, and note that the first term vanishes due to periodicity. We get

$$I_n = -\int_0^{2\pi} \frac{e^{-i\alpha_n\theta_n}}{-i\alpha_n}\ \frac{\partial}{\partial\theta_n} f(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n})\ \ d\theta_n. \tag{V.0.19}$$

We use the chain rule:

$$\frac{\partial}{\partial\theta_n} f(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}) = \frac{\partial f}{\partial\zeta_n}(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}) \cdot z_n i e^{i\theta_n}.$$

After simplifying we get

$$I_n = z_n \int_0^{2\pi} \frac{e^{-i(\alpha_n - 1)\theta_n}}{\alpha_n}\ \frac{\partial f}{\partial\zeta_n}(z \odot e^{i\theta})\ \ d\theta_n. \tag{V.0.20}$$

Using integration by parts again in the same way we have

$$I_n = z_n^2 \int_0^{2\pi} \frac{e^{-i(\alpha_n - 2)\theta_n}}{\alpha_n(\alpha_n - 1)} \; \frac{\partial^2 f}{\partial \zeta_n^2}(z \odot e^{i\theta}) \; d\theta_n. \tag{V.0.21}$$

Recall that we set $\beta_n = 2$ for $\alpha_n \neq 0, 1$. Therefore (V.0.21) can be rewritten (in terms of β_n) as

$$I_n = z_n^{\beta_n} \int_0^{2\pi} \frac{e^{-i(\alpha_n - 2)\theta_n}}{\alpha_n(\alpha_n - 1)} \; \frac{\partial^{\beta_n} f}{\partial \zeta_n^{\beta_n}}(z \odot e^{i\theta}) \, d\theta_n. \tag{V.0.22}$$

We substitute the expressions (V.0.20) or (V.0.22) for I_n in (V.0.16) (depending on the values of α_n, and therefore β_n). Rearranging the terms and the integrals we write (V.0.16) as

$$c_\alpha z^\alpha = \frac{z_n^{\beta_n}}{(2\pi)^n} \int\limits_{\theta_n} U(\alpha_n, \theta_n) \int\limits_{\theta_1} \cdots \left(\int\limits_{\theta_{n-1}} \frac{\frac{\partial^{\beta_n} f}{\partial \zeta_n^{\beta_n}}(z \odot e^{i\theta})}{\exp(i\alpha_{n-1}\theta_{n-1})} d\theta_{n-1} \right) \frac{d\theta_{n-2} \cdots d\theta_1}{\exp(i(\alpha_1\theta_1 + \cdots + \alpha_{n-2}\theta_{n-2}))} d\theta_n \tag{V.0.23}$$

Let I_{n-1} be the integral inside the parenthesis in (V.0.23). That is

$$I_{n-1} = \int_0^{2\pi} e^{-i\alpha_{n-1}\theta_{n-1}} \; \frac{\partial^{\beta_n} f}{\partial \zeta_n^{\beta_n}}(z \odot e^{i\theta}) \, d\theta_{n-1}. \tag{V.0.24}$$

Depending on the values of α_{n-1} we repeat the earlier procedure. If $\alpha_{n-1} = 0, 1$, we write the expression of I_{n-1} in terms of β_{n-1} from (V.0.24), otherwise we integrate I_{n-1} in (V.0.24) by parts with respect to the variable θ_{n-1} as previous. We substitute the expressions for I_{n-1} in (V.0.23) and so on. To prove that our claim (V.0.14) is true, we repeat the same procedure $(n-2)$ more times.

For $w \in \mathbb{C}^n$, define the torus

$$T(w) = \left\{ (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n : |\zeta_j| = |w_j| \;\text{ for } 1 \leqslant j \leqslant n \right\}. \tag{V.0.25}$$

We tane absolute values on both sides in (V.0.14). We get

$$|c_\alpha z^\alpha| = \left| \frac{z^\beta}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left(\prod_{j=1}^n U(\alpha_j, \theta_j) \right) \frac{\partial^{[\beta]} f}{\partial \zeta_1^{\beta_1} \cdots \partial \zeta_n^{\beta_n}} (z \odot e^{i\theta}) \; d\theta_n \cdots d\theta_1 \right|$$

$$\leqslant R^\beta \cdot \prod_{j=1}^n |U(\alpha_j,\theta_j)| \cdot \left\| \frac{\partial^{|\beta|} f}{\partial \zeta_1^{\beta_1} \cdots \partial \zeta_n^{\beta_n}} \right\|_{T(z)} \leqslant \left(R^\beta \prod_{j=1}^n \mu_{\alpha_j} \right) \cdot |||f|||_{2,\mathcal{P}} \tag{V.0.26}$$

where $R^\beta = R_1^{\beta_1} \cdots R_n^{\beta_n}$, $|U(\alpha_j,\theta_j)| = \mu_{\alpha_j}$, and $\|\cdot\|_{T(z)}$ denotes the sup norm on the set $T(z)$, which is a subset of $\mathcal{P}\backslash W$. Also, we note that $R^\beta \leqslant \prod_{j=1}^n (1+R_j^2)$. So, it follows from (V.0.26) that for $z \in \mathcal{P}\backslash W$,

$$|c_\alpha z^\alpha| \leqslant \left(\prod_{j=1}^n \mu_{\alpha_j} \right) \cdot \prod_{j=1}^n (1+R_j^2) \cdot |||f|||_{2,\mathcal{P}} = \left(M_{\alpha-0} \cdot \prod_{j=1}^n (1+R_j^2) \right) \cdot |||f|||_{2,\mathcal{P}} \tag{V.0.27}$$

where we note that $0 = (0,\cdots,0) \in \mathbb{N}^n$, $M_{\alpha-0} = \prod_{j=1}^n \mu_{\alpha_j}$.

In (V.0.27) we have proved that the function $z \mapsto c_\alpha z^\alpha$ is bounded on the set $\mathcal{P}\backslash W$. By the Riemann removable singularity theorem (see [27, p. 32]) we can extend the function holomorphically to $\mathcal{P}$ and the extended function admits the same bound. This proves our result for the case $\gamma = 0$.

Now, let $\gamma \in \mathbb{N}^n$. Recall the operator D^γ defined in (V.0.3). We multiply by z^α and apply D^γ in both sides of (V.0.13) to get,

$$\begin{aligned}
D^\gamma(c_\alpha z^\alpha) &= D^\gamma\left(\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(z \odot e^{i\theta})}{\exp(i\langle \alpha,\theta\rangle)} \; d\theta_n \cdots d\theta_1 \right) \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} D^\gamma\big(f(z \odot e^{i\theta})\big) \; e^{-i\langle \alpha,\theta\rangle} \; d\theta_n \cdots d\theta_1 \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (D^\gamma f)(z \odot e^{i\theta}) \; e^{i\langle \gamma,\theta\rangle} \; e^{-i\langle \alpha,\theta\rangle} \; d\theta_n \cdots d\theta_1 \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (D^\gamma f)(z \odot e^{i\theta}) \; e^{-i\langle \alpha-\gamma,\theta\rangle} \; d\theta_n \cdots d\theta_1 \\
&= c_{\alpha-\gamma}(D^\gamma f) z^{\alpha-\gamma} \qquad \text{by (V.0.13).}
\end{aligned} \tag{V.0.28}$$

Now, we use the bound for the case $\gamma = 0$ and the fact that $D^\gamma f \in \mathcal{A}^\infty(\mathcal{P})$ to get,

$$|D^\gamma(c_\alpha z^\alpha)| = \left| c_{\alpha-\gamma}(D^\gamma f) z^{\alpha-\gamma} \right| \leqslant \left(M_{\alpha-\gamma} \; \prod_{j=1}^n (1+R_j^2) \right) \cdot |||D^\gamma f|||_{2,\mathcal{P}}. \tag{V.0.29}$$

To complete the proof, we take supremum over all $z \in \mathcal{P}$. $\square$

Proof of Theorem V.0.3. First we assume Ω to be bounded. Let $z \in \Omega$. By Lemma IV.0.1, there exists a polyannulus $\mathcal{P} \subset \Omega$ such that $z \in \mathcal{P}$, where, for every $1 \leqslant j \leqslant n$, either $\{r_j < |z_j| < R_j\}$ or $\{|z_j| < R_j\}$ and $0 < r_j < R_j < \infty$ are real numbers. Let $f = \sum_{\eta \in \mathbb{Z}^n} c_\eta e_\eta$ be the Laurent series of the function $f \in \mathcal{A}^\infty(\Omega) \subset \mathcal{A}^\infty(\mathcal{P})$. Let $\alpha \in \mathbb{Z}^n$ and $\gamma \in \mathbb{N}^n$. Let $\mu_{\alpha_j - \gamma_j}$ and $M_{\alpha-\gamma}$ be as in the statement of Proposition V.0.1. Now, it follows by Proposition V.0.1 that,

$$\|D^\gamma(c_\alpha e_\alpha)\|_{\mathcal{P}} \leqslant \left(M_{\alpha-\gamma} \cdot \prod_{j=1}^{n}(1+R_j^2)\right) \cdot |||D^\gamma f|||_{2,\mathcal{P}} \tag{V.0.30}$$

Since $\mathcal{P} \subset \Omega$, $|||D^\gamma f|||_{2,\mathcal{P}} \leqslant |||D^\gamma f|||_{2,\Omega}$. Also, since Ω is bounded, there exists a finite number $B > 0$ (independent of the polyannulus $\mathcal{P}$, as long as $\mathcal{P} \subset \Omega$), such that $\prod_{j=1}^n (1+R_j^2) \leqslant B$. Therefore

$$\|D^\gamma(c_\alpha e_\alpha)\|_{\mathcal{P}} \leqslant \left(M_{\alpha-\gamma} \cdot B\right) \cdot |||D^\gamma f|||_{2,\Omega} \tag{V.0.31}$$

The right hand side of (V.0.31) is independent of $\mathcal{P}$. Now, we take supremum in the left of (V.0.31) over all $\mathcal{P}$'s contained in Ω to get,

$$\|D^\gamma(c_\alpha e_\alpha)\|_{\Omega} \leqslant \left(M_{\alpha-\gamma} \cdot B\right) \cdot |||D^\gamma f|||_{2,\Omega}. \tag{V.0.32}$$

We claim that,

$$\sum_{\alpha \in \mathbb{Z}^n} M_{\alpha-\gamma} = \lim_{N \to \infty} \sum_{\left\{|\alpha_j| \leqslant N : 1 \leqslant j \leqslant n\right\}} M_{\alpha-\gamma} < \infty. \tag{V.0.33}$$

We write,

$$\sum_{\alpha \in \mathbb{Z}^n} M_{\alpha-\gamma} = \sum_{\alpha \in \mathbb{Z}^n} \prod_{j=1}^{n} \mu_{\alpha_j-\gamma_j} = \prod_{j=1}^{n} \sum_{\alpha_j=-\infty}^{\infty} \mu_{\alpha_j-\gamma_j}, \tag{V.0.34}$$

Now, for every $1 \leqslant j \leqslant n$,

$$\sum_{\alpha_j=-\infty}^{\infty} \mu_{\alpha_j-\gamma_j} = \sum_{\alpha_j=-\infty}^{\gamma_j-1} \mu_{\alpha_j-\gamma_j} + \mu_0 + \mu_1 + \sum_{\alpha_j=\gamma_j+2}^{\infty} \mu_{\alpha_j-\gamma_j} \tag{V.0.35}$$

By definition, $\mu_0 = 1 = \mu_1$. For the other two sums on the right hand side of (V.0.35), let us make a change of variable $\nu = \alpha_j - \gamma_j$. Then,

$$\sum_{\alpha_j=-\infty}^{\infty} \mu_{\alpha_j-\gamma_j} = \sum_{\nu=-\infty}^{-1} \frac{1}{|\nu(\nu-1)|} + 2 + \sum_{\nu=2}^{\infty} \frac{1}{|\nu(\nu-1)|} = \sum_{\nu=1}^{\infty} \frac{1}{|\nu(\nu+1)|} + 2 + \sum_{\nu=2}^{\infty} \frac{1}{|\nu(\nu-1)|}, \tag{V.0.36}$$

where the second equality in (V.0.36) occurs after we switch the indices ν to $-\nu$, when ν is negative. Now, both the series over ν are telescoping series which add up to 1. Therefore from (V.0.34), $\sum_{\alpha\in\mathbb{Z}^n} M_{\alpha-\gamma} = \prod_{j=1}^{n}(1+2+1)^n = 4^n$ and our claim in (V.0.33) is proved.

Now, to complete the proof of the Theorem, let $m \in \mathbb{N}$ and $0 \leqslant \gamma_j \leqslant m$ for all j. Then

$$\begin{aligned}
|||D^\gamma f|||_{2,\Omega} = \sup_{z\in\Omega}\left\{ |D^\eta(D^\gamma f)(z)| : |\eta|_\infty \leqslant 2\right\} &= \sup_{z\in\Omega}\left\{ \left|D^\beta f(z)\right| : \beta_j \leqslant \gamma_j + 2 \text{ for all } j\right\} \\
&\leqslant \sup_{z\in\Omega}\left\{ \left|D^\beta f(z)\right| : |\beta|_\infty \leqslant |\gamma|_\infty + 2\right\} \\
&= |||f|||_{|\gamma|_\infty+2,\Omega} \leqslant |||f|||_{m+2,\Omega},
\end{aligned} \tag{V.0.37}$$

where we recall $|\gamma|_\infty := \max\left\{ |\gamma_j|, 1 \leqslant j \leqslant n\right\}$. We have now proved that for all $\gamma \in \mathbb{N}^n$ such that $0 \leqslant \gamma_j \leqslant m$,

$$\sum_{\alpha\in\mathbb{Z}^n} \|D^\gamma(c_\alpha e_\alpha)\|_\Omega \leqslant \left(B \cdot |||f|||_{m+2,\Omega}\right) \sum_{\alpha\in\mathbb{Z}^n} M_{\alpha-\gamma} = 4^n \cdot B \cdot |||f|||_{m+2,\Omega}. \tag{V.0.38}$$

Taking supremum over all such γ where $0 \leqslant \gamma_j \leqslant m$,

$$\sum_{\alpha\in\mathbb{Z}^n} |||c_\alpha e_\alpha|||_{m,\Omega} \leqslant 4^n \cdot B \cdot |||f|||_{m+2,\Omega}. \tag{V.0.39}$$

Let $\ell \geqslant 1$ be an integer and let $\tau : \mathbb{N} \to \mathbb{Z}^n$ be the bijection constructed in (IV.0.36) by taking $k = n$ as a special case. Then it follows from (IV.0.37) that

$$\sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{\ell}-1}{2} \right\rfloor} |||c_\alpha e_\alpha|||_{m,\Omega} < \sum_{j=0}^{\ell} \left|\left|\left|c_{\tau(j)} e_{\tau(j)}\right|\right|\right|_{m,\Omega} < \sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{\ell}-1}{2} \right\rfloor + 1} |||c_\alpha e_\alpha|||_{m,\Omega}, \tag{V.0.40}$$

where $|\alpha|_\infty := \max\left\{|\alpha_j|, 1 \leqslant j \leqslant n\right\}$. Now, from (V.0.39),

$$\lim_{\ell\to\infty} \sum_{|\alpha|_\infty \leqslant \left\lfloor \frac{\sqrt[k]{\ell}-1}{2} \right\rfloor} |||c_\alpha e_\alpha|||_{m,\Omega} \leqslant 4^n \cdot B \cdot |||f|||_{m+2,\Omega}. \tag{V.0.41}$$

Therefore from (V.0.40),

$$\sum_{j=0}^{\infty} \left|\left|\left|c_{\tau(j)} e_{\tau(j)}\right|\right|\right|_{m,\Omega} = \lim_{\ell\to\infty} \sum_{j=0}^{\ell} \left|\left|\left|c_{\tau(j)} e_{\tau(j)}\right|\right|\right|_{m,\Omega} \leqslant 4^n \cdot B \cdot |||f|||_{m+2,\Omega},$$

and the quantity $4^n \cdot B \cdot |||f|||_{m+2,\Omega}$ is a finite quantity. The absolute convergence follows from here.

Now we have proved that the series in (V.0.8) converges absolutely in $\mathcal{A}^\infty(\Omega)$, therefore unconditionally in $\mathcal{A}^\infty(\Omega)$ (see Lemma II.0.1). So, the series $\sum_{j=0}^{\infty} c_{\tau(j)} e_{\tau(j)}$ converges in $\mathcal{A}^\infty(\Omega)$. Let

$$g = \lim_{N\to\infty} \sum_{j=0}^{N} c_{\tau(j)} e_{\tau(j)} \tag{V.0.42}$$

in $\mathcal{A}^\infty(\Omega)$. Since the inclusion map $\mathcal{A}^\infty(\Omega) \xhookrightarrow{i} \mathcal{O}(\Omega)$ is continuous, the convergence in Equation (V.0.42) holds on $\mathcal{O}(\Omega)$ as well. But, by Theorem V.0.1, the Laurent series of a holomorphic function f on a Reinhardt domain Ω converges to f unconditionally. Therefore $f = g$, and the proof is complete, when the domain Ω is bounded.

Now, let Ω be unbounded. Recall from Section V.0.2 that we can write $\Omega = \bigcup_{\nu=0}^{\infty} \Omega_\nu$, where each Ω_ν is bounded. Also, recall that the Fréchet topology of $\mathcal{A}^\infty(\Omega)$ is generated by the collection of seminorms $\{\|f\|_{m,\Omega_\nu} : m, \nu \in \mathbb{N}\}$, where

$$\|f\|_{m,\Omega_\nu} := \sup\left\{ |D^\alpha f(z)| : z \in \Omega_\nu, [\alpha] \leqslant m \right\}. \tag{V.0.43}$$

We have already proved that for all integers $m, \nu \geqslant 0$, $\sum_{j=0}^{\infty} \left|\left|\left|c_{\tau(j)} e_{\tau(j)}\right|\right|\right|_{m,\Omega_\nu} < \infty$, where $|||\cdot|||_{m,\Omega_\nu}$ is as in (V.0.7) and τ is the bijection constructed in (IV.0.36). It is important to note that τ does not depend on ν. Since $\left\|c_{\tau(j)} e_{\tau(j)}\right\|_{m,\Omega_\nu} \leqslant \left|\left|\left|c_{\tau(j)} e_{\tau(j)}\right|\right|\right|_{m,\Omega_\nu}$, it follows that for all integers $m, \nu \geqslant 0$,

$$\sum_{j=0}^{\infty} \left\| c_{\tau(j)} e_{\tau(j)} \right\|_{m,\Omega_\nu} < \infty.$$

This proves that the Hartogs-Laurent series of f converges absolutely in the topology of $\mathcal{A}^\infty(\Omega)$. The rest of the proof is similar to the bounded case. $\square$

V.0.3. Density result on Bergman space

Let Ω be a Reinhardt domain in $\mathbb{C}^n$. Let $\omega : \Omega \to (0, \infty)$ be a continuous function on Ω such that $\omega(z_1, \cdots, z_n) = \omega(|z_1|, \cdots, |z_n|)$ for all $z \in \Omega$, then ω is called a *radial weight function*. For $1 \leqslant p < \infty$ and for a radial weight function ω, let $L^p_\omega(\Omega)$ be the weighted L^p space, a Banach space with the norm $\|\cdot\|_{\omega,p}$ defined by

$$\|f\|^p_{\omega,p} = \int_\Omega |f(z)|^p \omega(z) dV(z), \tag{V.0.44}$$

where $f \in L^p_\omega(\Omega)$ and dV is the volume (Haar) measure.

Define $A^p_\omega(\Omega) := \{f : f \in \mathcal{O}(\Omega) \bigcap L^p_\omega(\Omega)\}$. We call $A^p_\omega(\Omega)$ to be the *Weighted Bergman space* on Ω. Using the Bergman inequality, one can show $A^p_\omega(\Omega)$ is a closed subspace of $L^p_\omega(\Omega)$ (in the subspace topology), and therefore a Banach space (for a special case $p = 2$, see [25, p. 50]). In this section we will prove the following result.

Theorem V.0.4. *For $1 \leqslant p < \infty$, Laurent polynomials are dense in $A^p_\omega(\Omega)$.*

Proof. Let σ be the standard representation of $\mathbb{T}^n$ on $A^p_\omega(\Omega)$, that is $\sigma_\lambda(f)(z) = f(\lambda \odot z)$, where $f \in A^p_\omega(\Omega)$, $\lambda \in \mathbb{T}^n$ and $z \in \Omega$. By Proposition III.0.3, it is sufficient to show that σ is continuous. To show the continuity of σ, we use Proposition III.0.1. We show the norm $\|\cdot\|_{p,\omega}$ defined in (V.0.44) is invariant with respect to σ, and if $\{\lambda_j\} \subset \mathbb{T}^n$ be such that $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, then for all $f \in A^p_\omega(\Omega)$, $\sigma_{\lambda_j}(f) \to f$ in the topology of $A^p_\omega(\Omega)$. Let $\lambda \in \mathbb{T}^n$. Then,

$$\|\sigma_\lambda(f)\|^p_{\omega,p} = \int_\Omega |f(\lambda \odot z)|^p \omega(z) \; dV(z)$$

$$= \int_\Omega |f(\zeta)|^p \omega(\zeta) \cdot \det(J_\lambda) \cdot dV(\zeta) = \int_\Omega |f(\zeta)|^p \omega(\zeta)\, dV(\zeta) = \|f\|_{\omega,p}^p,$$

where we have used the change of variable $\zeta = \lambda \odot z$. Since ω is radial, $\omega(\zeta) = \omega(\lambda \odot z) = \omega(|\lambda \odot z|) = \omega(|z|) = \omega(z)$. The quantity J_λ is the $n \times n$ diagonal matrix with diagonal entries $|\lambda_j|^{-2}, 1 \leqslant j \leqslant n$, and $\det(J_\lambda)$ denotes the determinant of J_λ. The invariance of norms can ne proved by taking p-th roots in the both sides.

Let $f \in A^p_\omega(\Omega)$ and $\epsilon > 0$. Since f can be approximated by continuous compactly supported functions, there exists $g_\epsilon \in \mathcal{C}_c(\Omega)$ such that

$$\|f - g_\epsilon\|_{\omega,p} < \epsilon/3. \tag{V.0.45}$$

Let $\{\mathcal{P}_j\}_{j\in\mathbb{N}}$ be the countable collection of polyannuli that covers Ω (such a collection exists by Lemma IV.0.1). So, there exists a $j \in \mathbb{N}$ such that $\mathrm{supp}(g_\epsilon) \subset \overline{\mathcal{P}}_j$. In fact this g_ϵ is uniformly continuous on $\overline{\mathcal{P}}_j$. So, there exists a $\delta > 0$ such that,

$$\text{whenever } |z - w| < \delta, \quad |g_\epsilon(z) - g_\epsilon(w)| < \left(\frac{\epsilon/3}{M \cdot \mathrm{Vol}(\overline{\mathcal{P}}_j)}\right)^{1/p} \tag{V.0.46}$$

where $M = \max_{z \in \overline{\mathcal{P}}_j}(\omega(z))$ and $\mathrm{Vol}(\overline{\mathcal{P}}_j)$ denotes the volume of the compact set $\overline{\mathcal{P}}_j$. Let $z \in \Omega$, then there exists $m \in \mathbb{N}$ such that $z \in \mathcal{P}_m$. Let $A = \sup\{|z|, z \in \overline{\mathcal{P}}_m\}$. Since $\lambda_j \to \mathbb{1}$ in $\mathbb{T}^n$, for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that whenever $j \geqslant N$,

$$|\lambda_j - \mathbb{1}| < \frac{\delta}{\sqrt{n}A}. \tag{V.0.47}$$

So, for $z \in \Omega$, whenever $j \geqslant N$,

$$\begin{aligned} |\lambda_j \odot z - z| &= \left|(\lambda_1^j z_1 - z_1), \cdots, (\lambda_n^j z_n - z_n)\right| \\ &= \sqrt{\left|\lambda_1^j - 1\right|^2 \cdot |z_1|^2 + \cdots + \left|\lambda_n^j - 1\right|^2 \cdot |z_n|^2} \\ &< \sqrt{\frac{\delta^2}{nA^2} \cdot A^2 + \cdots + \frac{\delta^2}{nA^2} \cdot A^2} = \delta. \end{aligned} \tag{V.0.48}$$

So, it follows from (V.0.46), (V.0.47) and (V.0.48) that for all $j \geqslant N$,

$$\left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_{\omega,p} = \int_{K_j} |g_\epsilon(\lambda_j z) - g_\epsilon(z)|^p \, \omega(z) \, dV(z) < \epsilon/3. \tag{V.0.49}$$

Therefore, whenever $j \geqslant N$,

$$\begin{aligned}
&\left\|\sigma_{\lambda_j}(f) - f\right\|_{\omega,p} \\
&= \left\|\sigma_{\lambda_j}(f - g_\epsilon) + \sigma_{\lambda_j}(g_\epsilon) - g_\epsilon + g_\epsilon - f\right\|_{\omega,p}, \quad \text{since } \sigma_{\lambda_j} \text{ is linear,} \\
&\leqslant \left\|\sigma_{\lambda_j}(f - g_\epsilon)\right\|_{\omega,p} + \left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_{\omega,p} + \|g_\epsilon - f\|_{\omega,p} \\
&\leqslant \|f - g_\epsilon\|_{\omega,p} + \left\|\sigma_{\lambda_j}(g_\epsilon) - g_\epsilon\right\|_{\omega,p} + \|g_\epsilon - f\|_{\omega,p} \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned} \tag{V.0.50}$$

□

The following result is a corollary to the above density result. The result has been proved in a completely different method in [16]. Let $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$. It is easy to check that B_n is Reinhardt.

Corollary V.0.5. *Holomorphic polynomials are dense in $A^p_\omega(B_n)$.*

V.0.4. Principle of Missing Monomials and Applications

Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and let $X \subset \mathcal{O}(\Omega)$ be a quasi-complete LCTVS, and let σ be the standard representation of $\mathbb{T}^n$ on X. We have proved in Theorem V.0.1 that the abstract Fourier series of an element of X with respect to σ is the classical Laurent series. For $\alpha \in \mathbb{Z}^n$, recall from Section III.0.5 that we defined α-th Fourier mode X^σ_α, a closed invariant (with respect to σ) subspace of X, as

$$X^\sigma_\alpha = \{f \in X : \sigma_\lambda(f) = \lambda^\alpha f, \text{ for all } \lambda \in \mathbb{T}^n\}. \tag{V.0.51}$$

In this section we first show that for each α, X^σ_α is an one dimensional subspace of X generated by the monomial functions e_α, where $e_\alpha(z) = z^\alpha$. Then we discuss some important applications. Recall from Section III.0.5 that Π^σ_α,defined in (III.0.11), is a continuous linear projection from X onto X^σ_α.

Lemma V.0.2. *Let Ω be a Reinhardt domain in $\mathbb{C}^n$. Let $X \subset \mathcal{O}(\Omega)$ be a quasi-complete LCTVS and let σ be the standard representation of $\mathbb{T}^n$ on X. Then the following hold.*

(a) For each $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$,

$$X^\sigma_\alpha = \left\{ f \in X : f = f_0 \cdot e_\alpha \text{ on } \Omega \right\} \tag{V.0.52}$$

where e_α is the monomial function: $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and f_0 is a constant function Ω.

(b) (Principle of missing monomial) If $e_\alpha \notin X$ for some $\alpha \in \mathbb{Z}^n$, then $X^\sigma_\alpha = 0$.

Proof. Since Reinhardt domains are a special type of connected Hartogs open sets, part (a) follows directly from Theorem IV.0.2.

Now, let us prove part (b). Let $e_\alpha \notin X$ for some $\alpha \in \mathbb{Z}^n$ and $X^\sigma_\alpha \neq 0$. Then there is an $f \in X^\sigma_\alpha$ such that $f \neq 0$. By part (a), $f = c \cdot e_\alpha$ for some nonzero constant $c \in \mathbb{C}$. This shows $e_\alpha = \frac{1}{c} \cdot f$, i.e. $e_\alpha \in X^\sigma_\alpha$ and $X^\sigma_\alpha \subset X$. So, $e_\alpha \in X$, a contradiction. □

V.0.5. Applications

We first revisit the following result as an application of the principle of missing monomials. The result was proved by Sibony (see [30]) for Hartogs triangle $\{(z,w) \in \mathbb{C}^2 : |z| < |w| < 1\}$ in $\mathbb{C}^2$. Later the result was extended by Chakrabarti (see [9]) to Reinhardt domains.

Theorem V.0.6. *(Chakrabarti, [9]) Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain such that the origin is a boundary point of Ω. Then there is a neighborhood of the origin such that each function* *in $\mathcal{A}^\infty(\Omega)$ extends to this neighborhood holomorphically. (Sibony observed this result for the Hartogs triangle, see [30].)*

Proof. Let $X = \mathcal{A}^\infty(\Omega)$. If $\alpha_j < 0$ for some $1 \leqslant j \leqslant n$, taking sufficient holomorphic derivatives of e_α contradicts the continuity of e_α up to the boundary. So, whenever $\alpha_j < 0$ for some $1 \leqslant j \leqslant n$, $e_\alpha \notin X$, therefore $X^\sigma_\alpha = 0$. As a result the Laurent series of each $f \in X$ reduces to the Taylor series of f. Then we apply the Abel's lemma (see [27, p. 14]) to complete the proof. □

Next we revisit a well-known result due to Hartogs.

Theorem V.0.7. *(Hartogs extension theorem) Let P_1 and $P_{1/2}$ be two polydiscs in $\mathbb{C}^n, n \geqslant 2$, with centers at 0 and polyradii 1 and 1/2 respectively. Then every holomorphic function on $P_1 \backslash \overline{P_{1/2}}$ extends holomorphically to P_1.*

Proof. Let $\Omega = P_1 \backslash \overline{P_{1/2}}$ and $f \in \mathcal{O}(\Omega)$. If $\alpha_j < 0$ for some j, then e_α blows up on the set $Z_j = \{z \in \Omega : z_j = 0\}$. So, $X^\sigma_\alpha = 0$ when $\alpha_j < 0$ for some j. As a result the Laurent series of each $f \in \mathcal{O}(\Omega)$ reduces to the Taylor series of f. We apply the Abel's lemma to complete the proof. □

CHAPTER VI

SERIES IN DOLBEAULT COHOMOLOGY SPACES

Let Ω be a Reinhardt domain in $\mathbb{C}^n$. The goal of this section is to represent the Dolbeault cohomology classes of the domain Ω as Laurent-type series by using the abstract theory outlined in Chapter III. We first discuss the Dolbeault and Čech cohomology groups of Ω and their respective topologies.

VI.0.1. Dolbeault cohomology

Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and for $p,q \in \mathbb{N}$, let ω be a differential form on Ω of bi-degree (p,q), where $0 \leqslant p,q \leqslant n$, that is ω is of the form

$$\omega = \sideset{}{'}\sum_{|\alpha|=p,|\beta|=q} w_{\alpha\beta}\ dz^{\alpha} \wedge d\overline{z}^{\beta},$$

where $|\alpha| = p$ means $\alpha = (\alpha_1, \cdots, \alpha_p)$ is an increasing multi-index of size p, that is $1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_p \leqslant n$, and similarly $\beta = (\beta_1, \cdots, \beta_q)$ is an increasing multi-index of size q, the primed sum means the summation is taken over increasing multi-indices α and β, and $dz^{\alpha} = dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p}$ and similarly for $d\overline{z}^{\beta}$. Let $\Lambda^{p,q}(\Omega)$ denote the topological vector space of all smooth (p,q)-forms on Ω, with the usual $\mathcal{C}^{\infty}$-topology, which makes $\Lambda^{p,q}(\Omega)$ a Fréchet space. Consider the operator $\overline{\partial} : \Lambda^{p,q}(\Omega) \to \Lambda^{p,q+1}(\Omega)$ which takes (p,q) forms of Ω to $(p,q+1)$ forms. Denote by $Z^{p,q}(\Omega)$ the topological vector space of $\overline{\partial}$-closed (p,q) forms, that is $Z^{p,q}(\Omega)$ is the space of forms $\omega \in \Lambda^{p,q}(\Omega)$ such that $\overline{\partial}\omega = 0$. The space $Z^{p,q}(\Omega)$ is a subspace of $\Lambda^{p,q}(\Omega)$ with the induced subspace topology. Also, let $B^{p,q}(\Omega)$ be the $\overline{\partial}$-exact (p,q) forms, that is the image of $\overline{\partial} : \Lambda^{p,q-1}(\Omega) \to \Lambda^{p,q}(\Omega)$, a form $\omega \in \Lambda^{p,q}(\Omega)$ is called exact if there exists a form $\eta \in \Lambda^{p,q-1}(\Omega)$ such that $\overline{\partial}\eta = \omega$. Since for every $\omega \in \Lambda^{p,q}(\Omega)$, $\overline{\partial}^2\omega = 0$, one gets $B^{p,q}(\Omega) \subset Z^{p,q}(\Omega)$. We define the (p,q)-th *Dolbeault cohomology group* $H^{p,q}(\Omega)$ as,

$$H^{p,q}(\Omega) = \frac{Z^{p,q}(\Omega)}{B^{p,q}(\Omega)}, \tag{VI.0.1}$$

endowed with the quotient topology of the induced Fréchet topology of $Z^{p,q}(\Omega)$.

Let E be an arbitrary LCTVS with a topology τ and let E_1 be a subspace of E. Note that the quotient space E/E_1 is not always Hausdorff; however, E/E_1 is Hausdorff if and only if E_1 is closed in (E,τ). As a result, the quotient space

$$Z^{p,q}(\Omega)\big/\overline{B^{p,q}(\Omega)} \subset H^{p,q}(\Omega) \tag{VI.0.2}$$

is Hausdorff and is called the *reduced part* of $H^{p,q}(\Omega)$ and is denoted by $H^{p,q}_{\text{red}}(\Omega)$, where $\overline{B^{p,q}(\Omega)}$ is the closure of $B^{p,q}(\Omega)$. It can be proved that the topology of the quotient space

$$\overline{B^{p,q}(\Omega)}\big/B^{p,q}(\Omega) \subset H^{p,q}(\Omega). \tag{VI.0.3}$$

is indiscrete and is denoted by $H^{p,q}_{\text{ind}}(\Omega)$, called the *indiscrete part* of $H^{p,q}(\Omega)$. By the third group isomorphism theorem (see [15, p. 98]), $H^{p,q}(\Omega)$ is isomorphic to the algebraic direct sum $H^{p,q}_{\text{ind}}(\Omega) \oplus H^{p,q}_{\text{red}}(\Omega)$. Note that the projection map from $H^{p,q}(\Omega)$ onto $H^{p,q}_{\text{ind}}(\Omega)$, with null space $H^{p,q}_{\text{red}}(\Omega)$, is continuous (since any linear map onto an indiscrete space is continuous), and therefore by Theorem 5.10 in [23, p. 41], $H^{p,q}(\Omega)$ is homeomorphic to the direct sum $H^{p,q}_{\text{ind}}(\Omega) \oplus H^{p,q}_{\text{red}}(\Omega)$.

VI.0.2. Čech cohomology

Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and let $\underline{\mathcal{U}} = \{U_\alpha\}_{\alpha\in\Gamma}$ (Γ is a countable index set) be the countable cover of Ω by polyannuli, as in Lemma IV.0.1. Since polyannuli are pseudoconvex, the cover $\underline{\mathcal{U}}$ is a countable Leray cover of Ω. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on Ω. For $q \in \mathbb{N}$, denote by $\check{C}^q(\underline{\mathcal{U}},\mathcal{O})$ the q-th cochain group with respect to the cover $\underline{\mathcal{U}}$ and the sheaf $\mathcal{O}$, defined as

$$\bigoplus_{\{\alpha_0\neq\alpha_1\neq\cdots\neq\alpha_q\}} \mathcal{O}\big(U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}\big) \quad \text{where } \alpha_j \in \Gamma, 0 \leqslant j \leqslant q. \tag{VI.0.4}$$

The space $\check{C}^q(\underline{\mathcal{U}},\mathcal{O})$, being a countable product of Fréchet spaces, is a Fréchet space (see Lemma II.0.4). Let $\delta : \check{C}^q(\underline{\mathcal{U}},\mathcal{O}) \to \check{C}^{q+1}(\underline{\mathcal{U}},\mathcal{O})$ be the coboundary operator. A q-cochain f

is called a *cocyle* if $\delta f = 0$. Denote by $\check{Z}^q(\underline{\mathcal{U}}, \mathcal{O})$ the subspace of q-cocyles with the subspace topology. A q-cochain f is a *coboundary* if $f = \delta g$, for some $g \in \check{C}^{q-1}(\underline{\mathcal{U}}, \mathcal{O})$. It is easy to see that $\delta^2 = 0$. The q-th Čech cohomology group (with respect to the cover $\underline{\mathcal{U}}$) $\check{H}^q(\underline{\mathcal{U}}, \mathcal{O})$ is the quotient $\check{Z}^q(\underline{\mathcal{U}}, \mathcal{O})/\delta(\check{C}^{q-1}(\underline{\mathcal{U}}, \mathcal{O}))$. The cohomology group carries the natural quotient topology. Finally $\check{H}^q(\Omega, \mathcal{O})$, the q-th Čech cohomology group of Ω with respect to the sheaf $\mathcal{O}$ is the direct limit (with respect to refinement of covers) of the Čech groups $\check{H}^q(\underline{\mathcal{U}}, \mathcal{O})$. The q-th Čech cohomology group $\check{H}^q(\Omega, \mathcal{O})$ is endowed with the strong topology, i.e. the finest topology with respect to which the natural map $\check{H}^q(\underline{\mathcal{U}}, \mathcal{O}) \hookrightarrow \check{H}^q(\Omega, \mathcal{O})$ is continuous for every Leray cover $\underline{\mathcal{U}}$ of Ω. We define the reduced and indiscrete parts of the Čech cohomology similar to the case in Dolbeault groups. The following results are useful.

Result 1 (Laufer, 1967)**.** With respect to the topologies described above, there is a linear homeomorphism between the topological groups $\check{H}^q(\underline{\mathcal{U}}, \mathcal{O})$ and $\check{H}^q(\Omega, \mathcal{O})$.

Proof. See Section 2 in [26]. □

Result 2. Let $\Omega \subset \mathbb{C}^n$ be Reinhardt. With respect to the topology described above, there is a linear homeomorphism at the cohomology level between the topological groups

$$H^{p,q}(\Omega) \cong \bigoplus_{j=1}^{\binom{n}{p}} H^{0,q}(\Omega)$$

Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and let $\underline{\mathcal{U}} = \{U_\alpha\}_{\alpha \in \Gamma}$ be a countable Leray cover of Ω. In the next result, we state the well-known Dolbeault theorem.

Result 3. With respect to the topologies described above, there is a linear homeomorphism (called the Dolbeault isomorphism) between the topological groups $H^{0,q}(\Omega)$ and $\check{H}^q(\underline{\mathcal{U}}, \mathcal{O})$.

Remark. It follows from the above three results that the topologies of the vector spaces $H^{p,q}_{\mathrm{red}}(\Omega)$, $\bigoplus_{j=1}^{\binom{n}{p}} H^{0,q}_{\mathrm{red}}(\Omega)$, $\bigoplus_{j=1}^{\binom{n}{p}} \check{H}^q_{\mathrm{red}}(\underline{\mathcal{U}}, \mathcal{O})$ and $\bigoplus_{j=1}^{\binom{n}{p}} \check{H}^q_{\mathrm{red}}(\Omega, \mathcal{O})$ are essentially the same (and therefore the corresponding reduced (Hausdorff) parts of the cohomologies get mapped onto each other homeomorphically).

VI.0.3. Representation of the torus group on product and quotient topological vector spaces

The next two results are very useful and will be used to prove the existence of a series representation of Dolbeault cohomology on a Reinhardt domain Ω.

Lemma VI.0.1. *For $j \in \mathbb{N}$, let X_j be a Fréchet space and let $\sigma^j : \mathbb{T}^n \times X_j \to X_j$ be a continuous representation of $\mathbb{T}^n$ on X_j. Consider the product space $X = \prod_{j\in\mathbb{N}} X_j$ with the product topology. Let $\lambda \in \mathbb{T}^n$ and $x = (x_j) := (x_1, x_2, \cdots) \in X$, where $x_j \in X_j$. Define $\sigma : \mathbb{T}^n \times X \to X$ by,*

$$\sigma(\lambda, x) = (\sigma^j(\lambda, x_j)). \tag{VI.0.5}$$

Then σ is a continuous representation of $\mathbb{T}^n$ on X.

Proof. First we show that σ is representation of $\mathbb{T}^n$ on X. For all $\lambda \in \mathbb{T}^n$ and $x = (x_j) \in X$, we write $\sigma(\lambda, x) = \sigma_\lambda(x)$. We show that if σ is defined as in (VI.0.5), then $\sigma_\lambda \in \mathrm{Aut}(X)$ and the function $\lambda \mapsto \sigma_\lambda$ is a group homomorphism from $\mathbb{T}^n$ to $\mathrm{Aut}(X)$. For every $j \in \mathbb{N}$, we write $\sigma^j(\lambda, x_j) = \sigma^j_\lambda(x_j)$, that is $\sigma_\lambda(x) = (\sigma^j_\lambda(x_j))$. Since σ^j is continuous for every j, $\sigma^j_\lambda \in \mathrm{Aut}(X_j)$ and $\lambda \mapsto \sigma^j_\lambda$ is a group homomorphism from $\mathbb{T}^n$ to $\mathrm{Aut}(X_j)$. Let $x = (x_j)$ and $y = (y_j)$ in X. Then

$$\begin{aligned}
\sigma_\lambda(x+y) = \Big(\sigma^j_\lambda(x_j+y_j)\Big) &= \Big(\sigma^j_\lambda(x_j) + \sigma^j_\lambda(y_j)\Big) \quad \text{(Since } \sigma^j_\lambda \text{ is linear)}\\
&= \Big(\sigma^j_\lambda(x_j)\Big) + \Big(\sigma^j_\lambda(y_j)\Big)\\
&= \sigma_\lambda(x) + \sigma_\lambda(y).
\end{aligned} \tag{VI.0.6}$$

This proves σ_λ is linear. Since for each j, σ^j_λ is a bijection from X_j to itself, it follows easily that σ_λ is a bijection from X to itself. Since the map σ^j_λ is continuous on X_j for all j, the map σ_λ is continuous on X. By the open mapping theorem (see [29, Corollary 2.12, p. 49]), σ_λ^{-1} is also continuous. Hence, $\sigma_\lambda \in \mathrm{Aut}(X)$.

Now, let $\lambda,\mu\in\mathbb{T}^n$ and $x=(x_j)\in X$, then

$$\sigma_{\lambda\cdot\mu}(x)=\big(\sigma^j_{\lambda\cdot\mu}(x_j)\big)=\Big(\big(\sigma_\lambda\circ\sigma_\mu\big)^j(x_j)\Big)=\sigma_\lambda\circ\sigma_\mu(x). \tag{VI.0.7}$$

where the second equality is due to the fact that $\lambda\mapsto\sigma^j_\lambda$ is a group homomorphism. This shows that the map $\lambda\mapsto\sigma_\lambda$ is a group homomorphism from $\mathbb{T}^n$ to $\operatorname{Aut}(X)$. Now, to prove the continuity of σ, we use Proposition III.0.1. In Lemma II.0.4 we saw that the set of seminorms $P=\{p_{j,\ell}\circ\Pi_j\}_{j,\ell\in\mathbb{N}}$ generates the topology of X, where $p_{j,\ell}$ and Π_j are as in the statement of Lemma II.0.4. Observe that for all $x=(x_j)\in X$, $\lambda\in\mathbb{T}^n$ and $j,\ell\in\mathbb{N}$,

$$\begin{aligned}(p_{j,\ell}\circ\Pi_j)(\sigma_\lambda(x))&=p_{j,\ell}\big(\sigma^j_\lambda(x_j)\big)\\&=p_{j,\ell}(x_j)\quad\text{(since } p_{j,\ell}\text{ is an invariant seminorm on } X_j),\\&=p_{j,\ell}\circ\Pi_j(x).\end{aligned}\tag{VI.0.8}$$

This shows that the topology of X is generated by continuous invariant seminorms. Next we prove that for all $x=(x_j)\in X$, the map $\lambda\mapsto\sigma_\lambda(x)$ is continuous at $\lambda=\mathbb{1}$. Note that,

$$\begin{aligned}(p_{j,\ell}\circ\Pi_j)\big(\sigma_\lambda(x)-x\big)&=(p_{j,\ell}\circ\Pi_j)\Big(\big(\sigma^j_\lambda(x_j)\big)_j-(x_j)_j\Big)\\&=(p_{j,\ell}\circ\Pi_j)\Big(\big(\sigma^j_\lambda(x_j)-(x_j)\big)_j\Big)\\&=p_{j,\ell}\big(\sigma^j_\lambda(x_j)-x_j\big).\end{aligned}\tag{VI.0.9}$$

Let $\epsilon>0$. Since σ^j is continuous for every j, the map $\lambda\mapsto\sigma^j_\lambda(x_j)$ is continuous at $\lambda=\mathbb{1}$. So, there exists $\delta>0$ such that

$$\text{whenever }\ |\lambda-\mathbb{1}|<\delta,\ \ p_{j,\ell}\big(\sigma^j_\lambda(x_j)-x_j\big)<\epsilon.$$

The result now follows from (VI.0.9). □

Lemma VI.0.2. *Let σ be a continuous representation of $\mathbb{T}^n$ on an LCTVS X and let Y be a closed invariant subspace of X, that is $\sigma_\lambda(Y)\subset Y$. Endow X/Y with the quotient topology.*

Define $\widehat{\sigma} : \mathbb{T}^n \times X/Y \to X/Y$ *by,*

$$\widehat{\sigma}(\lambda, x+Y) = \sigma_\lambda(x) + Y. \tag{VI.0.10}$$

Then $\widehat{\sigma}$ *is a continuous representation of* $\mathbb{T}^n$ *on* X/Y*.*

Proof. The proof will be given in several steps. First, we need to check that the map $\widehat{\sigma}$ is well-defined. Let $x_1 + Y, x_2 + Y \in X/Y$, where $x_1 - x_2 \in Y$. Then

$$\begin{aligned}
\big(\sigma_\lambda(x_1) + Y\big) - \big(\sigma_\lambda(x_2) + Y\big) &= \sigma_\lambda(x_1) - \sigma_\lambda(x_2) + Y \\
&= \sigma_\lambda(x_1 - x_2) + Y \quad (\text{since } \sigma_\lambda \text{ is linear}) \\
&\in Y. \quad (\text{since } Y \text{ is invariant under } \sigma_\lambda)
\end{aligned}$$

Therefore $\widehat{\sigma}(\lambda, x_1 + Y) = \widehat{\sigma}(\lambda, x_2 + Y)$ and $\widehat{\sigma}$ is well-defined. For $\lambda \in \mathbb{T}^n$ and $x + Y \in X/Y$, write $\widehat{\sigma}(\lambda, x+Y) = \widehat{\sigma}_\lambda(x+Y)$. We prove that $\widehat{\sigma}_\lambda \in \mathrm{Aut}(X/Y)$. We start by showing that $\widehat{\sigma}_\lambda$ is linear. Let $x_1, x_2 \in X$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}
\widehat{\sigma}_\lambda(x_1 + Y + \alpha(x_2 + Y)) &= \widehat{\sigma}_\lambda(x_1 + \alpha x_2 + Y) \\
&= \sigma_\lambda(x_1 + \alpha x_2) + Y \\
&= \sigma_\lambda(x_1) + \alpha \cdot \sigma_\lambda(x_2) + Y + \alpha Y \quad (\text{by linearity of } \sigma_\lambda) \\
&= \sigma_\lambda(x_1) + Y + \alpha(\sigma_\lambda(x_2) + Y) \\
&= \widehat{\sigma}_\lambda(x_1 + Y) + \alpha \cdot \widehat{\sigma}_\lambda(x_2 + Y).
\end{aligned} \tag{VI.0.11}$$

Now, we show $\widehat{\sigma}_\lambda$ is a bijection from X/Y to itself. For $x_1, x_2 \in X$, let $\widehat{\sigma}_\lambda(x_1 + Y) = \widehat{\sigma}_\lambda(x_2 + Y)$. This implies $\sigma_\lambda(x_1) - \sigma_\lambda(x_2) \in Y$ (applying (VI.0.10)). Since σ_λ is one to one, $x_1 - x_2 \in Y$ and so $x_1 + Y = x_2 + Y$. Therefore, $\widehat{\sigma}_\lambda$ is one to one. Let $x + Y \in X/Y$. Then there exists $\sigma_{\lambda^{-1}}(x) \in X$ (it exists since σ_λ is onto) such that

$$\widehat{\sigma}_\lambda\big(\sigma_{\lambda^{-1}}(x) + Y\big) = \sigma_\lambda(\sigma_{\lambda^{-1}}(x)) + Y = \sigma_{\mathbb{1}}(x) + Y = x + Y.$$

Therefore $\widehat{\sigma}_\lambda$ is onto. Consequently, the map $\widehat{\sigma}_\lambda^{-1}$ exists. It remains to show that the maps $\widehat{\sigma}_\lambda$ and $\widehat{\sigma}_\lambda^{-1}$ are continuous. Let $\phi : X \to X/Y$ be the quotient map. Observe the following diagram.

$$\begin{tikzcd} X \arrow[r, "\sigma_\lambda"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ X/Y \arrow[r, "\widehat{\sigma}_\lambda"] & X/Y \end{tikzcd}$$

Since ϕ is continuous and open, and since σ_λ and σ_λ^{-1} are continuous, then it follows easily that the maps $\widehat{\sigma}_\lambda = \phi \circ \sigma_\lambda \circ \phi^{-1}$ is continuous and so is $\widehat{\sigma}_\lambda^{-1}$. We have now proved that $\widehat{\sigma}_\lambda \in \mathrm{Aut}(X/Y)$.

To prove that $\widehat{\sigma}$ in (VI.0.10) is a representation, one additionally needs to prove that the map $\lambda \mapsto \widehat{\sigma}_\lambda$ is a group homomorphism from $\mathbb{T}^n$ to X/Y. Let $\lambda, \mu \in \mathbb{T}^n$ and $x + Y \in X/Y$. Then

$$\widehat{\sigma}_{\lambda\cdot\mu}(x+Y) = \sigma_{\lambda\cdot\mu}(x) + Y = \sigma_\lambda(\sigma_\mu(x)) + Y = \widehat{\sigma}_\lambda(\sigma_\mu(x) + Y) = \widehat{\sigma}_\lambda(\widehat{\sigma}_\mu(x+Y)),$$

where we have used the fact that $\lambda \mapsto \sigma_\lambda$ is a group homomorphism from $\mathbb{T}^n$ to X. This proves that $\widehat{\sigma}$ in (VI.0.10) is a representation of $\mathbb{T}^n$ on X.

It remains to show that $\widehat{\sigma}$ is continuous. We use Proposition III.0.1 to show the continuity part. Let p be a continuous seminorm on X and define $\overline{p}$ on X/Y as in (II.0.14). Then $\overline{p}$ is continuous and by Lemma II.0.5, the set

$$\overline{P} = \{\overline{p} : \ p \text{ is a continuous seminorm on } X\}$$

generates the quotient topology on X/Y. Let $\overline{p} \in \overline{P}$. We show that $\overline{p}$ is an invariant seminorm. Let $\lambda \in \mathbb{T}^n$ and $\overline{x} = x + Y \in X/Y$. Then

$$\begin{aligned} \overline{p}(\widehat{\sigma}_\lambda(\overline{x})) &= \inf_{\phi(\sigma_\lambda(x)) = \widehat{\sigma}_\lambda(\overline{x})} p(\sigma_\lambda(x)) \\ &= \inf_{\phi(x) = \overline{x}} p(x) \quad \text{(since } p \text{ is invariant)} \\ &= \overline{p}(\overline{x}). \end{aligned} \tag{VI.0.12}$$

Next we show that for all $\overline{x} \in X/Y$, the map $\lambda \mapsto \widehat{\sigma}_\lambda(\overline{x})$ is continuous at $\lambda = \mathbb{1}$. Let $\epsilon > 0$. Note that

$$\overline{p}(\widehat{\sigma}_\lambda(\overline{x}) - \overline{x}) = \inf p(\sigma_\lambda(x) - x) \leqslant p(\sigma_\lambda(x) - x). \tag{VI.0.13}$$

Since the representation σ is continuous, there exists $\delta > 0$ such that whenever $|\lambda - \mathbb{1}| < \delta$, $p(\sigma_\lambda(x) - x) < \epsilon$. The result then follows from (VI.0.13). □

VI.0.4. Laurent series in Dolbeault cohomology

Let $\Omega \subset \mathbb{C}^n$ be Reinhardt. In this section we show that the standard representation of the torus group $\mathbb{T}^n$ on $\mathcal{O}(\Omega)$ (defined as in Equation (IV.0.5)) gives rise to a continuous representation of $\mathbb{T}^n$ on $H^{p,q}_{\mathrm{red}}(\Omega)$.

Let $\underline{\mathcal{U}} = \{U_\alpha\}_{\alpha\in\Gamma}$ be a countable Reinhardt Leray cover of Ω (by Lemma IV.0.1 a countable Reinhardt Leray cover exists). Let $\bigcap U_{\alpha_j} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$ and let $\sigma^{\mathcal{O}}$ be the standard representation of $\mathbb{T}^n$ on $\mathcal{O}\big(\bigcap U_{\alpha_j}\big)$; that is, $\sigma^{\mathcal{O}} : \mathbb{T}^n \times \mathcal{O}\big(\bigcap U_{\alpha_j}\big) \to \mathcal{O}\big(\bigcap U_{\alpha_j}\big)$ is defined by,

$$\sigma^{\mathcal{O}}(\lambda, f)(z) = f(\lambda \odot z), \tag{VI.0.14}$$

where $\lambda \in \mathbb{T}^n$, $f \in \mathcal{O}\big(\bigcap U_{\alpha_j}\big)$, $z \in \bigcap U_{\alpha_j}$, and $\lambda \odot z = (\lambda_1 z_1, \cdots, \lambda_n z_n)$. By Corollary IV.0.1, the representation $\sigma^{\mathcal{O}}$ is continuous. By Lemma II.0.4, the q^{th} Čech cochain group

$$\check{C}^q(\underline{\mathcal{U}}, \mathcal{O}) = \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_q} \mathcal{O}\big(\bigcap U_{\alpha_j}\big). \tag{VI.0.15}$$

is a Fréchet space. Note that every cochain element $f \in \check{C}^q(\underline{\mathcal{U}}, \mathcal{O})$ can be written as

$$f = \big(f_{\alpha_0,\cdots,\alpha_q}\big)_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_q}}.$$

Define $\sigma^{\check{C}^q} : \mathbb{T}^n \times \check{C}^q(\underline{\mathcal{U}}, \mathcal{O}) \to \check{C}^q(\underline{\mathcal{U}}, \mathcal{O})$ by,

$$\sigma^{\check{C}^q}(\lambda, f) = \Big(\sigma^{\mathcal{O}}\big(\lambda, f_{\alpha_1,\cdots,\alpha_q}\big)\Big)_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_q}}, \tag{VI.0.16}$$

where $\lambda \in \mathbb{T}^n$ and $f \in \check{C}^q(\underline{\mathcal{U}},\mathcal{O})$. By Lemma VI.0.1, $\sigma^{\check{C}^q}$ is a continuous representation of $\mathbb{T}^n$ on $\check{C}^q(\underline{\mathcal{U}},\mathcal{O})$. Let $\delta : \check{C}^q(\underline{\mathcal{U}},\mathcal{O}) \to \check{C}^{q+1}(\underline{\mathcal{U}},\mathcal{O})$ be the coboundary operator. For $\lambda \in \mathbb{T}^n$ and $f \in \check{C}^q(\underline{\mathcal{U}},\mathcal{O})$, write $\sigma^{\check{C}^q}(\lambda, f) = \sigma_\lambda^{\check{C}^q}(f)$. We claim that $\delta \circ \sigma_\lambda^{\check{C}^q} = \sigma_\lambda^{\check{C}^{q+1}} \circ \delta$. Indeed, for every $f \in \check{C}^q(\underline{\mathcal{U}},\mathcal{O})$,

$$\begin{aligned}\delta \circ \sigma_\lambda^{\check{C}^q}(f) = \left(\delta\big(\sigma^{\mathcal{O}}(\lambda, f_{\alpha_0,\cdots,\alpha_q})\big)\right)_{\alpha_0,\cdots,\alpha_{q+1}} &= \sum_{j=0}^{q+1}(-1)^j \ \sigma^{\mathcal{O}}(\lambda, f_{\alpha_0,\cdots,\widehat{\alpha_j},\cdots,\alpha_{q+1}}) \\ &= \sum_{j=0}^{q+1}(-1)^j \ (f\circ\lambda)_{\alpha_0,\cdots,\widehat{\alpha_j},\cdots,\alpha_{q+1}}, \end{aligned} \tag{VI.0.17}$$

where the $(q+1)$-tuple $(\alpha_0,\cdots,\widehat{\alpha_j},\cdots,\alpha_{q+1})$ is derived from the $(q+2)$-tuple $(\alpha_0,\cdots,\alpha_{q+1})$ by removing the j-th entry α_j. Now by (VI.0.17),

$$\sum_{j=0}^{q+1}(-1)^j \ (f\circ\lambda)_{\alpha_0,\cdots,\widehat{\alpha_j},\cdots,\alpha_{q+1}} = \sigma_\lambda^{\check{C}^{q+1}}\left(\sum_{j=0}^{q+1}(-1)^j \ f_{\alpha_0,\cdots,\widehat{\alpha_j},\cdots,\alpha_{q+1}}\right) = \sigma_\lambda^{\check{C}^{q+1}} \circ \delta(f). \tag{VI.0.18}$$

It follows easily from our claim that,

$$\sigma_\lambda^{\check{C}^q}(\check{Z}^q(\underline{\mathcal{U}},\mathcal{O})) \subset \check{Z}^q(\underline{\mathcal{U}},\mathcal{O}) \text{ and } \sigma_\lambda^{\check{C}^q}(\overline{\delta(\check{C}^{q-1}(\underline{\mathcal{U}},\mathcal{O}))}) \subset \overline{\delta(\check{C}^{q-1}(\underline{\mathcal{U}},\mathcal{O}))}.$$

Therefore by Lemma VI.0.2, $\sigma^{\check{C}^q}$ induces a continuous representation of $\mathbb{T}^n$ on $\check{H}^q_{\text{red}}(\underline{\mathcal{U}},\mathcal{O})$, call it $\sigma^{\check{H}^q}$. Since $\check{H}^q_{\text{red}}(\underline{\mathcal{U}},\mathcal{O})$ is homeomorphic to $H^{0,q}_{\text{red}}(\Omega)$ and $H^{p,q}_{\text{red}}(\Omega)$ is homemorphic to the finite direct sum of $H^{0,q}_{\text{red}}(\Omega)$, $\sigma^{\check{H}^q}$ gives rise to a continuous representation of $\mathbb{T}^n$ on $H^{p,q}_{\text{red}}(\Omega)$, call it $\sigma^{H^{p,q}}$. It follows from Section III.0.4 that every cohomology class $F \in H^{p,q}_{\text{red}}(\Omega)$ admits an abstract Fourier series with respect to the representation $\sigma^{H^{p,q}}$ as

$$F \sim \sum_{\alpha\in\mathbb{Z}^n} F_\alpha. \tag{VI.0.19}$$

Since the standard representation of the torus group on the space of holomorphic functions on a subset of Ω stirs up the representation $\sigma^{H^{p,q}}$ of $\mathbb{T}^n$ on $H^{p,q}_{\text{red}}(\Omega)$, we call the abstract Fourier series in (VI.0.19) the Laurent series of Dolbeault cohomology on Ω. In this section our goal is to prove the following result.

Theorem VI.0.1. *Let Ω be a Reinhardt domain in $\mathbb{C}^n$ and let $q \in \mathbb{N}$. Then every cohomology class $F \in H^{p,q}_{red}(\Omega)$ admits a Laurent series as in* (VI.0.19) *with respect to the continuous representation $\sigma^{H^{p,q}}$ of the torus group on $H^{p,q}_{red}(\Omega)$ (described as above) and the series converges unconditionally to F in the topology of $H^{p,q}_{red}(\Omega)$.*

Since the topologies of the reduced Dolbeault cohomology groups and the reduced Čech coholomology groups are essentially the same, it is sufficient to prove the following result.

Theorem VI.0.2. *Let Ω be a Reinhardt domain in $\mathbb{C}^n$, let $\underline{\mathcal{U}} = \{U_\alpha\}_{\alpha\in\Gamma}$ be a countable Leray cover of Ω. Let $q \in \mathbb{N}$ and let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on Ω. Then for every cohomology element F in $\check{H}^q_{red}(\underline{\mathcal{U}}, \mathcal{O})$, the Laurent series*

$$F \sim \sum_{\alpha\in\mathbb{Z}^n} F_\alpha \tag{VI.0.20}$$

with respect to the continuous representation $\sigma^{\check{H}^q}$ of $\mathbb{T}^n$ on $\check{H}^q_{red}(\underline{\mathcal{U}}, \mathcal{O})$ converges unconditioally to F in the topology of $\check{H}^q_{red}(\underline{\mathcal{U}}, \mathcal{O})$.

The next Lemma is the key to proving Theorem VI.0.2.

For each $j \in \mathbb{N}$, let C_j be a Fréchet space and let σ^j be a continuous representation of $\mathbb{T}^n$ on C_j. Let $C = \prod_{j\in\mathbb{N}} C_j$. By Lemma II.0.4, C is a Fréchet space and by Lemma VI.0.1, σ^j's give rise a continuous representation σ^C of $\mathbb{T}^n$ on C. Let Z be a closed invariant (with respect to the representation σ^C) subspace of C. Then σ^C induces a continuous representation σ^Z (restriction of σ^C to $\mathbb{T}^n \times Z$) of $\mathbb{T}^n$ on Z. Now, let B be a closed invariant (with respect to the representation σ^Z) subspace of Z. Then, by Lemma VI.0.2, σ^Z gives rise to a continuous representation σ of $\mathbb{T}^n$ on the quotient Z/B. Now we are ready to state the Lemma.

Lemma VI.0.3. *For each $j \in \mathbb{N}$, let $x^j \in C_j$ and let the abstract Fourier series $x^j = \sum_{\alpha\in\mathbb{Z}^n} x^j_\alpha$ with respect to the continuous representation σ^j is absolutely convergent in the topology of*

C_j. Let $x = (x^j)_j \in Z \subset C$ and let $\overline{x} = x + B \in Z/B$. Then the Fourier series $\overline{x} = \sum_{\alpha \in \mathbb{Z}^n} \overline{x}_\alpha$ with respect to the continuous representation σ is absolutely convergent in the topology of Z/B.

Proof. Let $P_j = \{p_{j,\ell}\}_{\ell \in \mathbb{N}}$ be a set of continuous seminorm on C_j that generates the Fréchet topology of C_j. By Lemma II.0.4, the set $P = \{p_{j,\ell} \circ \Pi_j\}_{j,\ell \in \mathbb{N}}$ generates the Fréchet topology of C, where $\Pi_j : C \to C_j$ is the projection operator. Since Z is endowed with the subspace topology, the set P also generates the topology of Z. Let $\overline{P} = \{\overline{p} : p \in P\}$. By Lemma II.0.5, the set $\overline{P}$ generates the quotient topology of Z/B. Let $\overline{p} \in \overline{P}$. To prove the result, we show that there exists a bijection $\gamma : \mathbb{N} \to \mathbb{Z}^n$ such that $\sum_{k=0}^{\infty} \overline{p}(\overline{x}_{\gamma(k)}) < \infty$. Let $\pi : Z \to Z/B$ be the quotient map. Observe that for all $\overline{y} \in Z/B$,

$$\begin{aligned} \overline{p}(\overline{y}) &= \inf_{\pi(y) = \overline{y}} p_{j,\ell} \circ \Pi_j(y) \quad \text{(for some } j, \ell \in \mathbb{N}) \\ &= \inf_{\pi(y) = \overline{y}} p_{j,\ell}(y^j). \end{aligned} \tag{VI.0.21}$$

However, by hypothesis, there exists a bijection $\tau : \mathbb{N} \to \mathbb{Z}^n$ such that for all $j, \ell \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} p_{j,\ell}(x^j_{\tau(k)}) < \infty. \tag{VI.0.22}$$

Therefore, by (VI.0.21), for the same bijection γ, $\sum_{k=0}^{\infty} \overline{p}(\overline{x}_{\gamma(k)}) < \infty$. □

Proof of Theorem VI.0.2. Let $F = \text{class}(f) \in \check{H}^q_{\text{red}}(\Omega, \mathcal{O})$, that is

$$f = \left(f_{\alpha_0, \cdots, \alpha_q}\right)_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}} \in \check{C}^q(\underline{\mathcal{U}}, \mathcal{O}),$$

and $f_{\alpha_0, \cdots, \alpha_q}$ is a holomorphic function on $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$. It follows from part(d) of Theorem IV.0.2 that the Laurent series of $f_{\alpha_0, \cdots, \alpha_q}$ converges absolutely in the topology of $\mathcal{O}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q})$, that is, converges absolutely in the Fréchet topology of $\check{C}^q(\underline{\mathcal{U}}, \mathcal{O})$. Therefore it follows from Lemma VI.0.3 that the series in (VI.0.20) converges absolutely in the topology of $\check{H}^q_{\text{red}}(\Omega, \mathcal{O})$.

We have now proved that the series in (VI.0.20) converges absolutely (and hence unconditionally) in $\check{H}^q_{\mathrm{red}}(\Omega, \mathcal{O})$. So, the series

$$\sum_{j=0}^{\infty} F_{\tau(j)} \tag{VI.0.23}$$

converges, where $\tau : \mathbb{N} \to \mathbb{Z}^n$ is the bijection constructed in (IV.0.36) (Note that the same bijection works here). Let $G = \lim_{N\to\infty} \sum_{j=0}^{N} F_{\tau(j)}$ in $\check{H}^q_{\mathrm{red}}(\Omega, \mathcal{O})$. Therefore by Lemma II.0.3, the Cesàro means of the partial sums of the series in (VI.0.23) converge to G as well. But, by the Fejér theorem (Theorem III.0.1), the Cesàro means of the partial sums converge to F. Therefore $F = G$ $\square$

The following two lemmas will be useful in the actual computation at the end of this chapter.

Let X be a quasi-complete LCTVS. Recall from Section III.0.5 that for $\alpha \in \mathbb{Z}^n$, we defined the Fourier modes X^{σ}_{α} with respect to a continuous representation σ of $\mathbb{T}^n$ on X, and the trigonometric polynomials $TP_{\sigma}(X)$. Let Y be a closed subspace of X. Then X/Y is a quasi-complete LCTVS. By Lemma VI.0.2, σ induces a continuous representation $\widehat{\sigma}$ of $\mathbb{T}^n$ on X/Y. Likewise we can define the subspaces Y^{σ}_{α}, $TP_{\sigma}(Y)$, $(X/Y)^{\widehat{\sigma}}_{\alpha}$ and $TP_{\widehat{\sigma}}(X/Y)$. We first prove the following result.

Lemma VI.0.4. *The space $TP_{\widehat{\sigma}}(X/Y)$ is isomorphic to the quotient $TP_{\sigma}(X)/TP_{\sigma}(Y)$.*

Proof. Let $\phi : TP_{\sigma}(X) \to X/Y$ be defined as $\phi(x) = \mathrm{class}_Y(x)$, where $x \in TP_{\sigma}(X)$, $\mathrm{class}_Y(x) = x + Y \in X/Y$ and $x = \sum_{\alpha \in \mathbb{Z}^n} x_{\alpha}$, where $x_{\alpha} \in X_{\alpha}$ and all but finitely many x_{α}'s are zero. We prove that $\mathrm{Im}(\phi) = TP_{\hat{\sigma}}(X/Y)$ and $\ker(\phi) = TP_{\sigma}(Y)$, where Im and ker stand for them image and kernel respectively. The result then follows from the isomorphism theorem. First we prove $\mathrm{Im}(\phi) = TP_{\hat{\sigma}}(X/Y)$. Clearly $\mathrm{Im}(\phi) \subset TP_{\hat{\sigma}}(X/Y)$, because if $x \in TP_{\sigma}(X)$, then $\mathrm{class}_Y(x) = \sum_{\alpha \in \mathbb{Z}^n} \mathrm{class}_Y(x_{\alpha}) \in TP_{\hat{\sigma}}(X/Y)$. Now let $y \in TP_{\hat{\sigma}}(X/Y)$. So y can be written as $y = \sum_{\alpha \in \mathbb{Z}^n} y_{\alpha}$, where $y_{\alpha} \in (X/Y)_{\alpha}$ and $(X/Y)_{\alpha}$ is a closed (invariant with respect to $\hat{\sigma}$)

subspace of X/Y (similar to the subspace X_α of X); also note that all but finitely many y_α's are zero. For every α, $y_\alpha \in (X/Y)_\alpha \subset X/Y$, write $y_\alpha = \text{class}_Y(x)$, where $x \in X$. To prove $\text{Im}(\phi) \supset TP_{\hat{\sigma}}(X/Y)$, it is sufficient to show that $x \in X_\alpha$. Since X is quasi-complete LCTVS and σ is a continuous representation, by Theorem III.0.4, there is a net of trigonometric polynomials $(x_j) \subset X$ such that $x_j \to x$ in the topology of X, as $j \to \infty$. So we write

$$y_\alpha = \text{class}_Y(x) = \text{class}_Y(\lim_{j\to\infty} x_j) = \lim_{j\to\infty} (\text{class}_Y(x_j)) \tag{VI.0.24}$$

where the last equality in (VI.0.24) is due to the fact that the quotient map is continuous. Recall the projection map defined in (III.0.11) in Section III.0.4. For $\beta \in \mathbb{Z}^n$, let Π_β^X and $\Pi_\beta^{X/Y}$ be the projection maps of X onto X_α and X/Y onto $(X/Y)_\alpha$ respectively. Applying $\Pi_\beta^{X/Y}$ on both sides of (VI.0.24) we get,

$$\Pi_\beta^{X/Y}(y_\alpha) = \lim_{j\to\infty} \big(\text{class}_Y(\Pi_\beta^X x_j)\big) = \lim_{j\to\infty} \big(\text{class}_Y(x_{j_\alpha})\big) \tag{VI.0.25}$$

where the first equality is due to the fact that the projection map is continuous and $x_{j_\alpha} \in X_\alpha$. Now for $\beta = \alpha$,

$$y_\alpha = \lim_{j\to\infty} \big(\text{class}_Y(x_{j_\alpha})\big) = \text{class}_Y(\lim_{j\to\infty} x_{j_\alpha}) = \text{class}_Y(x') \tag{VI.0.26}$$

for some $x' \in X_\alpha$ (since X_α is closed). But $y_\alpha = \text{class}_Y(x)$, therefore $x \in X_\alpha$. Now we prove $\ker(\phi) = TP_\sigma(Y)$. We note that

$$\begin{aligned}\ker(\phi) = \{x \in TP_\sigma(X) : \phi(x) \in Y\} &= \{x \in TP_\sigma(X) : x + Y = Y\} \\ &= \{x \in TP_\sigma(X) : x \in Y\} \\ &= TP_\sigma(X) \bigcap Y \\ &= TP_\sigma(Y). \end{aligned} \tag{VI.0.27}$$

□

Let Ω be a Reinhardt domain in $\mathbb{C}^2$ and let $\underline{\mathcal{U}} = \{U_1, U_2\}$ be a Leray cover of Ω. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on Ω. Let σ and $\widehat{\sigma}$ be the representations of $\mathbb{T}^n$ on $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})$ and $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}$ respectively, which are induced by the standard representation of $\mathbb{T}^n$ on $\mathcal{O}(\Omega)$. We have proved in Section VI.0.4 that σ and $\widehat{\sigma}$ are continuous, and therefore it follows by the abstract Fejér theorem that, $TP_\sigma(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O}))$ is dense in $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})$, $TP_\sigma(\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})})$ is dense in $\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}$, and $TP_{\widehat{\sigma}}(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})})$ is dense in $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}$. Let e_α be the monomial in two variables, that is, $e_\alpha(z) = z_1^{\alpha_1} z_2^{\alpha_2}$. Then the following result holds.

Lemma VI.0.5. *Let $U_1 \bigcap U_2$ is connected. Let $TP_\sigma(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})) = span\{e_\alpha : \alpha \in A \subset \mathbb{Z}^2\}$ and $TP_\sigma(\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}) = span\{e_\alpha : \alpha \in B \subset A\}$. Also, let $f \mapsto [f]$ be the quotient map from $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})$ to $\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}$, where $[f]$ denotes the class of $f \in \check{Z}^1(\underline{\mathcal{U}},\mathcal{O})$. Then*

$$TP_{\widehat{\sigma}}(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}) = span\big\{[e_\alpha] : \alpha \in A \backslash B\big\}.$$

Proof. It follows from Lemma VI.0.4,

$$TP_{\widehat{\sigma}}(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}) = TP_\sigma(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O}))/TP_\sigma(\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}).$$

Since the quotient map $e_\alpha \mapsto [e_\alpha]$ is continuous,

$$\begin{aligned} TP_\sigma(\check{Z}^1(\underline{\mathcal{U}},\mathcal{O}))/TP_\sigma(\overline{\check{B}^1(\underline{\mathcal{U}},\mathcal{O})}) &= \mathrm{span}\big\{[e_\alpha] : \alpha \in A, e_\alpha = 0 \text{ when } \alpha \in B\big\} \\ &= \mathrm{span}\big\{[e_\alpha] : \alpha \in A \backslash B\big\} \end{aligned} \tag{VI.0.28}$$

□

VI.0.5. Some examples

In this section we apply our theory to represent some Dolbeault cohomology classes of Reinhardt domains in $\mathbb{C}^2$ in power series.

Example 1. Let $\Delta = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the unit polydisc in $\mathbb{C}^2$. Let $0 < r_1, r_2 < 1$ and define

$$\mathbb{H}_0 := \{z \in \Delta : |z_1| > r_1, \text{or} \ \ |z_2| < r_2\}.$$

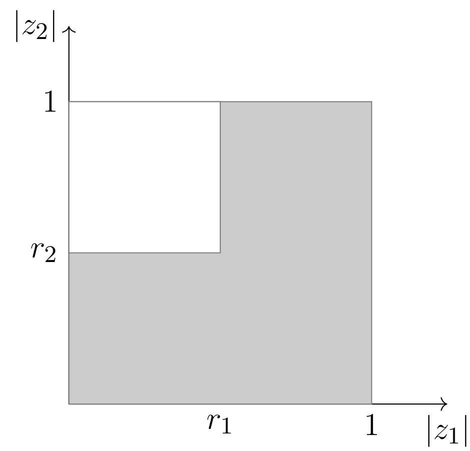

Figure 1. Reinhardt shadow of $\mathbb{H}_0$

The Reinhardt shadow of $\mathbb{H}_0$ is sketched in Figure 1 above. Using our result we can explicitly write the reduced Dolbeault cohomology group $H^{0,1}_{\text{red}}(\mathbb{H}_0)$ in terms of Laurent series. Let $U_1 = \{z \in \Delta : |z_1| < 1, |z_2| < r_2\}$ and $U_2 = \{z \in \Delta : r_1 < |z_1| < 1, |z_2| < 1\}$. Then $\underline{\mathcal{U}} = \{U_1, U_2\}$ is a Leray cover of $\mathbb{H}_0$. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\mathbb{H}_0$. Now

$$\check{H}^1_{\text{red}}(\mathbb{H}_0, \mathcal{O}) \cong \check{H}^1_{\text{red}}(\underline{\mathcal{U}}, \mathcal{O}) = \check{Z}^1(\underline{\mathcal{U}}, \mathcal{O}) / \overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})}. \tag{VI.0.29}$$

Note that $U_1 \bigcap U_2 = \{z \in \Delta : r_1 < |z_1| < 1, |z_2| < r_2\}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ and let $e_\alpha(z) = z_1^{\alpha_1} z_2^{\alpha_2}$ be a monomial in two variables. Now,

$$\begin{aligned} \check{Z}^1(\underline{\mathcal{U}}, \mathcal{O}) &= \{f : f = (f)_{12} \in \mathcal{O}(U_1 \bigcap U_2)\} \\ &= \overline{\text{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Big\{ e_\alpha : \alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0 \Big\}, \end{aligned} \tag{VI.0.30}$$

where the last equality holds since every holomorphic function on $U_1 \bigcap U_2$ can be approximated by Laurent polynomials, here $\overline{\text{span}}$ denotes the closure of the span. Similarly,

$$\overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})} = \overline{\text{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Big\{ \{e_\alpha : \alpha_1 \geqslant 0, \alpha_2 \geqslant 0\} \bigcup \{e_\alpha : \alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0\} \Big\}$$

$$= \overline{\operatorname{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Big\{ e_\alpha : \alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0 \Big\}. \tag{VI.0.31}$$

Now by Lemma VI.0.5,

$$\begin{aligned} \check{Z}^1(\underline{\mathcal{U}}, \mathcal{O})/\overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})} &= \overline{\operatorname{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Bigg\{ [e_\alpha] : \alpha \in \big\{\alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0\big\} \backslash \big\{\alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0\big\} \Bigg\} \\ &= \{0\}. \end{aligned} \tag{VI.0.32}$$

By Dolbeault's theorem, $H^{0,1}_{\text{red}}(\mathbb{H}_0) \cong \check{H}^1_{\text{red}}(\underline{\mathcal{U}}, \mathcal{O})$. Now, from Theorem VI.0.2, we conclude that the cohomology group $H^{0,1}_{\text{red}}(\mathbb{H}_0)$ can be represented by the Laurent series

$$\sum_{\alpha \in \mathbb{Z}^2} c_\alpha z^\alpha$$

and it follows from (VI.0.32) that $c_\alpha = 0$ for all $\alpha \in \mathbb{Z}^2$. Since the cohomology group $H^{0,1}(\mathbb{H}_0)$ is topologically isomorphic to the direct sum $H^{0,1}_{\text{red}}(\mathbb{H}_0) \oplus H^{0,1}_{\text{ind}}(\mathbb{H}_0)$ and since the reduced part vanishes, we can say that the cohomology group $H^{0,1}(\mathbb{H}_0)$ is indiscrete, in fact, it is infinite dimensional (see [8]).

Example 2. Let Δ be the unit polydisc in $\mathbb{C}^2$, and for $0 < r_1, r_2 < 1$, define

$$\mathbb{H}_1 := \{z \in \Delta : |z_1| > r_1, \text{or} \quad |z_2| > r_2\}$$

The Reinhardt shadow of $\mathbb{H}_1$ is sketched in Figure 2 below and using our result we explicitly write the cohomology group $H^{0,1}_{\text{red}}(\mathbb{H}_1)$ in terms of Laurent series.

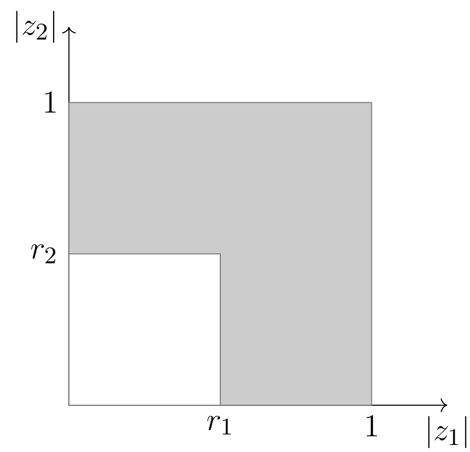

Figure 2. Reinhardt shadow of $\mathbb{H}_1$

Let $U_1 = \{z \in \Delta : |z_1| < 1, r_2 < |z_2| < 1\}$ and $U_2 = \{z \in \Delta : r_1 < |z_1| < 1, |z_2| < 1\}$. Then $\underline{\mathcal{U}} = \{U_1, U_2\}$ is a Leray cover of $\mathbb{H}_1$. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\mathbb{H}_1$. Note that $U_1 \bigcap U_2 = \{z \in \Delta : r_1 < |z_1| < 1, r_2 < |z_2| < 1\}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ and let e_α be the monomial in two variables, that is, $e_\alpha(z) = z_1^{\alpha_1} z_2^{\alpha_2}$. Now,

$$\begin{aligned} \check{Z}^1(\underline{\mathcal{U}}, \mathcal{O}) &= \{f : f = (f)_{12} \in \mathcal{O}(U_1 \textstyle\bigcap U_2)\} \\ &= \overline{\operatorname{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Big\{ e_\alpha : \alpha_1 \in \mathbb{Z}, \alpha_2 \in \mathbb{Z} \Big\}. \end{aligned} \tag{VI.0.33}$$

Similarly here we can write,

$$\overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})} = \overline{\operatorname{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \Big\{ \{e_\alpha : \alpha_1 \geqslant 0, \alpha_2 \in \mathbb{Z}\} \bigcup \{e_\alpha : \alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0\} \Big\}. \tag{VI.0.34}$$

Now by Lemma VI.0.5,

$$\check{Z}^1(\underline{\mathcal{U}}, \mathcal{O}) / \overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})} = \overline{\operatorname{span}}_{\mathcal{O}(U_1 \bigcap U_2)} \big\{ [e_\alpha] : \alpha \in A \backslash B \big\}, \tag{VI.0.35}$$

where $A = \big\{ \alpha \in \mathbb{Z}^2 : \alpha_1, \alpha_2 \in \mathbb{Z} \big\}$ and $B = \Big\{ \big\{ \alpha \in \mathbb{Z}^2 : \alpha_1 \geqslant 0, \alpha_2 \in \mathbb{Z} \big\} \bigcup \big\{ \alpha \in \mathbb{Z}^2 : \alpha_1 \in \mathbb{Z}, \alpha_2 \geqslant 0 \big\} \Big\}$, and therefore $A \backslash B = \big\{ \alpha \in \mathbb{Z}^2 : \alpha_1 < 0, \alpha_2 < 0 \big\}$.

By Dolbeault theorem, $H^{0,1}_{\text{red}}(\mathbb{H}_1) \cong \check{H}^1_{\text{red}}(\underline{\mathcal{U}}, \mathcal{O})$. By looking at (VI.0.35) and using Theorem V.0.1, we conclude that the cohomology group $H^{0,1}_{\text{red}}(\mathbb{H}_1)$ can be represented by the Laurent series

$$\sum_{\alpha_1=-\infty}^{-1} \sum_{\alpha_2=-\infty}^{-1} c_{\alpha_1,\alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}.$$

Example 3. Let $0 < r_1 < 1$, $0 < r_2 < r_3 < 1$ and let $\mathbb{H}_2$ be a domain in $\mathbb{C}^2$ defined as

$$\mathbb{H}_2 = D_1 \cup D_2 \cup D_3, \tag{VI.0.36}$$

where

$$D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < r_2\},$$

$$D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, r_3 < |z_2| < 1\}, \text{and}$$

$$D_3 = \{(z_1, z_2) \in \mathbb{C}^2 : r_1 < |z_1| < 1, |z_2| < 1\}.$$

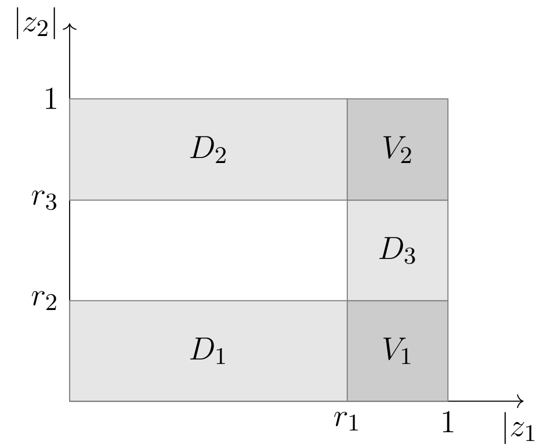

Figure 3. Reinhardt shadow of $\mathbb{H}_2$

We sketch the Reinhardt shadow of $\mathbb{H}_2$ in Figure 3 below. Using our result we can explicitly write the Dolbeault cohomology group $H^{0,1}_{\text{red}}(\mathbb{H}_2)$ in terms of Laurent series.

Let $U_1 = D_1 \cup D_2$ and $U_2 = D_3$, then $\underline{\mathcal{U}} = \{U_1, U_2\}$ is a Leray cover of $\mathbb{H}_2$. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\mathbb{H}_1$. Now

$$\check{H}^1_{\text{red}}(\mathbb{H}_2, \mathcal{O}) \cong \check{H}^1_{\text{red}}(\underline{\mathcal{U}}, \mathcal{O}) = \check{Z}^1(\underline{\mathcal{U}}, \mathcal{O}) / \overline{\check{B}^1(\underline{\mathcal{U}}, \mathcal{O})}. \tag{VI.0.37}$$

Let $V_1 = D_1 \cap D_3$ and $V_2 = D_2 \cap D_3$ and their shadows are sketched in the figure above. Observe that V_1 and V_2 are the two connected components of $U_1 \bigcap U_2$.

For $\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}^2$, let e_α be monomial in two variables: $e_\alpha(z)=z_1^{\alpha_1}z_2^{\alpha_2}$. Let

$$f_\alpha=\begin{cases} e_\alpha & \text{on } V_1 \\ 0 & \text{on } V_2 \end{cases} \tag{VI.0.38}$$

and let

$$g_\alpha=\begin{cases} e_\alpha & \text{on } V_2 \\ 0 & \text{on } V_1 \end{cases} \tag{VI.0.39}$$

and let $S=\{f_\alpha:\alpha\in\mathbb{Z}\times\mathbb{N}\}\bigcup\{g_\alpha:\alpha\in\mathbb{Z}^2\}$. Then $\overline{\operatorname{span}}\, S=\check{Z}^1(\underline{\mathcal{U}},\mathcal{O})$.

Let $T_1=\{\alpha\in\mathbb{Z}^2: f_\alpha\in\check{B}^1(\underline{\mathcal{U}},\mathcal{O})\}$. We claim that $T_1=\{\alpha:\alpha\in\mathbb{N}\times\mathbb{N}\}$. Let $f_\alpha\in\check{B}^1(\underline{\mathcal{U}},\mathcal{O})$, then

$$f_\alpha=h_1-h_2,\quad h_j\in\mathcal{O}(U_j), j=1,2. \tag{VI.0.40}$$

Now, it follows from Figure 3 that

$$h_1\big|_{D_1}\in\ \overline{\operatorname{span}}\{e_\alpha\big|_{D_1}:\alpha\in\mathbb{N}\times\mathbb{N}\}, \tag{VI.0.41}$$

$$h_1\big|_{D_2}\in\ \overline{\operatorname{span}}\{e_\alpha\big|_{D_2}:\alpha\in\mathbb{N}\times\mathbb{Z}\},\quad \text{and} \tag{VI.0.42}$$

$$h_2\in\ \overline{\operatorname{span}}\{e_\alpha:\alpha\in\mathbb{Z}\times\mathbb{N}\}. \tag{VI.0.43}$$

Since $f_\alpha=0$ on V_2, $h_1=h_2$ on V_2. Combining (VI.0.42), (VI.0.43) and the fact $h_1=h_2$ on V_2, we get

$$h_2\big|_{V_2}\in\ \overline{\operatorname{span}}\{e_\alpha:\alpha\in\mathbb{N}\times\mathbb{N}\}, \tag{VI.0.44}$$

and therefore h_2 can be extended to the entire bidisk $\{|z_1|<1,|z_2|<1\}$. Therefore it follows from (VI.0.40) and (VI.0.44) that the set $T_1=\{\alpha:\alpha\in\mathbb{N}\times\mathbb{N}\}$ and this proves our claim.

Let $T_2=\{\alpha\in\mathbb{Z}^2: g_\alpha\in\check{B}^1(\underline{\mathcal{U}},\mathcal{O})\}$. Following the same idea one can show that $T_2=\{\alpha:\alpha\in\mathbb{N}\times\mathbb{Z}\}$. Now, by Lemma VI.0.5, $\check{H}^1_{\text{red}}(\underline{\mathcal{U}},\mathcal{O})=\overline{\operatorname{span}}\ \widetilde{S}$, where

$$\widetilde{S}=\Big\{[f_\alpha]:\alpha\in(\mathbb{Z}\times\mathbb{N})\backslash T_1\Big\}\bigcup\Big\{[g_\alpha]:\alpha\in(\mathbb{Z}\times\mathbb{Z})\backslash T_2\Big\},$$

that is, $\widetilde{S} = \Big\{[f_\alpha] : \alpha_1 < 0, \alpha_2 \geqslant 0\Big\} \bigcup \Big\{[g_\alpha] : \alpha_1 < 0, \alpha_2 \in \mathbb{Z}\Big\}$. Therefore, every Dolbeault cohomology class can be represented as a Laurent series

$$\begin{cases} \sum_{\alpha_1 < 0, \alpha_2 \geqslant 0} c_\alpha z^\alpha & \text{on } V_1 \\ \sum_{\alpha_1 < 0, \alpha_2 \in \mathbb{Z}} d_\alpha z^\alpha & \text{on } V_2. \end{cases} \tag{VI.0.45}$$

BIBLIOGRAPHY

[1] D. Barrett and S. Şahutoğlu. Irregularity of the Bergman projection on worm domains in $\mathbb{C}^n$. *Michigan Math. J.*, 61(1):187–198, 2012.

[2] D. E. Barrett. Irregularity of the Bergman projection on a smooth bounded domain in C^2. *Ann. of Math. (2)*, 119(2):431–436, 1984.

[3] D. E. Barrett. Behavior of the Bergman projection on the Diederich-Fornaess worm. *Acta Math.*, 168(1-2):1–10, 1992.

[4] V. I. Bogachev and O. G. Smolyanov. *Topological vector spaces and their applications*. Springer Monographs in Mathematics. Springer, Cham, 2017.

[5] N. Bourbaki. *Integration. I. Chapters 1–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.

[6] H. Cartan. Les fonctions de deux variables complexes et le problème de la représentation analytique. *Journal de Mathématiques Pures et Appliquées*, 10:1–116, 1931.

[7] D. Catlin. Boundary behavior of holomorphic functions on pseudoconvex domains. *J. Differential Geometry*, 15(4):605–625 (1981), 1980.

[8] D. Chakrabarti. Some non-pseudoconvex domains with explicitly computable non-Hausdorff Dolbeault cohomology. *Arch. Math. (Basel)*, 105(6):571–584, 2015.

[9] D. Chakrabarti. On an observation of Sibony. *Proc. Amer. Math. Soc.*, 147(8):3451–3454, 2019.

[10] D. Chakrabarti, L. D. Edholm, and J. D. McNeal. Duality and approximation of Bergman spaces. *Adv. Math.*, 341:616–656, 2019.

[11] K. Chandrasekharan. *A course on topological groups*, volume 9 of *Texts and Readings in Mathematics*. Hindustan Book Agency, New Delhi, 1996.

[12] M. Christ. Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worm domains. *J. Amer. Math. Soc.*, 9(4):1171–1185, 1996.

[13] A. Dawn. Laurent series of holomorphic functions smooth up to the boundary. *Complex Analysis and its Synergies*, 7(2):13, Apr 2021.

[14] K. Diederich and J. E. Fornaess. Pseudoconvex domains: an example with nontrivial Nebenhülle. *Math. Ann.*, 225(3):275–292, 1977.

[15] D. S. Dummit and R. M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[16] P. Duren and A. Schuster. *Bergman spaces*, volume 100 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[17] J. B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[18] M. Hakim and N. Sibony. Spectre de $A(\bar{\Omega})$ pour des domaines bornés faiblement pseudoconvexes réguliers. *J. Functional Analysis*, 37(2):127–135, 1980.

[19] F. Hartogs. Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten. *Math. Ann.*, 62(1):1–88, 1906.

[20] J. Horváth. *Topological vector spaces and distributions. Vol. I.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[21] R. A. Johnson. Representations of compact groups on topological vector spaces: some remarks. *Proc. Amer. Math. Soc.*, 61(1):131–136 (1977), 1976.

[22] M. I. Kadets and V. M. Kadets. *Series in Banach spaces*, volume 94 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997. Conditional and unconditional convergence, Translated from the Russian by Andrei Iacob.

[23] J. L. Kelley and I. Namioka. *Linear topological spaces*. Springer-Verlag, New York-Heidelberg, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing, Graduate Texts in Mathematics, No. 36.

[24] T. W. Körner. *Fourier analysis*. Cambridge University Press, Cambridge, second edition, 1989.

[25] S. G. Krantz. *Function theory of several complex variables*. AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1992 edition.

[26] H. B. Laufer. On Serre duality and envelopes of holomorphy. *Trans. Amer. Math. Soc.*, 128:414–436, 1967.

[27] R. M. Range. *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[28] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[29] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[30] N. Sibony. Prolongement des fonctions holomorphes bornées et métrique de Carathéodory. *Invent. Math.*, 29(3):205–230, 1975.

[31] V. S. Vladimirov. *Methods of the theory of functions of many complex variables.* Translated from the Russian by Scripta Technica, Inc. Translation edited by Leon Ehrenpreis. The M.I.T. Press, Cambridge, Mass.-London, 1966.

[32] K. H. Zhu. Duality of Bloch spaces and norm convergence of Taylor series. *Michigan Math. J.*, 38(1):89–101, 1991.